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Doctoral Thesis

Sequent Calculi for Three non-Fregean Theories

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Declaration of Authorship

I, Agata Tomczyk, declare that this thesis titled, "Sequent Calculi for Three non-Fregean Theories" and the work presented in it are my own. I confirm that:

- This work was done wholly or mainly while in candidature for a research degree at this University.
- Where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated.
- Where I have consulted the published work of others, this is always clearly attributed.
- Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work.
- I have acknowledged all main sources of help.
- Where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself.

Signed:

Date:

"*... keep the Fregean axiom hidden in your pocket when entering the gate of NFL and be ready to use it at once, when you feel a confusing headache. Formally, you will be collapsing NFL into FL. Informally, you will be expelling yourself from a logical paradise into the rough, necessary world.* "

Roman Suszko
Abolition of Fregean Axiom

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Contents

Introduction

The aim of the work is to present three sequent calculi for three axiomatic extensions of non-Fregean logic called *Sentential Calculus with Identity* (SCI). Non-Fregean logics have been introduced and formalized by Polish logician, Roman Suszko, in the 60s and 70s of the XX$^{\text{th}}$ century. We will go through the historical steps that lead to their development in the next chapter, but it is worth mentioning that elements from Suszko's work appeared in other formalizations before; we will list some of them in the dissertation. Non-Fregean logics are studied mostly with regard to their semantic content, which can be presented both on the grounds of algebra or Kripke approach, but since their introduction and particularly now we notice the increase of work concerned with proof theory, too. The landscape of proof systems for non-Fregean logics of Suszko consists mostly of systems for Suszko's weakest[1] non-Fregean logic SCI: sequent-calculus style proof systems [4; 28; 35; 45; 46; 66] as well as tableaux systems [14; 16; 19; 42]. We can also find proof systems for intuitionistic variants of SCI, ISCI, in the form of sequent calculus [6; 11; 60], and natural deduction [5]. In this work we join this ongoing trend of developing structural proof systems for non-Fregean theories, thus paying special matter to non-Fregean theories stronger than SCI.

Structure of the thesis:

- Chapter 1: we sketch philosophical motivations that lead to the development of non-Fregean logics;
- Chapter 2: formal foundations for the susbsequent chapters; we introduce all notational conventions, a plethora of standard definitions rooted in algebra, as well as Classical Propositional Calculus and accompanying it sequent calculus G3cp;
- Chapter 3: a general information regarding our basic non-Fregean system Sentential Calculus with Identity and sequent calculus which will provide a base for further modifications for three extensions;
- Chapters 4 to 6: have similar structure, we will consecutively examine extensions called WB, WT and WH. We shall start with their syntactic and semantic description and follow up with sequent calculi, for which we will discuss their structural characteristics, along with certain resulting challenges;
- Chapter 7: we discuss certain limitations regarding structural approach to proving cut elimination. It appears that even though cut elimination

[1]The notion of weak/strong logic will be discussed in subsequent chapters.

has been proven for sequent calculus $\mathsf{G3}_{\mathsf{SCI}}$, which serves as the base for three calculi for axiomatic extensions examined in the thesis, the three mentioned systems do not posses the same characteristic—the cut rule can be eliminated but not without any cost. We discuss ways in which the original proof strategy fails and then discuss several problematic formulae. We also propose ways to overcome this issue;

- Chapter 8: conclusions and ideas for future work. Proof theory for non-Fregean theories is still not that well developed. There is, as a result, a plethora of ways of proceeding with research in different, non-classical context of non-Fregean logics.

Chapter 1

Philosophical background

The notion of identity has a rich background of philosophical and formal analysis, particularly (but not exclusively) developed in the second half of the 20^{th} century, which constitutes the period of interests to us.[1] Kaczmarek [31, p. 107] states that, in general, we can approach identity in one of two ways:

(a) We start with sentences that are syntactically the same, and then show that some sentences differ in shape but express the same judgment, and therefore describe the same situation.

(b) We start with equivalence of judgments and then show that there are some equivalent judgments that do not refer to the same situation.

The first strategy has been adopted by i.a. Carnap or Church. Carnap stated that in order to speak of two sentences as identical, we ought to make sure their intensions remain the same [3]. Intensions are identified with judgments and, overall, the way to study them is through truth-functions. Carnap then expanded his initial idea through the notion of intensional isomorphism, in which we can (to put it very simply) obtain two isomorphic sentences by way of replacing certain components with ones having the same intension. Church modified this theory, and instead of the sameness of intensions, proposed to utilize the notion of synonymity, thus developing synonymous isomorphism [7].

The second way of approaching identity has in turn been adopted by, for example, Vanderveken, who proposed a stronger version of implication: ϕ strongly implies χ provided the set of atomic judgments expressed in χ is contained in the set of atomic judgments expressed in ϕ. Strong implication in both directions entails both that ϕ materially implies χ and is also a better depiction of the natural use of language [63; 64].

Following the second approach, in this section we shall examine the course of philosophical events that led to the formulation of non-Fregean systems. These formulations concentrate on different treatment of the identity connective (we will be referring to the mentioned connective as identity, although it will mostly demonstrate different ways we can formalize similarity and resemblance). Ways of interpreting identity are a mixture of two approaches mentioned by Kaczmarek [31]. Naturally, we will begin with Frege's theory of meaning, based on which Suszko formulated the so-called *Fregean Axiom*. Frege's theory can be mostly attributed to approach (b). Logical equivalence provides

[1] It is worth mentioning that we will not focus in detail on formalizations, which are fundamentally not non-Fregean. We refer to the sources for more context.

a benchmark for the notion of identity. Then we will introduce the ontology of situations introduced in Wittgenstein's *Tractatus Logico-Philosophicus* and conclude with Suszko's formalization of this particular ontology. These two approaches present more of a marriage of (a) and (b). Suszko, in particular, builds axiomatic extensions of SCI through the gradual addition of equations of formulae non-identical in shape while still rooting its theory in the notion of sameness of situations denoted by sentences.

1.1 Frege's Philosophy of Language

In this section we will introduce and discuss Frege's approach to the notion of identity. We shall focus on works from throughout Frege's life, with the main emphasis on *Sense and Reference.*

> Equality gives rise to challenging questions which are not altogether easy to answer. Is it a relation? A relation between objects, or between names or signs of objects? [10, p. 56]

In *Sense and Reference* Frege investigates the conditions under which we would be able to assert whether two names/sentences are identical or not. If we were to compare two sentences—$a = a$ and $a = b$—with no controversy, then we are able to accept the validity of the former. However, if we additionally knew that the "a" and "b" refer to the same thing, ergo $a = b$ holds as well, then we would notice the difference in the cognitive value of the two sentences we compared. The validity of $a = a$ can be easily asserted, whereas the same for $a = b$ requires us to seek supplementary data. The question that arises is: what makes the latter statement hold? It does appear that both a and b, although syntactically different, denote the same object or process, but we ought to know how exactly (syntactically, semantically, etc.) we should compare a to b in order to decide the validity of $a = b$. Frege distinguishes between the sense and the reference (which we will also refer to as a semantic correlate) of the names/signs. The sense of the name is intertwined with the way we perceive, that is, with the objective result of our cognition's activity. Sense is understood by users of a given language who are familiar with its inner structure and symbolism. Take the names "Evening Star" and "Morning Star". Both of these may refer to the same object, the planet Venus, but their senses differ from each other. After all, our understanding of both these names is based upon the context in which they function. Every name (sign) is linked with a certain sense. What has to be underlined is the fact that sense, although linked with our cognitive processes, is objective. Frege does, however, highlight separate property—an associated idea—a mental representation built upon our experiences, emotions and cognitive functions. Its clarity (and clarity of its components) may differ for different people. But, as Frege is interested in the general definition of objective properties of names and sentences, an associated idea does not play a vital role in his considerations.

Sense, as we have mentioned, is a universal property. On the other hand, we can easily provide an example of a name that has a sense, but does not designate any object. Take, for instance, "the least rapidly convergent series". It expresses a sense, however there is no series satisfying the expressed conditions. Frege underlines that what matters is what a given name/sentence denotes. Therefore, following Frege's theory, the sense of the proper name is the certain thought associated with the said name, whereas the reference of the proper name is the designated object. Nonetheless, if we were to identify the reference of a sentence, it would not be a set of objects designated by names contained in it. Frege stresses that the existence of names' references allows us to determine the reference of sentences, which are truth values. Such "truth values" refer to two objects: the Truth and Falsity of a sentence. As a result, a sentence becomes a proper name for one of two truth values. Moreover, as we are comparing two sentences with regard to their semantic correlates, we can substitute any true sentence for any other of the same truth value, as all of those serve as proper names for Truth. This particular property, extensionality as *salva veritate* exchangeability, holds for any sentence that has one of two references.

However, it does not hold if we were to consider the components of any sentence. Frege provides a simple example of anomalies that such generalization would lead to—the case of subordinate sentences. If we were to analyze the following three sentences:

(a) Copernicus believed that the planetary orbits were circles.

(b) Copernicus believed that the apparent motion of the Sun was produced by the real motion of the Earth.

(c) Copernicus believed that orbits were elliptical.

we have to consider two truth values—that of the main sentence, and that of the subordinate sentences. If we were to substitute the subordinate sentence of (a) with the subordinate sentence of (b), the truth value of either main or subordinate clause would not change. However, if we were to perform the same procedure on sentences (a) and (c), we would change the truth value of the main sentence. One ought to take into consideration the reference of the two subordinate clauses ahead of substituting one for another. In the case of subordinate sentences beginning with verbs like "to believe" or "to think", Frege adopts sense (a thought) as the reference of the subordinate sentence.

In *Function and Concept*, Frege explores the differences between a general definition of a function and a function applied to a specific argument. Once more we notice that different names can denote the same object—take, for instance, "7" and "$5+2$". Both of those names denote the same object, namely the number 7. However, the first is not a function whereas we can recognize an applied function in the latter. How do we differentiate a function from a simple name of the object? Frege provides a simple way to distinguish the two: *an object is anything that is not a function, so that an expression for it does not contain any empty place* [10, p. 32]. Going back to our two names of the object 7—as an alternative to the Arabic numbers, we could have used Greek numbers, but the denoted object will not change. Frege notes that differences between names do not entail difference in the denoted objects. We can use this fact in analysis of the notion of function. Arguments of functions are nothing more than the proper names of certain objects. An argument is, as Frege notes, *a whole complete in itself*, whereas a function requires an argument to be complete. Then we can state that two given functions are equal, that is

$$(x^2 - 4x) = x(x - 4)$$

by means of their values for the same arguments.[2] The symbol “$=$” is used for the further analysis of functional expressions, as well as “$>$” and “$<$”. Now, Frege underlines that the logical value of a sentence is its truth function. As it is the case that $2^2 = 4$ and $2 > 1$ are both True, hence

$$(2^2 = 4) = (2 > 1)$$

is True as well. Of course, one could argue whether the use of “$=$” could be appropriate to both comparison of the result of the function application (which could be viewed as names) and comparison of the two equations (similarly, sentences).

Frege's theory can be depicted in the diagram below where sentences are denoted by ψ, referents (denotations) of ψ by $r(\psi)$, senses of ψ by $s(\psi)$ and the logical value by $t(\psi)$.[3]

$$\begin{tikzcd}
s(\psi) \arrow[dr] & \psi \arrow[l] \arrow[r, dotted] \arrow[d] & t(\psi) \\
 & r(\psi) \arrow[ur, dotted] &
\end{tikzcd}$$

1.2 Wittgenstein's ontology

> 6.13 Logic is not a theory but a reflexion of the world.

If we were to utilize Wittgenstein's conceptual framework, we would say that Frege's logical space consisted of two elements: Truth and Falsity. Wittgenstein disagreed with this statement, and defined a world as a logical space filled with facts, also referred to as a certain *states of affairs*. In *Tractatus*, Wittgenstein analyzes the relationship between language and the said world. And, although we may find theses in which Wittgenstein describes such a relation as a reflection, Wolniewicz [72] underlines that it is not an isomorphism (which could be deduced from the mirror analogy), but a homomorphism. A homomorphism would be more like a shadow—we may recognize the overall picture, but specific details are still obscured to our eyes. Therefore we can say that language's shadow is visible on the world and *vice versa*.

> 4.014 The gramophone record, the musical thought, the score, the waves of sound, all stand to one another in that pictorial internal

[2]The same goes for the curve we get from both of these functions.

[3]Additional comments (and the diagram itself) on relations shown on the scheme can be found in [51].

> relation, which holds between language and the world. To all of them the logical structure is common.

Wittgenstein presents the ontology of situations: a theory that stands in opposition to Frege's understanding of semantic correlates of sentences containing non-empty names. We cannot say that a given sentence ϕ denotes truth values. Wittgenstein, by focusing on facts as certain kinds of building blocks of the world, provides a better reference point as to what the sentences relate to. And what do true sentences refer to? This would be answered years later by Suszko through his interpretation of Wittgenstein's ontology.

> 4.022 A proposition shows its sense. A proposition shows how things stand if it is true. And it says that they do so stand.

> 4.03 A proposition must use old expressions to communicate a new sense. A proposition communicates a situation to us, and so it must be essentially connected with the situation. And the connection is precisely that it is its logical picture. A proposition states something only in so far as it is a picture.

Of course, we can not ignore Wittgenstein's attitude towards the identity sign, which was far from welcoming. Wittgenstein did not see a need to utilize the identity sign at all but, at the same time, did not reject the overall notion of identity.

> 5.53 Identity of the object I express by identity of the sign and not by means of a sign of identity. Difference of the objects by difference of the signs.

Identity of objects/situations can be expressed by way of using the same sign and not through the use of the sign of identity. This way we can formulate certain facts in a different manner, for example $F(a, b) \wedge a = b$ can be replaced by $F(a, a)$ and $F(a, b) \wedge a \neq b$ is replaced by $F(a, b)$[4]. This particular procedure also requires us to adapt the exclusive reading of variables. Hintikka showed that First Order Logic without the equality sign would still be just as expressive, however he did not see many benefits in Wittgenstein's approach over a more inclusive reading of variables (which, in turn, would require a presence of equality sign) [22].

1.3 Suszko's abolition of the Fregean Axiom

A (simplified) version of Frege's denotational theory can be portrayed by the following statement:

$$(\phi \leftrightarrow \chi) \leftrightarrow (\phi \equiv \chi) \tag{FA}$$

[4] If we were to move to First Order Logic, we could e.g. replace $\exists_x \exists_y (F(x, y) \wedge x = y)$ with $\exists_x F(x, x)$ and replace $\exists_x \exists_y (F(x, y) \wedge x \neq y)$ with $\exists_x \exists_y F(x, y)$.

which signifies that the sameness of the truth values of two given statements shall also entail identity of their denotations. Suszko, spurred by Wittgenstein's ontology, unequivocally rejected it. He called it the "Fregean axiom" (FA) and decided to construct a formalization of *Tractatus* which would include a notion of situation (as a denotation of a given sentence), thus introducing a new system—non-Fregean logic (NFL). Suszko became acquainted and infatuated with Wittgenstein's ontology through Bogusław Wolniewicz and his work. He reviewed a work by Wolniewicz, entitled *Study of Wittgenstein's Philosophy* [55]; his review was untypically lengthy, and established a new subject matter to which Suszko would remain faithful until his death. Suszko, in his goal to formalize *Tractatus*, could not accept well formed formulae as names for the truth values. Following Wittgenstein he proposed a number of NFLs by adding a predicate and binary connective expressing the identity of two situations, namely "$\equiv$"[5]. Although the new systems were called "non-Fregean", Suszko did not reject Frege's accomplishments; he kept the two-valuedness, although in Suszko's case two-valuedness is not a conceptualization of its ontological interpretation. Suszko proposed a more general approach in defining NFLs—the set of situations has to consist of at least two elements, which is depicted by the law:

$$\neg(\phi \equiv \neg\phi). \tag{\#}$$

Moreover, Suszko distinguished two different categories of variables: one, already mentioned, consisting of sentential variables running through the set of different situations, and the second of nominal variables, running through the universe of objects. Furthermore, Suszko claimed that the so-called "Fregean Logic" (FL) is merely an instance of an NFL with the set of situations consisting of two objects. To illustrate this, we can say that any non-Fregean theory in which all of the below theorems hold:

(1.1) $(\phi \equiv \chi) \vee (\phi \equiv \psi) \vee (\chi \equiv \psi)$;

(1.2) $(\phi \leftrightarrow \chi) \leftrightarrow (\phi \equiv \chi)$;

(1.3) $(\phi \equiv \chi) \equiv (\phi \leftrightarrow \chi)$

is called a Fregean theory within NFL.

Suszko proposed a number of $\mathsf{NFL}s$. We will later on focus on three extensions of the logic SCI (*Sentential Calculus with Identity*), however it has to be noted that there are uncountably many different extensions of the basic NFL theory (which are independent of each other) [13; 18]. The base of the said systems is SCI, which has been obtained from the Classical Propositional Calculus by the introduction of identity and the axioms characterizing it:

($\equiv_1$) $\phi \equiv \phi$

($\equiv_2$) $(\phi \equiv \chi) \rightarrow (\neg\phi \equiv \neg\chi)$

[5]Suszko used the same symbol for both the predicate and binary connective; the context of the sentence predetermines which of the two logical constants is being used [38].

($\equiv_3$) $(\phi \equiv \chi) \to (\phi \leftrightarrow \chi)$

($\equiv_4$) $((\phi \equiv \psi) \wedge (\chi \equiv \omega)) \to ((\phi \otimes \chi) \equiv (\psi \otimes \omega))$

where $\otimes \in \{\wedge, \vee, \to, \leftrightarrow, \equiv\}$ with the following interpretation of the above axioms:

($\equiv_1$): The denotation of every formula is identical with itself.

($\equiv_2$): If the denotations of formulae ϕ and χ are identical, then so too are the denotations of their negations.

($\equiv_3$): The identity connective is stronger than equivalence.

($\equiv_4$): Identity of denotations of arguments of a binary connective entails identity of denotations of the complex formula.

From the above interpretations it is clear that identity is a congruence with regard to unary and binary connectives.

The first axiom is the only one with the identity as the main connective. It illustrates the strength of the theory: the identity is strictly syntactical (in the sense of proposition 5.53 from Wittgenstein's *Tractatus*).

Woleński questions in [71] the notion of (FA). He argues that Frege himself was quite vague on the matter. If we were to investigate the conditions under which two proper names would be said to be identical/equal, those conditions would require far stronger foundations than equivalence of two names functioning as names for the truth values. Woleński recalls examples Frege provides in *Function and Concept* to illustrate the different meaning of the sign "=". In the example we mentioned above

$$(2^2 = 4) = (2 > 1) \tag{$\sharp$}$$

$(2^2 = 4)$ and $(2 > 1)$ express different thoughts, but the same meaning, which is their logical value. The equality sign is used to compare both kinds of entities: numbers and logical values. However, it does seem that the two names require stronger equality, although Woleński states that Frege would not say that those two names (from equation $\sharp$) are identical, since identity is a relation between objects and not names.

We also have to underline the role of non-Fregean logic in the context of linguistics. In [39] Omyła examines the relation between natural language and non-Fregean logics. Naturally, the introduction of the identity connective and formalization of Wittgenstein's ontology allows us to better depict the actual use of language. Not all sentences of the same logical value describe the same situation. Therefore the core philosophical foundation of NFL is a better fit for intuitive use of natural language. Moreover, through addition of the identity connective we obtain more expressive language. There are issues, though. In natural language there are certain intensional phrases that cannot be formalized in NFL. The same situation may be described by two different sentences ϕ and χ and someone may believe ϕ but at the same time not believe χ. Moreover, certain NFLs correspond to certain modal systems, but the above observation regarding natural language may provide a starting point in the research of merging two approaches to better encapsulate formalization of natural language use. We may also take into consideration the *framing effect*[6] [62], which also circles around the notion of our perception of identity of situation (e.g. a glass can be either half full or half empty, but the situation remains the same). Nevertheless, the two approaches to the notion of identity we listed at the beginning of the chapter can both be utilized within this process of more accurate formalization of the notion of identity. In three extensions we will examine in consecutive chapters we will show how the two methods are jointly used to gradually introduce more valid equations.

[6]Framing effect relates to one of the cognitive biases when people tend to differently interpret the same situation depending on the way it was presented to them.

Chapter 2

Logical preliminaries and notation

In this section we will introduce the notation and general definitions and theories that will be referred to throughout the thesis. Suszko proposed an algebraic approach to the semantics of non-Fregean logics, and we will follow in his footsteps. We start with the below list encompassing notation used in the thesis:

- lowercase Latin alphabet letters $p, q, ...$ will denote propositional variables;
- lowercase Greek alphabet letters $\phi, \chi, ...$ will denote formulae;
- uppercase Greek alphabet letters such as $\mathbf{\Phi}$, $\mathbf{\Psi}$ will denote sets of formulae;
- uppercase Greek alphabet letters such as $\Gamma, \Delta, ...$ will denote multisets of formulae;
- uppercase Latin alphabet letters $\mathcal{A}, \mathcal{B}, ..., \mathcal{L}, ...$ will denote algebraic structures;
- uppercase Latin alphabet letters $A, B, ..., L, ...$ will denote universes in algebraic structures $\mathcal{A}, \mathcal{B}, ..., \mathcal{L}, ...$, whereas lowercase $a, b, ...$ will denote elements of the mentioned universes;
- the following symbols $\wedge, \vee, \rightarrow, \leftrightarrow, \equiv$ refer to binary syntactical operations on formulae, respectively: conjunction, disjunction, implication, equivalence, identity; $\neg$ is the only unary operation on formulae considered in this work and two constants utilized in the work are: $\top, \bot$;
- the symbols $\dot{\cap}, \dot{\cup}, \dot{\rightarrow}, \dot{\leftrightarrow}, \dot{\equiv}$ refer to semantic equivalents of the above symbols, whereas $\dot{\neg}$ corresponds to $\neg$;
- by "$A \subseteq B$" we will mean that A is a subset of B and by "$A \subset B$" we will mean A is a proper subset of B, whereas power set of B will be denoted by "2^B";
- the sign "$=$" of identity will be used throughout the thesis in numerous contexts (both syntactic and semantic), e.g. to compare two numerical expressions, two elements of a given algebra, as definitional equivalence ($=_{df}$) and so on. We will additionally underline the desired use of this sign if it is not clear from the context;

- for a given functions f and g, their inverse functions will be written as f^{-1} and g^{-1} and the composition of f and g will be written as $f \circ g$; restriction of function f to set Ψ will be written as $f|_{\Psi}$.

2.1 Algebra

We will stay faithful to the algebraic approach Suszko adopted for describing his non-Fregean systems. We shall begin with basic algebraic constructions, some of which will later be adapted to a given non-Fregean theory. Most of the definitions and theorems shown below can be originally found in [38] and [43]. We will omit the detailed proofs of some of the following theorems and lemmas, as they can be found in the source material, however some of the proofs will be fully included.

Definition 1 (Abstract algebra [43, p. 22])**.** *A pair $\langle A, \{o_i\}_{i \in X} \rangle$ where A is a non-empty set and for every i, o_i is an operation in A, is called an abstract algebra (or: algebra).*

The cardinality of the set X can be arbitrary, finite or infinite, but later in the work we will consider finite sets of operations. Consequently, we will refer to a given algebra as an ordered tuple and write it as $\langle A, o_1, ..., o_n \rangle$ to simplify the notation.

Definition 2 (Similar algebras [43, p. 23])**.** *Two algebras*

$$\mathcal{A} = \langle A, o_1, o_2, ..., o_n \rangle \text{ and } \mathcal{B} = \langle B, o_1^*, o_2^*, ..., o_m^* \rangle$$

are called similar iff $n = m$ and for any $j = 1, 2, ..., n$ operations o_j and o_j^* have the same arity, which is denoted by v_j. Then $\langle v_1, ..., v_n \rangle$ is called similarity type of the algebras.

Definition 3 (Homomorphism [38, p. 12])**.** *Let $\mathcal{A} = \langle A, o_1, o_2, ..., o_n \rangle$ and $\mathcal{B} = \langle B, o_1^*, o_2^*, ..., o_n^* \rangle$ be similar algebras. Homomorphism from $\mathcal{A}$ to $\mathcal{B}$ is a function from A to B, such that for all $i = 1, 2, ..., n$ and for any sequence of elements: $a_1, a_2, ..., a_{v_i} \in A$ the following condition is met:*

$$h(o_i(a_1, a_2, ..., a_{v_i})) = o_i^*(h(a_1), h(a_2), ..., h(a_{v_i})).$$

Homomorphisms are also called structure-preserving functions. By $Hom(\mathcal{A}, \mathcal{B})$ we mean the set of homomorphisms from $\mathcal{A}$ to $\mathcal{B}$. Elements of $Hom(\mathcal{A}, \mathcal{A})$ are called endomorphisms of $\mathcal{A}$.

Definition 4 (Equivalence relation [43, p. 20])**.** *Let $\sim$ stand for a binary relation in a non-empty set A. $\sim$ is an equivalence relation in A if it is reflexive, symmetric and transitive in A, i.e., for arbitrary elements $a, b, c \in A$:*

- *$a \sim a$,*
- *if $a \sim b$, then $b \sim a$,*
- *if $a \sim b$ and $b \sim c$, then $a \sim c$.*

If $\sim$ is an equivalence relation in A, then by $|a| = \{b : a \sim b\}$ we will mean the equivalence class of a. By $A/_\sim = \{|a| : a \in A\}$ we understand the set of all equivalence classes of elements of A.

Definition 5 (Congruence relation [43, p. 25])**.** *Let $\mathcal{A} = \langle A, o_1, ..., o_n\rangle$ be an algebra and let $\sim$ be an equivalence relation in A. $\sim$ is said to be a congruence relation in $\mathcal{A}$ if conditions $a_1 \sim a'_1, ..., a_{v_i} \sim a'_{v_i}$ imply $o_i(a_1, ..., a_{v_i}) \sim o_i(a'_1, ..., a'_{v_i})$ for each $o_i, 1 \leq i \leq n$.*

Definition 6 (Quotient algebra [43, p. 26])**.** *Let $\mathcal{A} = \langle A, o_1, ..., o_n\rangle$ be an algebra and $\sim$ be a congruence relation in $\mathcal{A}$. The quotient algebra of $\mathcal{A}$ by $\sim$, symbolically $\mathcal{A}/_\sim = \langle A/_\sim, o^*_1, ..., o^*_n\rangle$, is the algebra whose universe is the set $A/_\sim$ of all equivalence classes $|a|$ $(a \in A)$ and whose all operations o^*_i are defined by*

$$o^*_i(|a_1|, ..., |a_{v_i}|) = |(o_i(a_1, \ldots, a_{v_i}))|.$$

Definition 7 (Ordering [43, p. 32])**.** *A binary relation $\leq$ in a set A is said to be an ordering in A if it is reflexive, antisymmetric and transitive in A, i.e., if for arbitrary $a, b, c \in A$*

1. *$a \leq a$,*
2. *if $a \leq b$ and $b \leq a$, then $a = b$,*
3. *if $a \leq b$ and $b \leq c$, then $a \leq c$.*

If $a \leq b$, then we say that a is included in b. Instead of writing $a \leq b$, we will sometimes write $b \geq a$.

Definition 8 (Ordered set [43, p. 32])**.** *If A is a non-empty set and $\leq$ a fixed ordering in A, then $\mathcal{A} = \langle A, \leq\rangle$ is called an ordered set.*

Definition 9 (Linear ordering [43, p. 33])**.** *An ordering $\leq$ defined in a set A is said to be a linear ordering in A provided it also satisfies the additional condition: for arbitrary $a, b \in A$, either $a \leq b$ or $b \leq a$.*

Definition 10 (Chain [43, p. 33])**.** *An ordered set $\mathcal{A} = \langle A, \leq\rangle$ is called a chain provided $\leq$ is a linear ordering relation in A.*

Definition 11 (Upper bound, lower bound [43, p. 34])**.** *Let $\mathcal{A} = \langle A, \leq\rangle$ be an ordered set and let S be a non-empty subset of A. An element $a_0 \in A$ is said to be an upper (lower) bound of S in $\mathcal{A}$ provided $a \leq a_0$ $(a \geq a_0)$ for all $a \in S$. If the set of all upper (lower) bounds of S contains the least (greatest) element, it is called the least upper bound (the greatest lower bound) of S in $\mathcal{A}$ and denoted by $l.u.b.S$ $(g.l.b.S)$.*

Definition 12 (Lattice [43, p. 34])**.** *An ordering $\leq$ in a set $\mathcal{A} = \langle A,\leq\rangle$ is said to be a lattice ordering if, for each $a, b \in A$, the elements $l.u.b.(a, b)$ and $g.l.b.(a, b)$ exist. Then the ordered set $\mathcal{A}$ is said to be a lattice; the least upper bound of $a, b \in A$ will be denoted by $a \dot{\cup} b$ and called the join of elements a, b, and the greatest lower bound of $a, b \in A$ will be denoted by $a \dot{\cap} b$ and called the meet of a, b.*

Definition 13 (The unit element, the zero element [43, p. 37])**.** *The greatest (least) element of a lattice $\mathcal{A}$, if it exists, will be called the unit element (the zero element) of $\mathcal{A}$ and will be denoted by* 1 *(by* 0*).*

If $\mathcal{A} = \langle A, \leq \rangle$ is a lattice with $0, 1 \in A$, then for every $a \in A$:

1. $a \leq 1$, $a \geq 0$,
2. $a \dot{\cup} 1 = 1$, $a \dot{\cap} 1 = a$,
3. $a \dot{\cup} 0 = a$, $a \dot{\cap} 0 = 0$.

Definition 14 (Filter [43, p. 44])**.** *A non-empty set F of elements of a lattice $\mathcal{A} = \langle A, \leq \rangle$ is said to be a filter in $\mathcal{A}$ provided for any elements $a, b \in A$: $a \dot{\cap} b \in F$ iff $a \in F$ and $b \in F$.*

A filter F in $\mathcal{A} = \langle A, \leq \rangle$ is said to be:

1. proper if it is a proper subset of A;
2. maximal in $\mathcal{A}$ provided it is proper and it is a maximal element in the ordered set of all proper filters in $\mathcal{A}$; F is then called an ultrafilter of $\mathcal{A}$;
3. prime iff for any $a, b \in F$: if $a \dot{\cup} b \in F$ then $a \in F$ or $b \in F$.

Definition 15 (Distributive lattice [43, p. 48])**.** *A lattice $\mathcal{A} = \langle A, \leq \rangle$ is said to be distributive if, for all $a, b, c \in A$,*

$$a \dot{\cap} (b \dot{\cup} c) = (a \dot{\cap} b) \dot{\cup} (a \dot{\cap} c) \quad \textit{and} \quad a \dot{\cup} (b \dot{\cap} c) = (a \dot{\cup} b) \dot{\cap} (a \dot{\cup} c).$$

For the following several theorems we will point to their sources, where the reader can find proofs. For the selected theorems that play a more crucial role in the main topic of the thesis, we will provide proofs (which, in some cases, were originally written in Polish).

For every fixed element $a_0 \in A$ (where $\mathcal{A}$ is a distributive lattice), the set of all elements $a \geq a_0$ is a filter called the principal filter generated by a_0 [43, p. 46].

Theorem 1. *Let $\mathcal{A} = \langle A, \leq \rangle$ be a lattice. For any fixed element $a_0 \in A$ and a filter F in $\mathcal{A}$, the class of all elements a such that*

$$a \geq a_0 \dot{\cap} c \textit{ for an element } c \in F$$

is the least filter in $\mathcal{A}$ containing a_0 and F.

The least filter containing a_0 and F, as described in the above theorem, will be called filter generated by the set $(a_0) \cup F$.[1]

Theorem 2. *The union of any chain of filters in a lattice $\mathcal{A}$ is a filter in $\mathcal{A}$. The union of any chain of proper filters in a lattice $\mathcal{A}$ having the zero element is a proper filter.*[2]

[1] Theorem and comments can be found in [43, p. 46].

[2] As above, [43, p. 46].

Now we will provide two different, yet equivalent definitions for Boolean algebra. Later in the thesis we will utilize both approaches, one defining Boolean algebras as distributive lattices, and the other through several conditions to be met by its elements.

Definition 16 (Boolean algebra, [43, p. 68])**.** *A Boolean algebra is a distributive lattice $\mathcal{A} = \langle A, \leq \rangle$ in which with every $a \in A$ there is associated an element $\dot\neg a$ such that*

$$(a \mathbin{\dot\cap} \dot\neg a) \mathbin{\dot\cup} b = b \qquad \textit{and} \qquad (a \mathbin{\dot\cup} \dot\neg a) \mathbin{\dot\cap} b = b.$$

Definition 17 (Boolean algebra [38, p. 42])**.** *An algebra*

$$\mathcal{A} = \langle A, \dot\neg, \dot\cap, \dot\cup, \dot\rightarrow, \dot\leftrightarrow \rangle$$

of similarity type $\langle 1, 2, 2, 2, 2 \rangle$ is called a Boolean algebra iff A is an arbitrary but at least two-element set and operations satisfy the following conditions:

(1) $((a \mathbin{\dot\cap} b) \mathbin{\dot\cup} c) = ((b \mathbin{\dot\cup} c) \mathbin{\dot\cap} (a \mathbin{\dot\cup} c))$,

(2) $((a \mathbin{\dot\cup} b) \mathbin{\dot\cap} c) = ((b \mathbin{\dot\cap} c) \mathbin{\dot\cup} (a \mathbin{\dot\cap} c))$,

(3) $(a \mathbin{\dot\cup} (b \mathbin{\dot\cap} \dot\neg b)) = a$,

(4) $(a \mathbin{\dot\cap} (b \mathbin{\dot\cup} \dot\neg b)) = a$,

(5) $(a \mathbin{\dot\rightarrow} b) = (\dot\neg a \mathbin{\dot\cup} b)$,

(6) $(a \mathbin{\dot\leftrightarrow} b) = ((a \mathbin{\dot\rightarrow} b) \mathbin{\dot\cap} (b \mathbin{\dot\rightarrow} a))$.

Definition 18 (Congruence relation determined by filter [43, p. 63])**.** *Let $\mathcal{A}$ be a lattice and F a filter in $\mathcal{A}$. Let $\sim$ be a binary operation on elements of algebra $\mathcal{A}$. $\sim$ is a congruence relation determined by filter F in $\mathcal{A}$ provided we have the following condition satisfied:*

$$a \sim b \textit{ iff } (a \in F \textit{ iff } b \in F).$$

If $\sim$ is a congruence relation determined by filter F in $\mathcal{A}$, then by $\mathcal{A}/_F$ we will mean a quotient algebra $\mathcal{A}/_\sim$.

Every pseudo-Boolean algebra is also a Boolean algebra. As a result, we apply the following theorem to Boolean algebras (even though in [43, p. 49] it relates to the former).

Theorem 3 ([43, p. 66])**.** *The following conditions are equivalent for every filter F of a Boolean algebra $\mathcal{A}$:*

(a) the filter F is maximal;

(b) the filter F is prime;

(c) the filter F is proper and, for every $a \in A$, either a or $\dot\neg a$ belongs to F;

(d) *for every* $a \in A$, *exactly one of the elements* $a, \dot{\neg}a$ *belongs to* F;

(e) *the quotient algebra* $\mathcal{A}/_F$ *is a two-element Boolean algebra.*

For the Theorem 4 we will refer to the Kuratowski-Zorn Lemma, which is the following:

Lemma 1 (Kuratowski–Zorn Lemma)**.** *For a given ordered set* A*: if every chain of elements of* A *has an upper bound in* A*, then* A *contains a maximal element (or: for any* $a_0 \in A$ *there exists a maximal element* $a \geq a_0$*).*

Theorem 4. *For arbitrary elements* a, b *of a distributive lattice* $\mathcal{A}$*, if the relation* $b \leq a$ *does not hold, then there exists a prime filter* F *such that*

$$a \notin F \text{ and } b \in F. \tag{*}$$

Proof. The proof that follows can be found in [43, p. 49-50]. Let $\mathcal{A} = \langle A, \leq \rangle$ be a distributive lattice, and $a, b, c, d, ... \in A$ as well as $b \not\leq a$. Let $\mathcal{A}^* = \langle A^*, \subseteq \rangle$ be an ordered (by inclusion) set of all filters in distributive lattice $\mathcal{A}$ such that (*) holds. A^* is not empty as it contains the principal filter generated by b. From Theorem 2 we have that any chain of $\mathcal{A}^*$ has an upper bound in $\mathcal{A}^*$, so by Lemma 1 there exists maximal element of $\mathcal{A}^*$, F which satisfies (*).

We show that F is prime. Suppose it is not the case. Therefore there are two elements c, d such that $c \mathbin{\dot{\cup}} d \in F$, but at the same time $c \notin F$ and $d \notin F$. Let F_c be a filter generated by $(c) \cup F$ (respectively let F_d be generated by $(d) \cup F$). We show that one of F_c and F_d does not contain a. From Theorem 1 we have: if

$$a \in F_c \text{ and } a \in F_d, \tag{**}$$

then there are $e, f \in F$ such that $a \geq c \mathbin{\dot{\cap}} e$ and $a \geq d \mathbin{\dot{\cap}} f$. Let $g = e \mathbin{\dot{\cap}} f$. We have $g \in F$, and $a \geq c \mathbin{\dot{\cap}} g$ and $a \geq d \mathbin{\dot{\cap}} g$, and from that $a \geq (c \mathbin{\dot{\cap}} g) \mathbin{\dot{\cup}} (d \mathbin{\dot{\cap}} g) = (c \mathbin{\dot{\cup}} d) \mathbin{\dot{\cap}} g \in F$. Therefore $a \in F$, which contradicts the fact that F satisfies (*). Hence (**) does not hold. It follows that if F is not prime, than one of F_c, F_d does not contain a, and hence it satisfies (*). But since $c \notin F$ and $d \notin F$, F is not maximal. A contradiction. Thus F is prime. □

Naturally, as the above theorem refers to lattices, it also refers to Boolean algebras as every Boolean algebra is a lattice.

2.2 Classical Propositional Calculus

In the following section we will present sequent calculus (SC, for short) for Classical Propositional Calculus (CPC, for short)—G3cp[3], which will be used as a base for sequent calculi introduced in the following chapters. We start with introducing the language(s).[4] We will begin with the triple $\langle \mathsf{Var}, F, v \rangle$ which

[3] G3cp constitutes a multi-conclusion (that is, such that more than one formula is allowed to occur in the succedent of a sequent) sequent calculus examined i.a. by Negri and von Plato in [36]. See section 2.3.1.

[4] This particular approach can be found in [38, pp. 14–15].

will be called an alphabet. An alphabet provides the base for our language. Let $\mathsf{Var} = \{p_1, p_2, ...\}$ be a countably infinite set of propositional variables, let $F = \{F_1, F_2, ..., F_n\}$ be a set of connectives and v be a function assigning to every connective its arity. Set L^* of formulae in alphabet $\langle \mathsf{Var}, F, v\rangle$ is defined inductively as the smallest set such that:

1. $\mathsf{Var} \subseteq L^*$,
2. if $\phi_1, \phi_2, ..., \phi_{v_j} \in L^*$, then finite sequence $F_j\phi_1\phi_2...\phi_{v_j}$ belongs to L^* for any $j = 1, 2, ..., n$.

Following that we obtain algebra where connectives are treated as operations in L^*:

$$\mathcal{L}^* = \langle L^*, F_1, F_2, ..., F_n\rangle.$$

All languages introduced in this thesis will be defined according to the above scheme. We now define the language utilized in the subsequent sections.

Definition 19 (Language $\mathcal{L}_{\mathsf{CPC}}$ [38, p. 14–15])**.** *Language $\mathcal{L}_{\mathsf{CPC}}$ is the following algebra of similar type* $\langle 1, 2, 2, 2, 2\rangle$

$$\mathcal{L}_{\mathsf{CPC}} = \langle L_{\mathsf{CPC}}, \neg, \wedge, \vee, \rightarrow, \leftrightarrow\rangle.$$

In the subsequent theories considered in this thesis we will add other connectives, such as $\top$ and $\equiv$.

For the reader's convenience and contrary to the above definitions we will use binary connectives in an infix manner. We assume $\neg$ binds stronger than any binary connective $\otimes$. $\wedge$ and $\vee$ bind stronger than $\rightarrow$ and $\leftrightarrow$. We will utilize parentheses whenever there is a risk of confusion. We will also omit the most external parentheses of expressions.

We consider the Hilbert system for CPC consisting of Truth-Functional Axioms (TFA). We consider the following schemes:

1. $\phi \rightarrow (\chi \rightarrow \phi)$,
2. $(\phi \rightarrow (\chi \rightarrow \psi)) \rightarrow ((\phi \rightarrow \chi) \rightarrow (\phi \rightarrow \psi))$,
3. $\neg\phi \rightarrow (\phi \rightarrow \chi)$,
4. $(\phi \rightarrow \chi) \rightarrow ((\neg\phi \rightarrow \chi) \rightarrow \chi)$,
5. $(\phi \leftrightarrow \chi) \rightarrow (\phi \rightarrow \chi)$,
6. $(\phi \leftrightarrow \chi) \rightarrow (\chi \rightarrow \phi)$,
7. $(\phi \rightarrow \chi) \rightarrow ((\chi \rightarrow \phi) \rightarrow (\phi \leftrightarrow \chi))$,
8. $(\phi \wedge \chi) \leftrightarrow (\neg(\phi \rightarrow \neg\chi))$,
9. $(\phi \vee \chi) \leftrightarrow (\neg\phi \rightarrow \chi)$.

Additionally, we consider a singular inference rule, *modus ponens*

$$\frac{\begin{array}{l}\phi\\ \phi \rightarrow \chi\end{array}}{\chi}$$

Definition 20 (Derivation, proof)**.** *Let Φ stand for a set of formulae of $\mathcal{L}_{\mathsf{CPC}}$. A finite sequence $\phi_1, ..., \phi_n$ of formulae of $\mathcal{L}_{\mathsf{CPC}}$ is a derivation of ϕ from Φ provided $\phi_n = \phi$ and each formula $\phi_i, i \leq n$, either belongs to Φ or has been derived from some ϕ_{i_1}, ϕ_{i_2}, $(i_1, i_2 < i)$ through an application of modus ponens. If $\Phi = \mathsf{TFA}$, then derivation $\phi_1, ..., \phi_n$ is a proof of ϕ in the axiomatic system for CPC.*

If $\phi_1, ..., \phi_n$ is a proof of ϕ in the axiomatic system for CPC, then the length of this proof is n.

There are two central features of logical consequence, as introduced by Tarski [57; 58]: truth preservation and formality. In regard to the first feature, it expresses the fact that it cannot be the case that at the same time both ϕ follows from a set of formulae Φ and ϕ is simultaneously false with formulae from Φ being true. Formality in turn refers to the fact that a consequence relation cannot depend on or in any way be influenced by empirical knowledge; it is strictly interconnected with the form of the sentences. A consequence relation is also not affected by the process of replacing names with other ones, provided they refer to the same object (cf. [29]). We now present a general definition of a consequence operation.

Definition 21. *Let $\mathcal{L}$ be a sentential language with L being the set of its formulae. Function $Cn : 2^L \mapsto 2^L$ is a consequence operation on $\mathcal{L}$ provided it satisfies the following conditions:*

$(C1)$ $\Phi \subseteq Cn(\Phi)$, *for all* $\Phi \subseteq L$*;*

$(C2)$ *If* $\Phi \subseteq \Psi$, *then* $Cn(\Phi) \subseteq Cn(\Psi)$; *for all* $\Phi, \Psi \subseteq L$*;*

$(C3)$ $Cn(Cn(\Phi)) = Cn(\Phi)$, *for all* $\Phi \subseteq L$.

As we can see, we utilize the notion of “consequence operation” in Tarski’s sense. A consequence operation as defined in Definition 21 satisfies Tarski’s conditions, that is reflexivity $(C1)$, monotonicity theorem of operation Cn $(C2)$ and closure condition $(C3)$.

By a substitution e in $\mathcal{L}$ we will understand endomorphism $e \in Hom(\mathcal{L}, \mathcal{L})$.

Definition 22 (Structural consequence operation)**.** *Consequence operation Cn in a given language $\mathcal{L}$ is called a structural consequence operation in $\mathcal{L}$ if for any substitution e in language $\mathcal{L}$ and for any $\phi \in L$ and $\Phi \subseteq L$ the following condition holds:*

$$\text{if } \phi \in Cn(\Phi), \text{ then } e\phi \in Cn(e\Phi).$$

We will not focus on the consequence operation with regard to the CPC, as it is not the main topic of the thesis, we will however define it with regard to non-Fregean systems in the subsequent chapters.

As a deductive system we will, following Suszko, understand a pair $\langle \mathcal{L}, Cn \rangle$, where $\mathcal{L}$ is a language algebra and Cn is a structural consequence operation defined on $\mathcal{L}$.

As for the semantics, we will briefly introduce the few definitions using Boolean algebra structures introduced in the previous section.

Definition 23 (CPC-model of language $\mathcal{L}_{\mathsf{CPC}}$)**.** *A structure of the form $\langle \mathcal{A}, F\rangle$ is called a* <u>CPC*-model of language* $\mathcal{L}_{\mathsf{CPC}}$</u> *if and only if $\mathcal{A}$ is an algebra similar to the language and F is any subset of A such that for any $a, b \in A$ the following conditions are met:*

1. $a \in F \Leftrightarrow \dot{\neg} a \notin F$,
2. $a \dot{\cap} b \in F \Leftrightarrow a \in F$ *and* $b \in F$,
3. $a \dot{\cup} b \in F \Leftrightarrow a \in F$ *or* $b \in F$,
4. $a \dot{\rightarrow} b \in F \Leftrightarrow a \notin F$ *or* $b \in F$,
5. $a \dot{\leftrightarrow} b \in F \Leftrightarrow a, b \in F$ *or* $a, b \notin F$.

*Later on we will refer to this structure simply as the "*CPC*-model".*

Definition 24 (Valuation)**.** *Let $\mathcal{A} = \langle A, \dot{\neg}, \dot{\cap}, \dot{\cup}, \dot{\rightarrow}, \dot{\leftrightarrow}\rangle$ be an algebra similar to $\mathcal{L}_{\mathsf{CPC}}$.* <u>*Valuation of language $\mathcal{L}_{\mathsf{CPC}}$ into $\mathcal{A}$*</u> *is a homomorphism h from $\mathcal{L}_{\mathsf{CPC}}$ to $\mathcal{A}$, that is, a function from L_{CPC} to A fulfilling the following conditions:*

1. $h(\neg\phi) = \dot{\neg} h(\phi)$,
2. $h(\phi \vee \chi) = h(\phi) \dot{\cup} v(\chi)$,
3. $h(\phi \wedge \chi) = h(\phi) \dot{\cap} v(\chi)$,
4. $h(\phi \rightarrow \chi) = h(\phi) \dot{\rightarrow} h(\chi)$,
5. $h(\phi \leftrightarrow \chi) = h(\phi) \dot{\leftrightarrow} h(\chi)$.

Now we turn to the notions of the satisfiability, truth and validity of a given formula ϕ of $\mathcal{L}_{\mathsf{CPC}}$.

Definition 25. *Let $\mathcal{M} = \langle \mathcal{A}, F\rangle$ be an arbitrary* CPC*-model and let $h \in Hom(\mathcal{L}_{\mathsf{CPC}}, \mathcal{A})$. For an arbitrary formula ϕ:* <u>*h satisfies ϕ in $\mathcal{M}$*</u> *iff $h(\phi) \in F$.*

Definition 26. *Let $\mathcal{M} = \langle \mathcal{A}, F\rangle$ be an arbitrary* CPC*-model. Formula* <u>*ϕ is true in model $\mathcal{M}$*</u> *if and only if for every valuation $h \in Hom(\mathcal{L}_{\mathsf{CPC}}, \mathcal{A})$, h satisfies ϕ.*

Definition 27. *Formula ϕ of $\mathcal{L}_{\mathsf{CPC}}$ is* <u>*valid in* CPC</u> *provided it is true in all* CPC*-models.*

In defining validity of formulae of language $\mathcal{L}_{\mathsf{CPC}}$ it is sufficient to consider exactly one model $\langle \mathcal{A}, F\rangle$, where $\mathcal{A}$ is the two-element Boolean algebra whose elements are denoted $\{0, 1\}$, and $F = \{1\}$.

2.3 Structural Proof Theory

The introduction of sequent calculus and natural deduction systems by Gentzen followed Gödel's and Gentzen's simultaneous independent research focusing on the translation of Peano arithmetic into Heyting arithmetic [12; 26; 27; 65]. This particular result, paired with a proof of consistency of Heyting arithmetic would entail consistency of classical arithmetic. The former was achieved by Gentzen in his doctoral dissertation, in which he presented two proof systems—natural deduction and sequent calculus. His goal was to formalize the actual practise of mathematicians while constructing a proof. He noted that contrary to other systems, such as axiomatic theories, it is the practise of employing a number of different assumptions that is preferred by mathematicians.

At the same time Jaśkowski was independently developing a system originating from similar motivations [30]. Jaśkowski's system predated those proposed by Gentzen; in 1926 Łukasiewicz discussed the possibility of formalizing mathematicians' actual practise, and one year later Jaśkowski showed his initial results at the First Congress of Mathematics in Lviv [1], see [26]. The biggest difference between the systems of Jaśkowski and Gentzen lies in the way the proofs are represented. In the natural deduction developed by Gentzen proofs are presented as trees, whereas in Jaśkowski's natural deduction system proofs are written down as series of formulae. In Jaśkowski's linear[5] natural deduction, inferences are based on formulae themselves and not their occurrences (contrary to Gentzen's system). This means that additional mechanisms should be utilized to exclude parts of a proof closed through the employment of specific formulae as assumptions, be it through graphical representation or utilization of prefixes [26].

Sequent calculus, today often studied independently from natural deduction, was originally introduced as a practical tool to analyze the derivability relation in natural deduction. The main result of Gentzen's research, *Hauptsatz*—the cut elimination theorem—shows us that any formula with a proof containing the use of the cut rule can be proved without cut. We will elaborate on this particular theorem and the rule itself later in the chapter.

2.3.1 System G3cp

In this section we introduce a version of G3cp, originally presented in [36]. For simplicity we will utilize the same name. The differences lie in the set of rules: in the original system there are rules for conjunction, disjunction, implication and constant falsum. In the system proposed below we shall omit falsum (as it is not a part of the language) and add rules for classical negation and equivalence. Moreover, in the examined system we include a slight difference in the definition of axioms. G3cp introduced in this section consists of logical rules only. We can additionally show that in G3cp structural rules are admissible.

Definition 28 (Multiset)**.** *Multiset is a generalized type of set in which multiple occurrences of an element are permitted.*

[5] Jaśkowski's system is often referred to as linear deduction, but it would be more precise to underline its more hierarchical than linear structure [26].

A sequent is the following structure

$$\Gamma \Rightarrow \Delta$$

where Γ (an antecedent of a sequent), Δ (a succedent of a sequent) are finite, possibly empty (but not both at the same time) multisets of formulae[6] whereas "$\Rightarrow$" expresses a relation between two multisets. We can consider different interpretations of a sequent, depending on our predetermined concept of the sequent being true or false in some context. For the former one (*i.e.*, the truth of a sequent), we can talk about two interpretations. Operational interpretation of sequent refers to single succedent sequents (i.e. succedents consist of one formula only) $\Gamma \Rightarrow \phi$ and states that conclusion ϕ can be derived from assumptions in Γ. Natural deduction best embodies this interpretation [36]. Denotational interpretation can be referred to multi-succedent sequent $\Gamma \Rightarrow \Delta$; in light of this interpretation we will say that the conjunction of formulae in Γ implies the disjunction of formulae in Δ. In this way we can also state that a comma in a sequent works conjunctively in its antecedent and disjunctively in its succedent. On the other hand, if we were to consider a sequent being false in some semantic context, we can look at Γ as a multiset of formulae true under some valuation and at Δ as a multiset of formulae false under the same valuation.

Expression of the form $\Gamma, \phi, \Gamma' \Rightarrow \Delta, \chi, \Delta'$ will be understood as $\Gamma \cup \{\phi\} \cup \Gamma' \Rightarrow \Delta \cup \{\chi\} \cup \Delta'$, where $\cup$ is the sum defined with regard to the multisets.

Usually a set of axioms of $\mathsf{G3cp}$ is determined by two elements, consecutively one axiom schema and an axiom, that is:

$$p_i \Rightarrow p_i \qquad\qquad \bot \Rightarrow$$

However, these two axioms require the use of structural rules, for instance weakening. Instead, to minimize our dependence on structural rules, we will utilize the general version of axiom schemata, which can be seen below where ϕ is an arbitrary formula of $\mathcal{L}_{\mathsf{CPC}}$:

$$\Gamma, \phi \Rightarrow \phi, \Delta$$

In subsequent sections we will also consider the following two axiom schemata:

$$\Gamma \Rightarrow \Delta, \top \qquad\qquad \bot, \Gamma \Rightarrow \Delta$$

In this and subsequent sequent systems by classical rules we mean logical rules for truth-functional connectives. The classical rules (Table 2.1 and 2.2) are divided into left- (L-) and right- (R-) rules. For a logical operator $\otimes$, for instance $\otimes$ standing for $\to$, the symbols $L_\to$, $R_\to$ indicate the rules where a formula with

[6] We could, however, additionally consider other possibilities. Sequents may be built with their elements being sequences or sets of formulae as well as lists of fixed length, i.e., through restricting the number of formulae within it. The most notable examples of the latter are single-conclusion sequents in sequent calculus for intuitionistic propositional logic [25; 36; 61].

$\rightarrow$ as the main operator is introduced in the antecedent and in the succedent respectively.

Classical rules allow us to synthesize more complex formulae out of their components (looking top-down) or decompose complex formulae (looking bottom-up). Γ and Δ in the rules are referred to as contexts, formulae of the form $\phi \otimes \chi$ (as well as $\neg\chi$, in the case of the rules for negation) in the conclusion of a given logical rule are called principal (or main) formulae of the rule, and ϕ and χ (or only χ) in the premisses are called active formulae of the rule.

Further on we will bring up several theorems regarding structural rules' admissibility; we therefore can now define G3cp as the set of the following rules: $\{L_\wedge, R_\wedge, L_\vee, R_\vee, L_\rightarrow, R_\rightarrow, L_\leftrightarrow, R_\leftrightarrow, L_\neg, R_\neg\}$ (Table 2.1). Derivations in SC are trees, the following definition comes from [61, p. 61].

Definition 29 (Derivation of sequent $\Gamma \Rightarrow \Delta$ in G3cp)**.** *Derivation of sequent $\Gamma \Rightarrow \Delta$ in G3cp is a finite labelled tree with a single root carrying $\Gamma \Rightarrow \Delta$ and each node-label connected with the labels of the (immediate) successor nodes (if any) according to one of the rules $\{L_\wedge, R_\wedge, L_\vee, R_\vee, L_\rightarrow, R_\rightarrow, L_\leftrightarrow, R_\leftrightarrow, L_\neg, R_\neg\}$.*

Definition 30 (Proof in G3cp)**.** *Proof of sequent $\Gamma \Rightarrow \Delta$ in G3cp is a derivation of $\Gamma \Rightarrow \Delta$ in G3cp with axioms labelling all of the top nodes.*

Definition 29 states that nodes of the tree are labelled with sequents. In subsequent sections we will introduce SC with additional, separate labels of sequents, however we will not introduce a separate terminology for the two kinds of labelling, and our intended use of "labels" will be clear from the context.

TABLE 2.1: Rules of G3cp: classical rules

$$\frac{\Gamma \Rightarrow \chi, \Delta}{\neg\chi, \Gamma \Rightarrow \Delta}\, L_\neg \qquad \frac{\chi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg\chi}\, R_\neg$$

$$\frac{\phi, \chi, \Gamma \Rightarrow \Delta}{\phi \wedge \chi, \Gamma \Rightarrow \Delta}\, L_\wedge \qquad \frac{\Gamma \Rightarrow \Delta, \phi \quad \Gamma \Rightarrow \Delta, \chi}{\Gamma \Rightarrow \Delta, \phi \wedge \chi}\, R_\wedge$$

$$\frac{\phi, \Gamma \Rightarrow \Delta \quad \chi, \Gamma \Rightarrow \Delta}{\phi \vee \chi, \Gamma \Rightarrow \Delta}\, L_\vee \qquad \frac{\Gamma \Rightarrow \Delta, \phi, \chi}{\Gamma \Rightarrow \Delta, \phi \vee \chi}\, R_\vee$$

$$\frac{\Gamma \Rightarrow \Delta, \phi \quad \chi, \Gamma \Rightarrow \Delta}{\phi \rightarrow \chi, \Gamma \Rightarrow \Delta}\, L_\rightarrow \qquad \frac{\phi, \Gamma \Rightarrow \Delta, \chi}{\Gamma \Rightarrow \Delta, \phi \rightarrow \chi}\, R_\rightarrow$$

$$\frac{\phi, \chi, \Gamma \Rightarrow \Delta \quad \Gamma \Rightarrow \Delta, \phi, \chi}{\phi \leftrightarrow \chi, \Gamma \Rightarrow \Delta}\, L_\leftrightarrow \qquad \frac{\phi, \Gamma \Rightarrow \chi, \Delta \quad \chi, \Gamma \Rightarrow \phi, \Delta}{\Gamma \Rightarrow \Delta, \phi \leftrightarrow \chi}\, R_\leftrightarrow$$

Through examination of the rules in Table 2.1 we notice that, if we were to look at the derivations bottom-up, through the use of logical rules we ensure that the subformula property is satisfied (provided no other rules are added to G3cp).

Definition 31. *Let ϕ be a formula of $\mathcal{L}_{\mathsf{CPC}}$. $sub(\phi)$ is the smallest set of formulae closed under the rules:*

1. $\phi \in sub(\phi)$;
2. *if* $\neg\psi \in sub(\phi)$, *then* $\psi \in sub(\phi)$;
3. *if* $\psi \otimes \chi \in sub(\phi)$ *(where* $\otimes \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$*), then* $\psi, \chi \in sub(\phi)$.

Each element of $sub(\phi)$ is called a subformula of ϕ.

It has been shown that G3cp satisfies the subformula property, for example in [36, p. 57]. The changes included in the examined system do not interfere with this property.

Lemma 2 (Subformula property)**.** *All formulae in the proof of $\Rightarrow \phi$ in* G3cp *are elements of set $sub(\phi)$.*

The structural rules presented in Table 2.2 allow us to modify the content of a sequent, without modifying the internal structure of formulae.

Table 2.2: Rules of G3cp: structural rules

$$\frac{\Gamma \Rightarrow \Delta}{\chi, \Gamma \Rightarrow \Delta}\, L_{wk} \qquad \frac{\chi, \chi, \Gamma \Rightarrow \Delta}{\chi, \Gamma \Rightarrow \Delta}\, L_{ctr}$$

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \chi}\, R_{wk} \qquad \frac{\Gamma \Rightarrow \Delta, \chi, \chi}{\Gamma \Rightarrow \Delta, \chi}\, R_{ctr}$$

$$\frac{\Gamma \Rightarrow \Delta, \phi \quad \phi, \Theta \Rightarrow \Pi}{\Gamma, \Theta \Rightarrow \Delta, \Pi}\, cut$$

We will refer to formula ϕ in the cut rule as the cut-formula of application of the rule. The cut rule, with no disregard to its importance, introduces certain issues with regard to the process of building derivations. We can interpret it as a rule transcribing the property of transitivity of derivability, moreover, we can consider certain cases of cut that express *modus ponens* or hypothetical syllogism, see [25]. Nonetheless, in contrast to classical rules, we cannot ensure that a *cut*-formula we introduce to the derivation is a subformula of our initial problem.

The cut rule presented above is an example of a rule with independent contexts, in contrast to two-premiss logical rules with shared contexts. This particular feature will have its consequences in non-Fregean extensions in subsequent sections.

We will also make several comments regarding $\mathsf{G3cp}$ semantics, which will provide a base for other considerations in subsequent chapters. It has to be noted that we will omit the proofs of several theorems (*e.g.* invertibility or correctness of rules), as they can be found in [36], where their authors utilize different semantics based on valuations, but this particular approach is equivalent to Boolean algebra semantics utilized in this thesis.

Definition 32 (Satisfiability of a sequent)**.** *Let $\mathcal{M} = \langle \mathcal{A}, F \rangle$ be an arbitrary but fixed CPC-model and let $h \in Hom(\mathcal{L}, \mathcal{A})$. Sequent $\Gamma \Rightarrow \Delta$ is satisfied in $\mathcal{M}$ under h provided if all formulae from Γ are satisfied in $\mathcal{M}$ under h, then at least one formula from Δ is satisfied in $\mathcal{M}$ under h.*

Definition 33 (Truth of a sequent)**.** *Let $\mathcal{M} = \langle \mathcal{A}, F \rangle$ be an arbitrary CPC-model. Sequent $\Gamma \Rightarrow \Delta$ is true in $\mathcal{M}$ provided $\Gamma \Rightarrow \Delta$ is satisfied in $\mathcal{M}$ under every $h \in Hom(\mathcal{L}, \mathcal{A})$.*

Definition 34 (Validity of a sequent)**.** *Sequent $\Gamma \Rightarrow \Delta$ is valid in CPC, if it is true in each CPC-model.*

Theorem 5. *Sequent $\Gamma \Rightarrow \Delta$ has a proof in $\mathsf{G3cp}$ iff $\Gamma \Rightarrow \Delta$ is valid in CPC.*

Proof. As noted above, proof of completeness of $\mathsf{G3cp}$ can be found in [36]. □

This particular proof is expressed for valuation semantics (i.e., an assignment of truth values to formulae, based on the true values of their propositional variables), but, as it is known, a given formula ϕ is a tautology in valuation semantics iff it is a tautology in Boolean algebra semantics.

2.3.2 Structural rules' admissibility

We often interchangeably use the notions of admissibility and elimination of a given rule(s) R, however it has to be underlined that the two terms refer to different properties of sequent systems: the former applies to a rule set without R, while the latter applies to a rule set with R.

By $\vdash_{\mathsf{SC}} \Gamma \Rightarrow \Delta$ we mean that sequent $\Gamma \Rightarrow \Delta$ is derivable in a given SC.

Definition 35 (Admissibility of a rule in a system)**.** *Let SC be a sequent calculus without R in its rule set and let R be a rule acting on sequents, with $S_1 \ldots, S_n$ as premises schemata, and S—conclusion scheme. We say that R is admissible in SC iff: if instances of $S_1 \ldots, S_n$ are derivable in SC, then a relevant instance of S is derivable in SC.*

Definition 36 (Eliminability of a rule in a system)**.** *Let S denote an arbitrary sequent. Rule R is eliminable in $\mathsf{SC} \cup \{R\}$ iff: if $\vdash_{\mathsf{SC}\cup\{R\}} S$, then $\vdash_{\mathsf{SC}} S$.*

In conclusion, elimination is a converse of admissibility. For example, if we were to consider the cut rule, if $\mathsf{SC} \cup \{cut\}$ is a system with a given cut rule in the rule set and SC is the same system but without the cut, the cut rule can be eliminated in $\mathsf{SC} \cup \{cut\}$ iff cut is admissible in SC [25, p. 90].

The proof for the following theorem about the admissibility of weakening can be found in Chapter 7.

Theorem 6 (Admissibility of weakening *wk*). *If* $\vdash_{\mathsf{G3cp}} \Gamma \Rightarrow \Delta$*, then* $\vdash_{\mathsf{G3cp}} \phi, \Gamma \Rightarrow \Delta$ *and* $\vdash_{\mathsf{G3cp}} \Gamma \Rightarrow \Delta, \phi$*.*

The proof of admissibility of weakening is standard. However, the proof for the admissibility of height-preserving contraction differs from the standard approach due to the general form of the axioms. In this situation the rules are not height-preserving invertible. Nonetheless, contraction is still admissible, which will be shown in Chapter 7.

Theorem 7 (Admissibility of contraction *ctr*). *If* $\vdash_{\mathsf{G3cp}} \phi, \phi, \Gamma \Rightarrow \Delta$*, then* $\vdash_{\mathsf{G3cp}} \phi, \Gamma \Rightarrow \Delta$ *and if* $\vdash_{\mathsf{G3cp}} \Gamma \Rightarrow \Delta, \phi, \phi$*, then* $\vdash_{\mathsf{G3cp}} \Gamma \Rightarrow \Delta, \phi$*.*

In the standard proof we refer to the induction on the height of the derivation. The proof can be found in [36].

Theorem 8 (Admissibility of cut). *If* $\vdash_{\mathsf{G3cp}} \Gamma \Rightarrow \Delta, \phi$ *and* $\vdash_{\mathsf{G3cp}} \phi, \Theta \Rightarrow \Pi$*, then* $\vdash_{\mathsf{G3cp}} \Gamma, \Theta \Rightarrow \Delta, \Pi$*.*

The full proof can be found in [36], although we will also present the proof in Chapter 7.

Chapter 3

Sentential Calculus with Identity

Sentential Calculus with Identity (SCI) is the weakest non-Fregean logic proposed by Roman Suszko.[1] It is obtained from classical logic by means of adding a binary *identity connective* $\equiv$. Then, the expression "$\phi \equiv \chi$" (which we will refer to as an *equation*) expresses the identity of situations denoted by formulae ϕ and χ. We introduce SCI as the foundation for the three axiomatic extensions we investigate in the subsequent sections. Regarding semantics, we will stay faithful to the algebraic approach proposed by Suszko, as it provides the best formalization of Wittgenstein's ideas examined in *Tractatus.*

3.1 The Hilbert system for SCI

We will supplement CPC language with an introduction of the identity connective, thereby obtaining SCI-language. We consider the following structure:

Definition 37. *Language* $\mathcal{L}_{\mathsf{SCI}}$ *of theory* SCI *is the following algebra of similarity type* $\langle 1, 2, 2, 2, 2, 2\rangle$

$$\mathcal{L}_{\mathsf{SCI}} = \langle L_{\mathsf{SCI}}, \neg, \wedge, \vee, \to, \leftrightarrow, \equiv\rangle.$$

In $\mathcal{L}_{\mathsf{SCI}}$ we are unable to define constants $\top$ or $\bot$ (similarly we are unable to define $\neg\phi =_{df} \phi \to \bot$), as it would lead us to a different, stronger non-Fregean logic. By logic H_1 being stronger than logic H_2 we mean that H_1's set of theorems contains more elements than that of logic H_2.[2]

We distinguish two separate sets of formulae: Truth-Functional Axioms (TFA) and the set of formulae falling under axioms schemata characterizing the identity connective $\equiv$ (IDA) [51, p. 185] in $\mathcal{L}_{\mathsf{SCI}}$:

($\equiv_1$) $\phi \equiv \phi$

($\equiv_2$) $(\phi \equiv \chi) \to (\neg\phi \equiv \neg\chi)$

($\equiv_3$) $(\phi \equiv \chi) \to (\phi \leftrightarrow \chi)$

($\equiv_4$) $((\phi \equiv \psi) \wedge (\chi \equiv \omega)) \to ((\phi \otimes \chi) \equiv (\psi \otimes \omega))$,

[1] There are non-Fregean logics weaker than SCI proposed by other logicians, *e.g.,* Grzegorczyk's Minimal Non-Fregean Logic. [15; 17; 21]

[2] Suszko underlined that "genuine logic should be as weak as possible" [51, p.192], however it is reasonable to aim for a stronger entailment relation.

where $\otimes$ can be any one of the available binary connectives.

The first axiom schema, ($\equiv_1$), expresses the reflexivity of the identity connective. Moreover, it is the only validity of SCI with the identity as the main connective (see [2]). Additionally, axiom schemata ($\equiv_2$), ($\equiv_4$) are referred to as invariance axioms with regard to the identity connective. In [4] ($\equiv_2$) is written as $(\phi \equiv \chi) \to ((\phi \to \bot) \equiv (\chi \to \bot))$, but, as we mentioned above, we cannot utilize definitional equations in SCI without causing any harm to the theory: $\neg\phi \equiv (\phi \to \bot)$ is not a theorem of SCI. In [2, p. 293] Suszko writes that *The completeness theorem may be used to show that if* $\vee$ *(disjunction) is included in the set of primitive connectives then e.g.* $\neg(p \vee q) \equiv (\neg p \to q)$ *is consistent. Hence to construe* $p \vee q$ *as an abbreviation of* $\neg p \to q$ *is, in effect, to adopt* $p \vee q \equiv \neg p \to q$ *as an axiom. Similar remarks may be made about any abbreviation.* We should be somewhat cautious when translating between two languages, as we could inadvertently create a stronger logic. The axiom schema ($\equiv_3$) is called a special axiom for the identity connective [38, p. 86]: it expresses the fact that the equation entails equivalence, and not the other way around (which is expressed by the rejection of the Fregean Axiom).

Definition 38 (Derivation, proof)**.** *Let* Φ *stand for a set of formulae of* $\mathcal{L}_{\mathsf{SCI}}$*. A finite sequence* $\phi_1, ..., \phi_n$ *of formulae of* $\mathcal{L}_{\mathsf{SCI}}$ *is a* <u>*derivation of* ϕ *from* Φ</u> *provided* $\phi_n = \phi$ *and each formula* $\phi_i, i \leq n$*, either belongs to* Φ *or has been derived from some* ϕ_{i_1}, ϕ_{i_2}*,* $(i_1, i_2 < i)$ *through an application of modus ponens. If* $\Phi = \mathsf{TFA} \cup \mathsf{IDA}$*, then derivation* $\phi_1, ..., \phi_n$ *is a* <u>*proof of* ϕ *in the axiomatic system for* SCI</u>*.*

If $\phi_1, ..., \phi_n$ is a proof of ϕ in the axiomatic system for SCI, then the length of this proof is n.

We distinguish two types of consequence operation: syntactical (C) and semantical (C_M) [2]. We will focus on the former, understood as we defined it in the previous chapter.

Consequence operation C is defined by the set $\mathsf{TFA} \cup \mathsf{IDA}$ of axioms and a singular inference rule, *modus ponens* as follows: $\phi \in C(\Phi)$ iff there is a derivation of ϕ from Φ, as defined in Definition 38. If $\Phi = \mathsf{TFA} \cup \mathsf{IDA}$, then we shall say that ϕ is a <u>logical theorem of SCI</u>.

Theorem 9. *C has the following properties [2, p. 290–291][3]:*

($C1$) $\Phi \subseteq C(\Phi)$*, for all* $\Phi \subseteq L_{\mathsf{SCI}}$*,*

($C2$) *If* $\Phi \subseteq \Psi$*, then* $C(\Phi) \subseteq C(\Psi)$*; for all* $\Phi, \Psi \subseteq L_{\mathsf{SCI}}$*,*

($C3$) $C(C(\Phi)) = C(\Phi)$*, for all* $\Phi \subseteq L_{\mathsf{SCI}}$*,*

($C4$) $\phi \in C(\Phi \cup \{\psi\})$ *iff* $\psi \to \phi \in C(\Phi)$*,*

($C5$) $\phi \in C(\Phi)$ *iff* $\phi \in C(\Psi)$*, where* Ψ *is some finite subset of* Φ*.*

[3]In the original paper the authors examined language with negation, implication and identity connective only.

$(C1)$–$(C3)$ are the basic properties we listed for the consequence operation on CPC and are extended by $(C4)$ and $(C4)$ (also applicable for CPC). Condition $(C5)$ states that consequence operation C is finitary, whereas condition $(C4)$ states that it is closed under the deduction theorem.

Definition 39 (Consistent set [38, p. 21])**.** *Let $\mathcal{L}$ be a language and let C^* be a consequence operation in this language. Set Φ of formulae of $\mathcal{L}$ is consistent with regard to consequence operation C^* if and only if*

$$C^*(\Phi) \neq L.$$

Definition 40 (Complete theory [38, p. 21–22])**.** *Let $\mathcal{L}$ be a language and let C^* be a consequence operation in this language. A set Φ of formulae of $\mathcal{L}$ is called a C^*-theory iff*

$$C^*(\Phi) = \Phi.$$

If in language $\mathcal{L}$ there exist maximal consistent sets of formulae with regard to consequence operation C^*, we shall call them complete C^*-theories.

Definition 41 (Compact consequence operation [38, p. 22])**.** *Consequence operation C^* defined with regard to language $\mathcal{L}$ is called logically compact provided every inconsistent set X of formulae of language $\mathcal{L}$ contains a finite inconsistent subset.*

Definition 42 (Regular consequence operation [38, p. 24])**.** *Consequence operation C^* defined with regard to language $\mathcal{L}$ is called regular if and only if every C^*-theory T is the product of all complete C^*-theories containing theory T.*

We will also say with regard to consequence operation C the following: for an arbitrary substitution function e for propositional variables in $\mathcal{L}_{\mathsf{SCI}}$ the following conditions hold:

- $e(\mathsf{TFA} \cup \mathsf{IDA}) \subseteq (\mathsf{TFA} \cup \mathsf{IDA})$
- $e\beta \in C(\{e\alpha, e(\alpha \to \beta)\})$,

hence we know that C is a structural consequence operation. Moreover, C is regular, logically compact and finitary [38, p. 87].

Definition 43 (SCI-theory)**.** *Any set Φ of formulae of language $\mathcal{L}_{\mathsf{SCI}}$ such that $C(\Phi) = \Phi$ is called an SCI-theory. SCI-theory is invariant provided it is closed under the substitution rule.*

Definition 44. $\phi \in \mathsf{TFT}$ *iff* $\phi \in C(\mathsf{TFA})$ *(where TFT stands for Truth Functional Tautologies).*

Finally, it is worth mentioning that for Suszko a logic, or a deductive system, is a pair composed of a language algebra and a structural consequence operation defined on it. Hence, in keeping with Suszko, we define a deductive system for SCI:

Definition 45. *Pair $H_{\mathsf{SCI}} = \langle \mathcal{L}_{\mathsf{SCI}}, C\rangle$ is called Sentential Calculus with Identity,* SCI *for short.*

Suszko distinguished two different ways of extending a given logic: either by an addition of axioms (such extension is called *elementary*) or by different inference rules (*non-elementary* extensions) [23]. In this work we will focus on three elementary extensions of SCI: WB, WT and WH. These three theories are formalized as an interpretation of different theses from *Tractus* and can be semantically interpreted by means of different algebras: respectively, Boolean algebra, topological Boolean algebra and Henle algebra. In addition, let WF be a maximal consistent Fregean theory.

Definition 46 (Fregean theory)**.** *A given theory T is called a Fregean theory in language $\mathcal{L}_{\mathsf{SCI}}$, if among theorems of T there are all formulae of $\mathcal{L}_{\mathsf{SCI}}$ that are represented by the following formula schema: $(\phi \equiv \chi) \equiv (\phi \leftrightarrow \chi)$.*

We can organize the aforementioned theories in the following order (as was done in [13]):

$$\mathsf{SCI} \subset \mathsf{WB} \subset \mathsf{WT} \subset \mathsf{WH} \subset \mathsf{WF}.$$

Consequently, the following order of consequence operations describes their dependence with one another:

$$\mathsf{C}_{\mathsf{SCI}} \prec \mathsf{C}_{\mathsf{WB}} \prec \mathsf{C}_{\mathsf{WT}} \prec \mathsf{C}_{\mathsf{WH}} \prec \mathsf{C}_{\mathsf{WF}},$$

where $\prec$, which points to the relation of being a proper sublogic, is defined as follows [13, p. 27][4].

Definition 47 (Subtheory, sublogic, proper sublogic)**.** *Let L be a universe of any* SCI *language $\mathcal{L}$ and let A and B be* SCI*-theories:*

- *A is a subtheory of B provided $A \subseteq B$;*
- *We will say C^A is a sublogic of C^B provided for any $X \subset L$ the following holds:*

  $$C^A(X) \subseteq C^B(X).$$

  If C^A is a sublogic of C^B, then we write $C^A \preceq C^B$;
- *We will say that C^A is a proper sublogic of C^B provided for all $X \subset L$ the following holds:*

  $$C^A(X) \subseteq C^B(X)$$

  and for some $Y \subset L$ the following holds:

  $$C^B(Y) \nsubseteq C^A(Y).$$

  If C^A is a proper sublogic of C^B, we will write $C^A \prec C^B$.

[4]Fragment self-translated.

In [13] it is shown that the class of all extensions of SCI is partially ordered by $\preceq$, where a minimal element is SCI and the maximal one is a Fregean-theory WF. In [13] it is also proved that between SCI and WF there are uncountably many different extensions of SCI, see also [18].

3.2 Semantics of SCI

Certain constructions defined below will be used in the subsequent sections (for axiomatic extensions of SCI) whilst some will undergo specific modifications with the purpose of underlining the differences between the presented theories.

The following object, an SCI-algebra, allows us to depict an internal relation that holds between language and the universe of situations (or: the world); a relation that appears in *Tractatus* between language and certain elements of the world (*e.g.* one between the score and the musical thought).

Definition 48. *Any algebra of the form $\mathcal{A} = \langle A, \dot{\neg}, \dot{\cap}, \dot{\cup}, \dot{\to}, \dot{\leftrightarrow}, \dot{\equiv} \rangle$ similar to $\mathcal{L}_{\mathsf{SCI}}$, will be called an* SCI-*algebra.*

The decision to introduce "$\dot{\equiv}$" in place of symbol "$\circ$" (which is prevailing in literature) is due to the fact we use "$\circ$" to show function composition.

If $\mathcal{A} = \langle A, \dot{\neg}, \dot{\cap}, \dot{\cup}, \dot{\to}, \dot{\leftrightarrow}, \dot{\equiv} \rangle$ is an SCI-algebra, then set A is called a universe of situations while the other elements of the SCI-algebra are various operations we can apply to elements of the set A. The following definitions are based on ones from [38, p. 90]:

Definition 49 (SCI-model)**.** *A structure of the form $\langle \mathcal{A}, F \rangle$ (where $\mathcal{A} = \langle A, \dot{\neg}, \dot{\cap}, \dot{\cup}, \dot{\to}, \dot{\leftrightarrow}, \dot{\equiv} \rangle$) is called an* SCI-*model if and only if $\mathcal{A}$ is an* SCI-*algebra, and F is any subset of A such that for any $a, b \in A$ the following conditions are met:*

1. $a \in F \Leftrightarrow \dot{\neg} a \notin F$,
2. $a \mathbin{\dot{\cap}} b \in F \Leftrightarrow a \in F$ *and* $b \in F$,
3. $a \mathbin{\dot{\cup}} b \in F \Leftrightarrow a \in F$ *or* $b \in F$,
4. $a \mathbin{\dot{\to}} b \in F \Leftrightarrow a \notin F$ *or* $b \in F$,
5. $a \mathbin{\dot{\leftrightarrow}} b \in F \Leftrightarrow a, b \in F$ *or* $a, b \notin F$,
6. $a \mathbin{\dot{\equiv}} b \in F \Leftrightarrow a = b$.

If $\langle \mathcal{A}, F \rangle$ is an SCI-*model, then F is called a normal ultrafilter of algebra $\mathcal{A}$.*

Of course, an ultrafilter can be also defined separately from the structure of the SCI-model [38, p. 47]. This particular definition appears in Chapter 4 as well.

Definition 50. *Let $\mathcal{A}$ be an* SCI-*algebra. Ultrafilter F of $\mathcal{A}$ is called a normal ultrafilter, provided for any $a, b \in \mathcal{A}$:*

$$a \mathbin{\dot{\equiv}} b \in F \textit{ iff } a = b.$$

The following definition is a formalization of the shadow metaphor, proposed by Wolniewicz in [72]. Wolniewicz refers to the mirror metaphor proposed by Wittgenstein and proposes shadow as a better metaphor for homomorphism; the mirror metaphor is better suited for an isomorphism. Homomorphism allows situations in which some of the language's details are disguised by codenotations. Elements of algebra could thereby be understood as the shadow of a language, and not its actual reflection.

Definition 51 (Valuation)**.** *Valuation of language $\mathcal{L}_{\mathsf{SCI}}$ into its similar algebra $\mathcal{A} = \langle A, \dot{\neg}, \dot{\cap}, \dot{\cup}, \dot{\rightarrow}, \dot{\leftrightarrow}, \dot{\equiv} \rangle$ is a homomorphism h from $\mathcal{L}_{\mathsf{SCI}}$ to $\mathcal{A}$, hence it fulfills the following conditions:*

1. $h(\neg\phi) = \dot{\neg} h(\phi)$,
2. $h(\phi \vee \chi) = h(\phi) \mathbin{\dot{\cup}} h(\chi)$,
3. $h(\phi \wedge \chi) = h(\phi) \mathbin{\dot{\cap}} h(\chi)$,
4. $h(\phi \rightarrow \chi) = h(\phi) \mathbin{\dot{\rightarrow}} h(\chi)$,
5. $h(\phi \leftrightarrow \chi) = h(\phi) \mathbin{\dot{\leftrightarrow}} h(\chi)$,
6. $h(\phi \equiv \chi) = h(\phi) \mathbin{\dot{\equiv}} h(\chi)$.

Definition 52. *Let $\mathcal{M} = \langle \mathcal{A}, F \rangle$ be an arbitrary* SCI*-model and let $h \in Hom(\mathcal{L}_{\mathsf{SCI}}, \mathcal{A})$. For an arbitrary formula ϕ: h satisfies ϕ in $\mathcal{M}$ iff $h(\phi) \in F$.*

Definition 53. *Let $\mathcal{M} = \langle \mathcal{A}, F \rangle$ be an arbitrary* SCI*-model. Formula ϕ is true in model $\mathcal{M}$ if and only if for every valuation $h \in Hom(\mathcal{L}_{\mathsf{SCI}}, \mathcal{A})$ h satisfies formula ϕ in model $\mathcal{M}$.*

Definition 54. *Formula ϕ is valid in* SCI *provided it is true in all* SCI*-models.*

We define the semantic consequence operation C_M by means of the SCI-models. It follows Tarski's observation that a given formula ϕ follows from the set of formulae $\mathbf{\Phi}$ provided every model of $\mathbf{\Phi}$ is also a model of ϕ.

Definition 55. *Let A be a universe of algebra $\mathcal{A}$ and let D be a subset of A. $\phi \in C_M(\mathbf{\Phi})$ iff for all valuations $h \in Hom(\mathcal{L}, \mathcal{A})$ if $h(\mathbf{\Phi}) \subseteq D$, then $h(\phi) \in D$.*

Moreover, analogously to C, C_M satisfies the following conditions:

(C_M1) $\mathbf{\Phi} \subseteq C_M(\mathbf{\Phi})$, for all $\mathbf{\Phi} \subseteq L$;

(C_M2) If $\mathbf{\Phi} \subseteq \mathbf{\Psi}$, then $C_M(\mathbf{\Phi}) \subseteq C_M(\mathbf{\Psi})$; for all $\mathbf{\Phi}, \mathbf{\Psi} \subseteq L$.

(C_M3) $C_M(C_M(\mathbf{\Phi})) = C_M(\mathbf{\Phi})$, for all $\mathbf{\Phi} \subseteq L$;

We define standard semantic notions with regard to sequents in $\mathsf{G3}_{\mathsf{SCI}}$:

Definition 56 (Satisfiability of a sequent)**.** *Let $\mathcal{M} = \langle \mathcal{A}, F \rangle$ be an arbitrary but fixed* SCI*-model and let $h \in Hom(\mathcal{L}_{\mathsf{SCI}}, \mathcal{A})$. Sequent $\Gamma \Rightarrow \Delta$ is satisfied in $\mathcal{M}$ under h provided that, if all formulae from Γ are satisfied in $\mathcal{M}$ under h, that is $h(\chi) \in F$ (for all $\chi \in \Gamma$), then at least one formula in Δ is satisfied in $\mathcal{M}$ under h as well.*

Definition 57 (Truth of a sequent)**.** *Let $\mathcal{M} = \langle \mathcal{A}, F \rangle$ be an arbitrary* SCI*-model. Sequent $\Gamma \Rightarrow \Delta$ is* <u>*true in $\mathcal{M}$*</u> *provided that for each $h \in Hom(\mathcal{L}_{\mathsf{SCI}}, \mathcal{A})$, sequent $\Gamma \Rightarrow \Delta$ is satisfied in $\mathcal{M}$ under h.*

Definition 58 (Validity of a sequent)**.** *Sequent $\Gamma \Rightarrow \Delta$ is* <u>*valid in* SCI</u>*, if it is true in each* SCI*-model.*

Theorem 10. SCI *is the set of all and only formulae true in every* SCI*-model.*

Proof can be found in [38].

3.3 Sequent Calculi ℓG3$_{\mathsf{SCI}}$ and G3$_{\mathsf{SCI}}$

Indrzejczak in [28] notes that the introduction of identity in sequent calculus rules can be thought of as local (in a situation where we work on selected formulae within some identity-dedicated rules) or global (applying identity-dedicated rules results in some changes to the whole sequent). Following that, we can state that the introduction of identity in the sequent calculus we present in this section is local.

Sequent Calculus ℓG3$_{\mathsf{SCI}}$ has been obtained through employment of a strategy proposed by Negri in [36], which allows us to turn axioms into rules. It depends on the fact that if we can transform a given axiom into a formula of the following shape:

$$p_1 \wedge ... \wedge p_m \rightarrow q_1 \vee ... \vee q_n$$

we can obtain the rules of the following structure, both left-sided and right-sided:

$$\frac{q_1, p_1, ..., p_m, \Gamma \Rightarrow \Delta \quad ... \quad q_n, p_1, ..., p_m, \Gamma \Rightarrow \Delta}{p_1, ..., p_m, \Gamma \Rightarrow \Delta}\ L$$

or

$$\frac{\Gamma \Rightarrow \Delta, q_1, ..., q_n, p_1 \quad ... \quad \Gamma \Rightarrow \Delta, q_1, ..., q_n, p_m}{\Gamma \Rightarrow \Delta, q_1, ..., q_n}\ R$$

In [4] the author underlines the fact that the original strategy was defined for CPC and depended on the fact that we can work with propositional variables only. In the case of SCI we are unable to transform a given axiom into the above sequent structure, as we are unable to decompose equations, therefore the final product of two sequent calculi for SCI provided in [4] did not possess all of the desired properties, such as cut elimination (in the case of rG3$_{\mathsf{SCI}}$), subformula property (in both calculi) and so on.

For these reasons, in [4], to proceed with the strategy, the author additionally has to ensure that the closure condition is satisfied, where the closure condition is understood as follows.

Definition 59 (Closure condition [36, p. 130])**.** *If a system with non-logical rules has a rule, where a substitution instance in the atoms produces a rule of* *the form:*

$$\frac{q_1, p_1, ..., q_{m-2}, p, p, \Gamma \Rightarrow \Delta \quad ... \quad q_n, p_1, ..., p_{m-2}, p, p, \Gamma \Rightarrow \Delta}{p_1, ..., p_{m-2}, p, p, \Gamma \Rightarrow \Delta}$$

then it also has to contain the rule:

$$\frac{q_1, p_1, ..., q_{m-2}, p, \Gamma \Rightarrow \Delta \quad ... \quad q_n, p_1, ..., p_{m-2}, p, \Gamma \Rightarrow \Delta}{p_1, ..., p_{m-2}, p, \Gamma \Rightarrow \Delta}$$

Applying this condition results in the rule set in which contraction is admissible, which, paired with the fact that the overall strategy is built to ensure the admissibility of other structural rules, means that as a result $\ell\mathsf{G3_{SCI}}$ can be defined as the set of the following rules: $\{L_\wedge$, $R_\wedge$, $L_\vee$, $R_\vee$, $L_\to$, $R_\to$, $\mathbf{L}^1_\equiv$, $\mathbf{L}^2_\equiv$, $\mathbf{L}^3_\equiv$, $\mathbf{L}^{3*}_\equiv\}$ (Tables 3.1 and 3.2). In [4] besides $\ell\mathsf{G3_{SCI}}$ the author proposed right-sided calculus $r\mathsf{G3_{SCI}}$ (right- or left-sidedness relates to the identity-dedicated rules), however for the $r\mathsf{G3_{SCI}}$ there is no proof of cut elimination.

TABLE 3.1: $\ell\mathsf{G3_{SCI}}$: classical rules

$$\phi \Rightarrow \phi \ (ax) \qquad\qquad \bot, \Gamma \Rightarrow \Delta$$

$$\frac{\phi, \chi, \Gamma \Rightarrow \Delta}{\phi \wedge \chi, \Gamma \Rightarrow \Delta}\ L_\wedge \qquad \frac{\Gamma \Rightarrow \Delta, \phi \quad \Gamma \Rightarrow \Delta, \chi}{\Gamma \Rightarrow \Delta, \phi \wedge \chi}\ R_\wedge$$

$$\frac{\phi, \Gamma \Rightarrow \Delta \quad \chi, \Gamma \Rightarrow \Delta}{\phi \vee \chi, \Gamma \Rightarrow \Delta}\ L_\vee \qquad \frac{\Gamma \Rightarrow \Delta, \phi, \chi}{\Gamma \Rightarrow \Delta, \phi \vee \chi}\ R_\vee$$

$$\frac{\Gamma \Rightarrow \Delta, \phi \quad \chi, \Gamma \Rightarrow \Delta}{\phi \to \chi, \Gamma \Rightarrow \Delta}\ L_\to \qquad \frac{\phi, \Gamma \Rightarrow \Delta, \chi}{\Gamma \Rightarrow \Delta, \phi \to \chi}\ R_\to$$

Here we present a slightly modified version of the left-sided calculus, $\mathsf{G3_{SCI}}$, in which we utilize classical negation, equivalence, and we omit constant $\bot$ (Tables 3.3, 3.5, 3.4).

Let us observe that rule $L^1_\equiv$ of $\mathsf{G3_{SCI}}$ (Table 3.5) has no principal formula and rule $L^4_\equiv$ of $\mathsf{G3_{SCI}}$ (Table 3.5) has two principal formulae: $\phi \equiv \psi$ and $\chi \equiv \omega$.

$\mathsf{G3_{SCI}}$ is defined as the following set of rules: $\{L_\neg, R_\neg, L_\wedge$, $R_\wedge$, $L_\vee$, $R_\vee$, $L_\to$, $R_\to$, $L_\leftrightarrow, R_\leftrightarrow$, $L^1_\equiv$, $L^2_\equiv$, $L^3_\equiv$, $L^4_\equiv, cut\}$ Rules $L^1_\equiv$, $L^3_\equiv$, $L^4_\equiv$ come from the original system (with two numerical changes within the names), however rule $L^2_\equiv$ has been added in order to correspond to the axiom ($\equiv_2$).

Definition 60 (Derivation of sequent $\Gamma \Rightarrow \Delta$ in $\mathsf{G3_{SCI}}$). *Derivation of sequent $\Gamma \Rightarrow \Delta$ in $\mathsf{G3_{SCI}}$ is a finite tree with a single root labelled with sequent $\Gamma \Rightarrow \Delta$ and each node-label connected with the labels of the (immediate) successor nodes (if any) according to one of the rules of $\mathsf{G3_{SCI}}$.*

TABLE 3.2: $\ell\mathsf{G3}_{\mathsf{SCI}}$: identity-based rules

$$\frac{\phi \equiv \phi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ \mathbf{L}^1_{\equiv} \qquad \frac{\phi \equiv \chi, \Gamma \Rightarrow \Delta, \chi \quad \phi \equiv \chi, \phi, \Gamma \Rightarrow \Delta}{\phi \equiv \chi, \Gamma \Rightarrow \Delta}\ \mathbf{L}^2_{\equiv}$$

$$\frac{(\phi \otimes \psi) \equiv (\chi \otimes \omega), \phi \equiv \chi, \psi \equiv \omega, \Gamma \Rightarrow \Delta}{\phi \equiv \chi, \psi \equiv \omega, \Gamma \Rightarrow \Delta}\ \mathbf{L}^3_{\equiv}$$

$$\frac{(\phi \otimes \phi) \equiv (\chi \otimes \chi), \phi \equiv \chi, \Gamma \Rightarrow \Delta}{\phi \equiv \chi, \Gamma \Rightarrow \Delta}\ \mathbf{L}^{3*}_{\equiv}$$

Definition 61 (Proof in $\mathsf{G3}_{\mathsf{SCI}}$). *<u>Proof of a sequent $\Gamma \Rightarrow \Delta$ in $\mathsf{G3}_{\mathsf{SCI}}$</u> is a derivation of sequent $\Gamma \Rightarrow \Delta$ in $\mathsf{G3}_{\mathsf{SCI}}$ with axioms at all of the top nodes.*

In the original system this particular axiom was written with the definition $\neg\phi =_{df} \phi \to \bot$, which we reject, as it generates different logic, stronger than SCI, on which we commented earlier in the chapter. Also, rule $L^3_{\equiv}$ has been modified from a two-premiss to one-premiss rule. Moreover, in the original system rule $\mathbf{L}^{3*}_{\equiv}$ had been added to satisfy the closure condition. Its main goal was to show that contraction is admissible. This particular problem was illustrated by the formula $(\phi \equiv \psi) \to ((\phi \otimes \phi) \equiv (\psi \otimes \psi))$. $\mathbf{L}^{3*}_{\equiv}$ allows us to construct the following proof, in which we do not need two occurrences of $\phi \equiv \psi$ in the premiss. Apparently, rule $\mathbf{L}^3_{\equiv}$ is not sufficient to prove the sequent, as it requires two occurrences of $\phi \equiv \psi$ in the premise. But with the second occurrence we can only prove sequent $\phi \equiv \psi \Rightarrow (\phi \equiv \psi) \to ((\phi \otimes \phi) \equiv (\psi \otimes \psi))$.

$$\dfrac{\dfrac{(\phi \otimes \phi) \equiv (\psi \otimes \psi), \phi \equiv \psi \Rightarrow (\phi \otimes \phi) \equiv (\psi \otimes \psi)}{(\phi \equiv \psi) \Rightarrow ((\phi \otimes \phi) \equiv (\psi \otimes \psi))}\ \mathbf{L}^3_{\equiv}*}{\Rightarrow (\phi \equiv \psi) \to ((\phi \otimes \phi) \equiv (\psi \otimes \psi))}\ R_{\to}$$

But, it turns out, we can remove occurrences of $\phi \equiv \psi$ by means of other identity-dedicated rules:

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c} D \\ \vdots \\ \phi \equiv \psi, \Gamma^* \Rightarrow \phi \equiv \psi, \Delta \end{array}}{(\phi \equiv \psi) \leftrightarrow (\phi \equiv \psi), \phi \equiv \phi, \psi \equiv \psi, \delta, \phi \equiv \psi \Rightarrow (\phi \otimes \phi) \equiv (\psi \otimes \psi)}\ L_{\leftrightarrow}}{\delta, \phi \equiv \phi, \psi \equiv \psi, \phi \equiv \psi \Rightarrow (\phi \otimes \phi) \equiv (\psi \otimes \psi)}\ L^3_{\equiv}}{\phi \equiv \phi, \psi \equiv \psi, \phi \equiv \psi \Rightarrow (\phi \otimes \phi) \equiv (\psi \otimes \psi))}\ L^4_{\equiv}}{\Rightarrow (\phi \equiv \psi) \to ((\phi \otimes \phi) \equiv (\psi \otimes \psi))}\ L^1_{\equiv}(\times 2), R_{\to}(\times 1)$$

TABLE 3.3: $\mathsf{G3_{SCI}}$: classical rules

$$\phi, \Gamma \Rightarrow \Delta, \phi \ (ax)$$

$$\frac{\phi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg\phi} L_{\neg} \qquad \frac{\Gamma \Rightarrow \Delta, \phi}{\neg\phi, \Gamma \Rightarrow \Delta} R_{\neg}$$

$$\frac{\phi, \chi, \Gamma \Rightarrow \Delta}{\phi \wedge \chi, \Gamma \Rightarrow \Delta} L_{\wedge} \qquad \frac{\Gamma \Rightarrow \Delta, \phi \quad \Gamma \Rightarrow \Delta, \chi}{\Gamma \Rightarrow \Delta, \phi \wedge \chi} R_{\wedge}$$

$$\frac{\phi, \Gamma \Rightarrow \Delta \quad \chi, \Gamma \Rightarrow \Delta}{\phi \vee \chi, \Gamma \Rightarrow \Delta} L_{\vee} \qquad \frac{\Gamma \Rightarrow \Delta, \phi, \chi}{\Gamma \Rightarrow \Delta, \phi \vee \chi} R_{\vee}$$

$$\frac{\Gamma \Rightarrow \Delta, \phi \quad \chi, \Gamma \Rightarrow \Delta}{\phi \to \chi, \Gamma \Rightarrow \Delta} L_{\to} \qquad \frac{\phi, \Gamma \Rightarrow \Delta, \chi}{\Gamma \Rightarrow \Delta, \phi \to \chi} R_{\to}$$

$$\frac{\phi, \chi, \Gamma \Rightarrow \Delta \quad \Gamma \Rightarrow \Delta, \phi, \chi}{\phi \leftrightarrow \chi, \Gamma \Rightarrow \Delta} L_{\leftrightarrow} \qquad \frac{\phi, \Gamma \Rightarrow \chi, \Delta \quad \chi, \Gamma \Rightarrow \phi, \Delta}{\Gamma \Rightarrow \Delta, \phi \leftrightarrow \chi} R_{\leftrightarrow}$$

where D is the following derivation:

$$\frac{(\phi \otimes \phi) \equiv (\psi \otimes \psi), \Gamma \Rightarrow (\phi \otimes \phi) \equiv (\psi \otimes \psi)}{\Gamma \Rightarrow (\phi \otimes \phi) \equiv (\psi \otimes \psi)} L^4_{\equiv}$$

and $\delta = (\phi \equiv \psi) \equiv (\phi \equiv \psi)$, $\Gamma = \{\phi \equiv \psi \times 3, \phi \equiv \phi, \psi \equiv \psi, \delta\}$, $\Gamma^* = \{\delta, \phi \equiv \phi, \psi \equiv \psi\}$, $\Delta = \{\phi \equiv \psi, (\phi \otimes \phi) \equiv (\psi \otimes \psi)\}$

3.3.1 Completeness of $\mathsf{G3_{SCI}}$

We now present completeness through interpretation of H_{SCI} in $\mathsf{G3_{SCI}}$.

Theorem 11 (Interpretation of H_{SCI} within $\mathsf{G3_{SCI}}$)**.** *If formula ϕ is provable in axiomatic system H_{SCI}, then sequent $\Rightarrow \phi$ is provable in* $\mathsf{G3_{SCI}}$.

Proof. We show completeness as follows:

1. We show that for every axiom ψ, sequent $\Rightarrow \psi$ has a proof in $\mathsf{G3_{SCI}}$.
2. Based on the proof of a given formula ϕ in H_{SCI}, we show that sequent $\Rightarrow \phi$ has a proof in $\mathsf{G3_{SCI}}$ through simulation of *modus ponens* in sequent calculus.

We use induction with respect to the length of the proof of a given formula ϕ in deductive system H_{SCI}:

Base: If ϕ is a formula with a proof in H_{SCI} of length equal to 1, sequent $\Rightarrow \phi$ has a proof in $\mathsf{G3_{SCI}}$.

TABLE 3.4: $\ell\mathsf{G3}_{\mathsf{SCI}}$, $\mathsf{G3}_{\mathsf{SCI}}$: structural rules

$$\frac{\Gamma \Rightarrow \Delta, \phi \quad \phi, \Pi \Rightarrow \Theta}{\Gamma, \Pi \Rightarrow \Delta, \Theta}\ cut$$

$$\frac{\phi, \phi, \Gamma \Rightarrow \Delta}{\phi, \Gamma \Rightarrow \Delta}\ L_{ctr} \qquad \frac{\Gamma \Rightarrow \Delta, \phi, \phi}{\Gamma \Rightarrow \Delta, \phi}\ R_{ctr}$$

$$\frac{\Gamma \Rightarrow \Delta}{\phi, \Gamma \Rightarrow \Delta}\ L_{wk} \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \phi}\ R_{wk}$$

TABLE 3.5: $\mathsf{G3}_{\mathsf{SCI}}$: identity-based rules

$$\frac{\phi \equiv \phi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ L^1_{\equiv} \qquad \frac{\neg\phi \equiv \neg\chi, \phi \equiv \chi, \Gamma \Rightarrow \Delta}{\phi \equiv \chi, \Gamma \Rightarrow \Delta}\ L^2_{\equiv}$$

$$\frac{\phi \leftrightarrow \chi, \phi \equiv \chi, \Gamma \Rightarrow \Delta}{\phi \equiv \chi, \Gamma \Rightarrow \Delta}\ L^3_{\equiv}$$

$$\frac{(\phi \otimes \chi) \equiv (\psi \otimes \omega), \phi \equiv \psi, \chi \equiv \omega, \Gamma \Rightarrow \Delta}{\phi \equiv \psi, \chi \equiv \omega, \Gamma \Rightarrow \Delta}\ L^4_{\equiv}$$

Here we present proofs of axioms of SCI:

($\equiv_1$):

$$\frac{\phi \equiv \phi \Rightarrow \phi \equiv \phi}{\Rightarrow \phi \equiv \phi}\ L^1_{\equiv}$$

($\equiv_2$):

$$\dfrac{\dfrac{\neg\phi \equiv \neg\psi, \phi \equiv \psi \Rightarrow \neg\phi \equiv \neg\psi}{\phi \equiv \psi \Rightarrow \neg\phi \equiv \neg\psi}\ L^2_{\equiv}}{\Rightarrow (\phi \equiv \psi) \to (\neg\phi \equiv \neg\psi)}\ R_{\to}$$

($\equiv_3$):

$$\dfrac{\dfrac{\phi \leftrightarrow \psi, \phi \equiv \psi \Rightarrow \phi \leftrightarrow \psi}{\phi \equiv \psi \Rightarrow \phi \leftrightarrow \psi}\ L^3_{\equiv}}{\Rightarrow (\phi \equiv \psi) \to (\phi \leftrightarrow \psi)}\ R_{\to}$$

($\equiv_4$):

$$\cfrac{\cfrac{\cfrac{(\phi \otimes \chi) \equiv (\psi \otimes \omega), \phi \equiv \psi, \chi \equiv \omega \Rightarrow (\phi \otimes \chi) \equiv (\psi \otimes \omega)}{\phi \equiv \psi, \chi \equiv \omega \Rightarrow (\phi \otimes \chi) \equiv (\psi \otimes \omega)}\ L^4_{\equiv}}{(\phi \equiv \psi) \wedge (\chi \equiv \omega) \Rightarrow (\phi \otimes \chi) \equiv (\psi \otimes \omega)}\ L_{\wedge}}{\Rightarrow ((\phi \equiv \psi) \wedge (\chi \equiv \omega)) \to ((\phi \otimes \chi) \equiv (\psi \otimes \omega))}\ R_{\to}$$

Induction hypothesis: sequent $\Rightarrow \psi$ is provable in $\mathsf{G3}_{\mathsf{SCI}}$, if a given formula ψ has a proof in H_{SCI} of length at most n.

Suppose formula ϕ has a proof in H_{SCI} of length $(n+1)$. We then have the following: either (1) ϕ is an axiom of H_{SCI} or (2) ϕ is a conclusion of *modus ponens*. In the case of (1) we know sequent $\Rightarrow \phi$ has a proof, as shown above. In the case of (2) we build a derivation as shown on the derivation schema below (which shows that *modus ponens* is obtainable in $\mathsf{G3}_{\mathsf{SCI}}$):

$$\cfrac{\begin{matrix} D_1 \\ \vdots \\ \Rightarrow \chi \end{matrix} \qquad \cfrac{\begin{matrix} D_2 \\ \vdots \\ \Rightarrow \chi \to \phi \end{matrix} \qquad \cfrac{\chi \Rightarrow \chi \quad \phi \Rightarrow \phi}{\chi, \chi \to \phi \Rightarrow \phi}\ L_{\to}}{\chi \Rightarrow \phi}\ cut}{\Rightarrow \phi}\ cut$$

χ and $\chi \to \phi$ appear in a derivation of ϕ in H_{SCI}. Therefore their proofs in H_{SCI} are shorter than proofs of sequents $\Rightarrow \chi$ and $\Rightarrow \chi \to \phi$ in $\mathsf{G3}_{\mathsf{SCI}}$ (where D_1 and D_2 are proofs of the said sequents). As a result the above schema is a proof of sequent $\Rightarrow \phi$. $\square$

Theorem 12 (Completeness)**.** *If a sequent $\Rightarrow \phi$ is valid in* SCI*, it is provable in* $\mathsf{G3}_{\mathsf{SCI}}$.

Proof. From Theorem 10 we know that SCI is complete with regard to the algebraic semantics. This, paired with the Theorem 11 shows that for all formulae ϕ valid in SCI, sequent $\Rightarrow \phi$ is provable in $\mathsf{G3}_{\mathsf{SCI}}$. $\square$

3.3.2 Soundness of $\mathsf{G3}_{\mathsf{SCI}}$

In this section we examine $\mathsf{G3}_{\mathsf{SCI}}$ with regard to algebraic semantics. Even though utilization of algebraic semantics is a standard way of analyzing non-Fregean theories, the sequent calculi presented so far have not been the subject of this particular analysis.

We consider two properties of rules of $\mathsf{G3}_{\mathsf{SCI}}$: preservability of satisfiability of sequents from premiss to conclusion (correctness) and from conclusion to premiss (invertibility; particularly useful in proof-search from the root up). The logical rule $R_{\neg}, L_{\neg}, R_{\vee}, L_{\vee}, R_{\wedge}, L_{\wedge}, R_{\to}, L_{\to}, R_{\leftrightarrow}$, and $L_{\leftrightarrow}$ have both properties. Applying these rules to the premisses satisfied in a given SCI-model $\mathcal{A}$ under $h \in Hom(\mathcal{L}_{\mathsf{SCI}}, \mathcal{A})$ will result in a conclusion satisfied in $\mathcal{A}$ under h and the other way around. Their behavior is classical, therefore as the proofs are self-explanatory we will omit them in this section. We shall now focus on the correctness and invertibility of the four identity rules.

Definition 62 (Correctness of a rule). *Rule R is correct in* SCI *provided that for each* SCI*-model $\mathcal{M}$ and for every valuation h in $\mathcal{M}$, if the premiss(-es) of R is (are) satisfied in $\mathcal{M}$ under h, then so is the conclusion.*

Definition 63 (Invertibility of a rule). *Rule R is invertible in* SCI *provided that for each* SCI*-model $\mathcal{M}$ and for every valuation h in $\mathcal{M}$, if the conclusion of R is satisfied in $\mathcal{M}$ under h, then so is (are) its premiss(-es).*

For the following lemmas, $\mathcal{A}$ denotes an SCI-algebra, $\mathcal{M} = \langle \mathcal{A}, F \rangle$ is an arbitrary but fixed SCI-model and h is an arbitrary homomorphism from $\mathcal{L}_{\mathsf{SCI}}$ into $\mathcal{A}$.

Lemma 3. *$L^1_{\equiv}$ is correct in* SCI.

Proof. Suppose that (1) $\Gamma \Rightarrow \Delta$ is not satisfied in $\mathcal{M}$ under h and (2) $\phi \equiv \phi, \Gamma \Rightarrow \Delta$ is satisfied in $\mathcal{M}$ under h. From (1) we know that all of the formulae in the antecedent of the conclusion are satisfied and all formulae within the succedent are not satisfied in $\mathcal{M}$ under h. $\phi \equiv \phi$ is satisfied in $\mathcal{M}$ under h, because F is a normal ultrafilter. By assumption (2) there has to be at least one formula satisfied in Δ which stands in opposition to (1). □

Lemma 4. *$L^2_{\equiv}$ is correct in* SCI.

Proof. Suppose that (1) $\chi \equiv \phi, \Gamma \Rightarrow \Delta$ is not satisfied in $\mathcal{M}$ under h, and (2) $\neg\chi \equiv \neg\phi, \chi \equiv \phi, \Gamma \Rightarrow \Delta$ is satisfied in $\mathcal{M}$ under h. From (1) we have that $\chi \equiv \phi$ and all formulae from Γ are satisfied in $\mathcal{M}$ under h and no formula in Δ is satisifed in $\mathcal{M}$ under h. Therefore $h(\chi \equiv \phi) = h(\chi) \dot{\equiv} h(\phi) \in F$, and from that we have $h(\chi) = h(\phi)$. From (2) we know that if all formulae from the antecedent are satisfied in $\mathcal{M}$, then so is at least one formula from its succedent. Paired with (1) we know that $\neg\chi \equiv \neg\phi$ cannot be satisfied in $\mathcal{M}$. Then we have $h(\neg\chi \equiv \neg\phi) = h(\neg\chi) \dot{\equiv} h(\neg\phi)$, therefore $\dot{\neg} h(\chi) \neq \dot{\neg} h(\phi)$. But we know that $h(\chi) = h(\phi)$ from which it follows that $\dot{\neg} h(\chi) = \dot{\neg} h(\phi)$—a contradiction. □

Lemma 5. *$L^3_{\equiv}$ is correct in* SCI.

Proof. Suppose that (1) $\chi \equiv \phi, \Gamma \Rightarrow \Delta$ is not satisfied in $\mathcal{M}$ under h, and (2) $\chi \leftrightarrow \phi, \chi \equiv \phi, \Gamma \Rightarrow \Delta$ is satisfied in $\mathcal{M}$ under h. From (1) we know that $\chi \equiv \phi$ and formulae within Γ are satisfied in $\mathcal{M}$ and formulae from Δ are not. $\chi \equiv \phi$ being satisfied in $\mathcal{M}$ means that $h(\chi \equiv \phi) = h(\chi) \dot{\equiv} h(\phi) \in F$ and from that it follows that $h(\chi) = h(\phi)$. Hence it follows that $\chi \leftrightarrow \phi$ is satisfied in $\mathcal{M}$ under h, as there cannot be that simultaneously $h(\phi) \in F$ and $h(\chi) \notin F$. Therefore as the formulae in the antecedent of the premiss are satisfied in $\mathcal{M}$ under h then at least one formula in Δ is satisfied as well, which contradicts (1). □

Lemma 6. *$L^4_{\equiv}$ is correct in* SCI.

Proof. Suppose that (1) $\phi \equiv \psi, \chi \equiv \omega, \Gamma \Rightarrow \Delta$ is not satisfied in $\mathcal{M}$ under h and (2) $(\phi \otimes \chi) \equiv (\psi \otimes \omega), \phi \equiv \psi, \chi \equiv \omega, \Gamma \Rightarrow \Delta$ is satisfied in $\mathcal{M}$ under h. From (1) we know that both $h(\phi) \dot{\equiv} h(\psi) \in F$ and $h(\chi) \dot{\equiv} h(\omega) \in F$, and, consequently, $h(\phi) = h(\psi), h(\chi) = h(\omega)$. $\otimes \in \{\wedge, \vee, \rightarrow, \leftrightarrow, \equiv\}$, but in the case of classical connectives the reasoning is analogous, so we will focus on conjunction and identity connective:

1. ($\otimes = \wedge$) We will show that with the above information it cannot be that $(\phi \wedge \chi) \equiv (\psi \wedge \omega)$ is not satisfied in $\mathcal{M}$. For now let us assume that it is not satisfied in $\mathcal{M}$ under h. This would mean that $h(\phi \wedge \chi) \dot{\equiv} h(\psi \wedge \omega) \notin F$ and, accordingly, $h(\phi \wedge \chi) \neq h(\psi \wedge \omega)$. But from (1) we know that $h(\phi) = h(\psi)$ and $h(\chi) = h(\omega)$, therefore $h(\phi \wedge \chi) = h(\psi \wedge \omega)$, and then $h(\phi \wedge \chi) \dot{\equiv} h(\psi \wedge \omega) \in F$, which shows us that $(\phi \wedge \chi) \equiv (\psi \wedge \omega)$ is satisfied in $\mathcal{M}$ under h. As all the formulae from the antecedent are satisfied in $\mathcal{M}$ under h, then there has to be at least one formula in Δ which is satisfied as well. We arrived at the contradiction with (1).

2. ($\otimes = \equiv$) From (1) we have that $(\phi \equiv \chi)$ and $(\psi \equiv \omega)$ are satisfied in $\mathcal{M}$ under h. As above, let us assume the formula $(\phi \equiv \chi) \equiv (\psi \equiv \omega)$ is not satisfied in $\mathcal{M}$. It follows that $h(\phi \equiv \chi) \dot{\equiv} h(\psi \equiv \omega) \notin F$, which means that $h(\phi \equiv \chi) \neq h(\psi \equiv \omega)$. But from (1) we have $h(\phi) = h(\psi)$ and $h(\chi) = h(\omega)$. Therefore it cannot be the case that $(h(\phi) \dot{\equiv} h(\chi)) \neq (h(\psi) \dot{\equiv} h(\omega))$. As a result we have $h((\phi \equiv \chi) \equiv (\psi \equiv \omega)) \in F$, which contradicts (1).

The rule $L^4_{\equiv}$ is correct for all $\otimes \in \{\wedge, \vee, \rightarrow, \leftrightarrow, \equiv\}$. $\square$

We now move to invertibility of identity-dedicated rules.

Lemma 7. *$L^1_{\equiv}$ is invertible in* SCI.

Proof. Suppose that $\phi \equiv \phi, \Gamma \Rightarrow \Delta$ is not satisfied in $\mathcal{M}$ under h. That means that all formulae from Γ in conclusion of the rule are satisfied in $\mathcal{M}$ under h and no formula in Δ is satisfied in $\mathcal{M}$ under h, thus the conclusion is not satisfied in $\mathcal{M}$ under h. $\square$

Lemma 8. *$L^2_{\equiv}$ is invertible in* SCI.

Proof. Suppose that $\neg\chi \equiv \neg\phi, \chi \equiv \phi, \Gamma \Rightarrow \Delta$ is not satisfied in $\mathcal{M}$ under h. We then have that a conclusion $\chi \equiv \phi, \Gamma \Rightarrow \Delta$ of $L^2_{\equiv}$ is also not satisfied in $\mathcal{M}$ under h. $\square$

Lemma 9. *$L^3_{\equiv}$ is invertible in* SCI.

Proof. Suppose that $\chi \leftrightarrow \phi, \chi \equiv \phi, \Gamma \Rightarrow \Delta$ is not satisfied in $\mathcal{M}$ under h. We therefore know that $\chi \equiv \phi, \Gamma \Rightarrow \Delta$ is also not satisfied in $\mathcal{M}$ under h. $\square$

Lemma 10. *$L^4_{\equiv}$ is invertible in* SCI.

Proof. Suppose that $(\phi \otimes \chi) \equiv (\psi \otimes \omega), \phi \equiv \psi, \chi \equiv \omega, \Gamma \Rightarrow \Delta$ is not satisfied in $\mathcal{M}$ under h. Similarly as above, we can see that sequent $\phi \equiv \psi, \chi \equiv \omega, \Gamma \Rightarrow \Delta$ is not satisfied in $\mathcal{M}$ under h. $\square$

Through the above lemmas 7–10 we notice the semantic monotonicity. The addition of formulae to a given sequent satisfied in $\mathcal{M}$ under h will not alter the said property.

Theorem 13 (Soundness)**.** *If a sequent is provable in* $\mathsf{G3}_{\mathsf{SCI}}$*, it is valid in* SCI.

Proof. We show that all $\mathsf{G3}_{\mathsf{SCI}}$ axioms of the form $\phi, \Gamma \Rightarrow \Delta, \phi$ are valid in SCI. Let $\mathcal{M} = \langle \mathcal{A}, F \rangle$ be an arbitrary SCI-model. Then we have that for all $h \in Hom(\mathcal{L}_{\mathsf{SCI}}, \mathcal{A})$ if h satisfies ϕ, Γ in $\mathcal{M}$, then at least one formula in Δ, ϕ is also satisfied in $\mathcal{M}$ under h. We then examine an arbitrary branch of the proof tree and n^{th} and $n+1^{\text{st}}$ sequent (counting from the axiom). n^{th} sequent is then valid in SCI per induction hypothesis, $n+1^{\text{st}}$ sequent through Lemmas 3–6 and correctness of rules for classical connectives. □

Then from Theorems 12 and 13 we have the following:

Theorem 14 (Adequacy)**.** *Sequent* $\Rightarrow \phi$ *is provable in* $\mathsf{G3}_{\mathsf{SCI}}$ *iff* $\Rightarrow \phi$ *is valid in* SCI*.*

System $\ell\mathsf{G3}_{\mathsf{SCI}}$ has been shown in [4] to have all structural rules admissible; in Chapter 7 we will also show that making small changes to the language and, as a result, the rule set, does not affect this structural characteristic. We will then present the structure of the cut elimination proof for $\mathsf{G3}_{\mathsf{SCI}}$, which follows the steps of the one presented in [4].

In the following section we will exploit the relation between G3cp and $\mathsf{G3}_{\mathsf{SCI}}$ which will be utilized in subsequent chapters of the dissertation.

Theorem 15. *If sequent* $\Rightarrow \phi$ *has a proof in* G3cp*, then it also has a proof in* $\mathsf{G3}_{\mathsf{SCI}}$*.*

Proof. Language $\mathcal{L}_{\mathsf{SCI}}$ is an extension of $\mathcal{L}_{\mathsf{CPC}}$, therefore in derivation of $\Rightarrow \phi$ we can solely depend on rules from G3cp (which appear within the set of rules in $\mathsf{G3}_{\mathsf{SCI}}$). □

Now we will look at the following relations, which will be applicable in the subsequent chapters. We begin with a definition of function translating formulae between languages of CPC and SCI. By Eq we will mean the following set of formulae: $\mathsf{Eq} = \{\phi_1 \equiv \phi_2 : \phi_1, \phi_2 \in L_{\mathsf{SCI}}\}$.

A homomorphism $h \in Hom(\mathcal{L}_{\mathsf{SCI}}, \mathcal{L}_{\mathsf{SCI}})$ will be called atomic if for each variable $p_i \in \mathsf{VAR}$ the following holds: $h(p_i) = p_i$ or $h(p_i) \in \mathsf{Eq}$.

Let us recall that for a function f defined on a set $\mathbf{\Phi}$, by $f|_{\mathbf{\Psi}}$, where $\mathbf{\Psi} \subseteq \mathbf{\Phi}$, we mean the restriction of function f to set $\mathbf{\Psi}$.

Definition 64 (Translation from $\mathcal{L}_{\mathsf{CPC}}$ to $\mathcal{L}_{\mathsf{SCI}}$)**.** *Let* $h \in Hom(\mathcal{L}_{\mathsf{SCI}}, \mathcal{L}_{\mathsf{SCI}})$ *be an atomic homomorphism. The restriction* $h|_{L_{\mathsf{CPC}}}$ *of function* h *to the set* L_{CPC} *of formulae of language* $\mathcal{L}_{\mathsf{CPC}}$ *will be called a translation from* $\mathcal{L}_{\mathsf{CPC}}$ *to* $\mathcal{L}_{\mathsf{SCI}}$ *indicated by atomic homomorphism* h *or simply a translation from* $\mathcal{L}_{\mathsf{CPC}}$ *to* $\mathcal{L}_{\mathsf{SCI}}$*.*

Letter μ will be used to denote translations from $\mathcal{L}_{\mathsf{CPC}}$ to $\mathcal{L}_{\mathsf{SCI}}$.

Let us observe that from Definition 67 the following corollary follows.

Corollary 1. *If* μ *is a translation from* $\mathcal{L}_{\mathsf{CPC}}$ *to* $\mathcal{L}_{\mathsf{SCI}}$ *indicated by* h*, then*

1. *for each* $p_i \in \mathsf{VAR}$*,* $\mu(p_i) = p_i$ *or* $\mu(p_i) \in \mathsf{Eq}$*,*

2. *for each* $\phi \in L_{\mathsf{CPC}}$*,* $\mu(\neg\phi) = h(\neg\phi) = \neg h(\phi) = \neg\mu(\phi)$*,*

3. *for each* $\phi, \psi \in L_{\mathsf{CPC}}$, $\mu(\phi \otimes \psi) = h(\phi \otimes \psi) = h(\phi) \otimes h(\psi) = \mu(\phi) \otimes \mu(\psi)$.

Suppose ϕ is a formula of $\mathcal{L}_{\mathsf{CPC}}$ such that sequent $\Rightarrow \phi$ has a proof in $\mathsf{G3cp}$. Let us consider a translation μ from language $\mathcal{L}_{\mathsf{CPC}}$ into language $\mathcal{L}_{\mathsf{SCI}}$. The next theorem is a generalization of Theorem 15. It states that sequent $\Rightarrow \mu(\phi)$ has a proof in $\mathsf{G3}_{\mathsf{SCI}}$, where the only rules utilized are classical rules.

Theorem 16. *If sequent* $\Rightarrow \phi$ *has a proof in* $\mathsf{G3cp}$*, then for every translation* μ *from* $\mathcal{L}_{\mathsf{CPC}}$ *to* $\mathcal{L}_{\mathsf{SCI}}$ *sequent* $\Rightarrow \mu(\phi)$ *has a proof in* $\mathsf{G3}_{\mathsf{SCI}}$*, where the only rules utilized are classical rules.*

Proof. Let D be a proof of a sequent $\Rightarrow \phi$ in $\mathsf{G3cp}$. We consider all cases, that is D being an axiom or consisting of application of the classical rules. We show that in each case we obtain derivations "homomorphic" to the original ones; i.e., rules are applied to applications of function μ. Following that, by $\mu(\Gamma)$ we mean multiset obtained through application of function μ to all formulae in Γ.

$$(1) \quad \Gamma, \psi \Rightarrow \psi, \Delta \quad \rightsquigarrow \quad \mu(\Gamma), \mu(\psi) \Rightarrow \mu(\psi), \mu(\Delta)$$

$$(2) \quad \frac{\Gamma \Rightarrow \Delta, \chi}{\neg\chi, \Gamma \Rightarrow \Delta}\, L_{\neg} \quad \rightsquigarrow \quad \frac{\mu(\Gamma) \Rightarrow \mu(\Delta), \mu(\chi)}{\neg\mu(\chi), \mu(\Gamma) \Rightarrow \mu(\Delta)}\, L_{\neg}$$

$$(3) \quad \frac{\chi, \psi, \Gamma \Rightarrow \Delta}{\chi \wedge \psi, \Gamma \Rightarrow \Delta}\, L_{\wedge} \quad \rightsquigarrow \quad \frac{\mu(\chi), \mu(\psi), \mu(\Gamma) \Rightarrow \mu(\Delta)}{\mu(\chi) \wedge \mu(\psi), \mu(\Gamma) \Rightarrow \mu(\Delta)}\, L_{\wedge}$$

$$(4) \quad \frac{\chi, \Gamma \Rightarrow \Delta \quad \psi, \Gamma \Rightarrow \Delta}{\chi \vee \psi, \Gamma \Rightarrow \Delta}\, L_{\vee}$$

$$\rightsquigarrow$$

$$\frac{\mu(\chi), \mu(\Gamma) \Rightarrow \mu(\Delta) \quad \mu(\psi), \mu(\Gamma) \Rightarrow \mu(\Delta)}{\mu(\chi) \vee \mu(\psi), \mu(\Gamma) \Rightarrow \mu(\Delta)}\, L_{\vee}$$

$$(5) \quad \frac{\Gamma \Rightarrow \Delta, \chi \quad \psi, \Gamma \Rightarrow \Delta}{\chi \to \psi, \Gamma \Rightarrow \Delta}\, L_{\to}$$

$$\rightsquigarrow$$

$$\frac{\mu(\Gamma) \Rightarrow \mu(\Delta), \mu(\chi) \quad \mu(\psi), \mu(\Gamma) \Rightarrow \mu(\Delta)}{\mu(\chi) \to \mu(\psi), \mu(\Gamma) \Rightarrow \mu(\Delta)}\, L_{\to}$$

$$(6) \quad \frac{\chi, \psi, \Gamma \Rightarrow \Delta \quad \Gamma \Rightarrow \Delta, \chi, \psi}{\chi \leftrightarrow \psi, \Gamma \Rightarrow \Delta}\, L_{\leftrightarrow}$$

$$\rightsquigarrow$$

$$\frac{\mu(\chi), \mu(\psi), \mu(\Gamma) \Rightarrow \mu(\Delta) \quad \mu(\Gamma) \Rightarrow \mu(\Delta), \mu(\chi), \mu(\psi)}{\mu(\chi) \leftrightarrow \mu(\psi), \mu(\Gamma) \Rightarrow \mu(\Delta)}\, L_{\leftrightarrow}$$

$\square$

The following theorem examines reverse transformation, that is from the proof in $\mathsf{G3}_{\mathsf{SCI}}$ to the proof in $\mathsf{G3cp}$.

Theorem 17. *Let $\Rightarrow \phi$ be a sequent which has a proof in $\mathsf{G3}_{\mathsf{SCI}}$ such that in the proof we have applications of classical and structural rules only. Then there exists a formula χ of $\mathcal{L}_{\mathsf{CPC}}$ which is valid in CPC-models and a translation μ from $\mathcal{L}_{\mathsf{CPC}}$ to $\mathcal{L}_{\mathsf{SCI}}$ such that $\mu(\chi) = \phi$.*

Proof. Let us consider proof D of a sequent $\Rightarrow \phi$ in $\mathsf{G3}_{\mathsf{SCI}}$. We will substitute any occurrences of equations within D with variables in such a way that we will obtain a proof of a sequent $\Rightarrow \chi$ in $\mathsf{G3cp}$ (i.e. $\mu(\chi) = \phi$, where χ will be a formula resulting from ϕ by this elimination of equations).

Let $\mathsf{VAR}(D)$ be the set of all variables occurring in D and let $\mathsf{Eq}(D)$ be the set of all equations occurring in D. All elements $\psi_1, ..., \psi_j$ of $\mathsf{Eq}(D)$ will be assigned to variables $p_{i_1}, ..., p_{i_j}$ which are not elements of $\mathsf{VAR}(D)$. If a given equation appears more than once, all its occurrences will be substituted with the same variable; otherwise different equations are substituted with different variables. The outlined procedure indicates the definition of translation μ from $\mathcal{L}_{\mathsf{CPC}}$ to $\mathcal{L}_{\mathsf{SCI}}$. First we define $\mu^* : \mathsf{VAR} \longrightarrow \mathsf{VAR} \cup \mathsf{Eq}$

- $\mu^*(p_i) = p_i$: for $p_i \in \mathsf{VAR}(D)$;
- $\mu^*(p_{i_k}) = \psi_k$: for variables $p_{i_k} \notin VAR(D)$, for each i_k , $1 \leq k \leq j$ (if $\mathsf{Eq}(D)$ consists of j equations);
- $\mu^*(p_i) = p_i$: for other variables.

Function μ^* can be extended to a homomorphism $h \in Hom(\mathcal{L}_{\mathsf{SCI}}, \mathcal{L}_{\mathsf{SCI}})$ in a unique way. We can see that h is atomic, hence $\mu : L_{\mathsf{CPC}} \longrightarrow L_{\mathsf{SCI}}$ defined as a restriction of h to L_{CPC} is a translation from $\mathcal{L}_{\mathsf{CPC}}$ to $\mathcal{L}_{\mathsf{SCI}}$. Moreover, it is easy to see that μ is an injective function, therefore the inverse function μ^{-1} is well-defined. Let χ stand for $\mu^{-1}(\phi)$.

Now we consider proof D and tree D'. Let D' be a derivation tree obtained through employment of the above procedure. We want to show that D' is a proof of $\Rightarrow \chi$ in $\mathsf{G3cp}$.

Any axiom $\Gamma, \psi \Rightarrow \Delta, \psi$ of $\mathsf{G3}_{\mathsf{SCI}}$ in D corresponds to axiom $\mu^{-1}(\Gamma), \mu^{-1}(\psi) \Rightarrow \mu^{-1}(\Delta), \mu^{-1}(\psi)$ in $\mathsf{G3cp}$. The same goes for any classical rule application, for instance $R_\wedge$ of $\mathsf{G3}_{\mathsf{SCI}}$

$$\frac{\Gamma \Rightarrow \Delta, \psi_1 \quad \Gamma \Rightarrow \Delta, \psi_2}{\Gamma \Rightarrow \Delta, \psi_1 \wedge \psi_2}\, R_\wedge$$

corresponds to the application of $R_\wedge$ in $\mathsf{G3cp}$:

$$\frac{\mu^{-1}(\Gamma) \Rightarrow \mu^{-1}(\Delta), \mu^{-1}(\psi_1) \quad \mu^{-1}(\Gamma) \Rightarrow \mu^{-1}(\Delta), \mu^{-1}(\psi_2)}{\mu^{-1}(\Gamma) \Rightarrow \mu^{-1}(\Delta), \mu^{-1}(\psi_1) \wedge \mu^{-1}(\psi_2)}\, R_\wedge$$

where $(\mu^{-1}(\psi_1) \wedge \mu^{-1}(\psi_2)) = \mu^{-1}(\psi_1 \wedge \psi_2)$, i.e. we do not change the internal logical structure of the initial formula.

The application of the remaining rules in both calculi can be analyzed analogously. As a result we showed that $\Rightarrow \mu^{-1}(\phi)$ has a proof in G3cp and since $\mu^{-1}(\phi) = \chi$, from Theorem 5 we have that χ is valid in CPC. $\square$

Chapter 4

WB logic and sequent calculus $\mathsf{G3}_{\mathsf{WB}}$

WB is the first of three main axiomatic extensions of SCI we examine in the context of structural proof theory. It constitutes a set of formulae that are true in Boolean algebras, and is obtained through extending interpretation of the identity connective through Boolean algebra axioms. In this section we will begin by introducing the axiomatic system and semantics. We will show how identity is still stronger than equivalence. Then we will move to sequent calculus $\mathsf{G3}_{\mathsf{WB}}$ accompanied by analysis of its major characteristics.

4.1 Hilbert system for WB

For the Boolean extension of SCI, for the sake of simplicity we shall use the same language as in the case of SCI, in which we had both $\neg$ and $\leftrightarrow$: from now on we will denote the language algebra as $\mathcal{L}$, which will also apply for the subsequent chapters/extensions of SCI.

Definition 65. *The language of* WB *is an algebra of similarity type* $\langle 1, 2, 2, 2, 2, 2 \rangle$

$$\mathcal{L} = \langle L, \neg, \wedge, \vee, \rightarrow, \leftrightarrow, \equiv \rangle.$$

However, in contrast to SCI-language $\mathcal{L}_{\mathsf{SCI}}$, we can now also define two constants, falsum $\bot =_{df} p_1 \wedge \neg p_1$ and constant verum $\top =_{df} \neg\bot$. In SCI these definitions could not be added, as we would obtain a stronger logic, where $\neg\top \equiv \bot$ would be a theorem, which is not a theorem of SCI. It is, however, a theorem in WB. Moreover, an equation of $\top$ and any given axiom of WB ϕ will not hold in WB, but $\top \leftrightarrow \phi$ will.

WB is the smallest Boolean non-Fregean theory. We obtain it through an addition of the set of axioms WBA to TFA $\cup$ IDA, represented by the following axiom schemata [38, p. 103]:

($\equiv_5$) $((\phi \wedge \chi) \vee \psi) \equiv ((\chi \vee \psi) \wedge (\phi \vee \psi))$

($\equiv_6$) $((\phi \vee \chi) \wedge \psi) \equiv ((\chi \wedge \psi) \vee (\phi \wedge \psi))$

($\equiv_7$) $(\phi \vee (\chi \wedge \neg\chi)) \equiv \phi$

($\equiv_8$) $(\phi \wedge (\chi \vee \neg\chi)) \equiv \phi$

($\equiv_9$) $(\phi \to \chi) \equiv (\neg\phi \vee \chi)$

($\equiv_{10}$) $(\phi \leftrightarrow \chi) \equiv ((\phi \to \chi) \wedge (\chi \to \phi))$

The WB-identity is weaker than SCI-identity, however it is still not flattened to the equivalence. The set of WB axioms along with *modus ponens* allows us to prove equations of the form $\phi \equiv \chi$ provided $\phi \leftrightarrow \chi$ can be derived from the set TFA of axioms, which is depicted in the following, alternative to axiomatic, definition of WB:

$$\mathsf{WB} = C(\{\phi \equiv \chi : (\phi \leftrightarrow \chi) \in \mathsf{TFT}\}) \tag{WB}$$

Definition 66 (Derivation, proof)**.** *Let Φ stand for a set of formulae of $\mathcal{L}$. A finite sequence $\phi_1, ..., \phi_n$ of formulae of $\mathcal{L}$ is a derivation of ϕ from Φ provided $\phi_n = \phi$ and each formula $\phi_i, i \leq n$, either belongs to Φ or has been derived from some ϕ_{i_1}, ϕ_{i_2}, $(i_1, i_2 < i)$ through an application of modus ponens. If $\Phi = \mathsf{TFA} \cup \mathsf{IDA} \cup \mathsf{WBA}$, then derivation $\phi_1, ..., \phi_n$ is a proof of ϕ in the axiomatic system for WB.*

If $\phi_1, ..., \phi_n$ is a proof of ϕ from $\mathsf{TFA} \cup \mathsf{IDA} \cup \mathsf{WBA}$, then the length of this proof is n.

Consequence operation C_{WB} is defined by set $\mathsf{TFA} \cup \mathsf{IDA} \cup \mathsf{WBA}$ of axioms and a singular inference rule, *modus ponens*, that is $\phi \in C_{\mathsf{WB}}(\Phi)$ iff $\phi \in C(\Phi \cup \mathsf{TFA} \cup \mathsf{SCI} \cup \mathsf{WBA})$. Elements of $C_{\mathsf{WB}}(\emptyset)$ are called logical theorems of WB. In [38, p. 104] these facts are expressed as follows:

$$\alpha \in C_{\mathsf{WB}}(\Phi) \Leftrightarrow \alpha \in C(\mathsf{WB} \cup \Phi) \Leftrightarrow \alpha \in C(\mathsf{WBA} \cup \Phi).$$

Naturally, C_{WB} satisfies conditions specified in Theorem 9.

Finally, in keeping with Suszko we introduce the logic WB as a deductive system H_{WB}.

Definition 67. $H_{\mathsf{WB}} = \langle \mathcal{L}, C_{\mathsf{WB}} \rangle$.

4.2 Semantics of WB

Definition 68. *Algebra $\mathcal{A} = \langle A, \dot{\neg}, \dot{\cap}, \dot{\cup}, \dot{\to}, \dot{\leftrightarrow}, \dot{\equiv} \rangle$ of similarity type $\langle 1, 2, 2, 2, 2, 2 \rangle$ is called B-algebra provided that the following conditions are satisfied:*

(b_1) $((a \dot{\cap} b) \dot{\cup} c) = ((b \dot{\cup} c) \dot{\cap} (a \dot{\cup} c))$,

(b_2) $((a \dot{\cup} b) \dot{\cap} c) = ((b \dot{\cap} c) \dot{\cup} (a \dot{\cap} c))$,

(b_3) $(a \dot{\cup} (b \dot{\cap} \dot{\neg} b)) = a$,

(b_4) $(a \dot{\cap} (b \dot{\cup} \dot{\neg} b)) = a$,

(b_5) $(a \dot{\to} b) = (\dot{\neg} a \dot{\cup} b)$,

(b_6) $(a \dot{\leftrightarrow} b) = ((a \dot{\to} b) \dot{\cap} (b \dot{\to} a))$.

Set A is called a universe of situations.

Conditions (b_1)—(b_6) correspond to axioms $(\equiv_5)$—$(\equiv_{10})$. As we can see, a B-algebra is a Boolean algebra with additional binary operation $\doteqdot$ (see Definition 17). Given a B-algebra $\mathcal{A}$, by $0^{\mathcal{A}}$ and $1^{\mathcal{A}}$ we refer, respectively, to the zero and the unit of $\mathcal{A}$. (In the sequel we drop $\mathcal{A}$ in the superscript).

Definition 69. *Pair $\langle \mathcal{A}, F\rangle$ is called a B-model if and only if $\mathcal{A}$ is a B-algebra and F is a normal ultrafilter in $\mathcal{A}$.*

As a result, the above conditions (Definition 68) paired with the conditions listed in Definition 49 give us a bigger set of schemata of valid equations (in the case of SCI it consisted of the singular schema $\phi \equiv \phi$). With that, we can define the satisfiability, truth and validity of a given formula in WB.

Definition 70 (Satisfiability of a formula in a model)**.** *Let $\mathcal{M} = \langle \mathcal{A}, F\rangle$ be an arbitrary but fixed B-model, and let $h \in Hom(\mathcal{L}, \mathcal{A})$. Formula ϕ is satisfied in $\mathcal{M}$ under h if and only if $h(\phi) \in F$.*

Definition 71 (Truth of a formula in a model)**.** *Let $\mathcal{M} = \langle \mathcal{A}, F\rangle$ be an arbitrary but fixed B-model. Formula ϕ is true in $\mathcal{M}$ if and only if: for all $h \in Hom(\mathcal{L}, \mathcal{A})$ ϕ is satisfied in $\mathcal{M}$ under h.*

Definition 72 (Validity of a formula)**.** *Formula ϕ is valid in WB (in symbols $\models_{\mathsf{WB}} \phi$) iff ϕ is true in all B-models $\mathcal{M}$.*

We will also refer to formulae valid in WB as WB-tautologies.

Theorem 18 (Semantic modus ponens)**.** *Let $\mathcal{M} = \langle \mathcal{A}, F\rangle$ be a B-model. If $a \in F$ and $a \dot{\to} b \in F$, then $b \in F$.*

Proof. Suppose $a, a \dot{\to} b \in F$. From point $(b5)$ in Definition 68 and point 3 from Definition 49 we know that either $\dot{\neg} a \in F$ or $b \in F$. The former is impossible per point d from Theorem 3. $\square$

Theorem 18 warrants that *modus ponens* applied to formulae true in a given B-model always results in a formula which is true in the same model. Another theorem states that WB is both sound and complete with respect to the presented semantics.

Theorem 19. WB *is the set of all and only formulae true in every* B*-model [13; 38].*

The proof of the theorem can be found in [38]. In the proof we show that WB is a subset of the set of all formulae true in every B-model $\mathcal{M} = \langle \mathcal{A}, F\rangle$. We do it by showing that a given homomorphism $h \in Hom(\mathcal{L}, \mathcal{A})$ sends all WB axioms to F, accompanied by the Theorem 18. Then, for a given $\phi \notin \mathsf{WB}$ we show that there is a B-model such that ϕ is not an element of all formulae true in the said model.

4.3 Sequent Calculus $\mathsf{G3_{WB}}$

Sequent calculus $\mathsf{G3_{WB}}$ is based on the sequent calculus we described in the previous section, $\mathsf{G3_{SCI}}$. We extend $\mathsf{G3_{SCI}}$ by one rule, a singular right identity rule $R^{B}_{\equiv}$. $L^{2}_{\equiv}$, as mentioned earlier, has been added due to the utilization of classical negation $\neg\phi$ in the calculus, instead of defining it as $\phi \rightarrow \bot$. Four left-sided identity rules constitute a syntactic interpretation of identity connective in SCI. Rule $R^{B}_{\equiv}$ (see Table 4.3), on the other hand, is a formalized syntactic depiction of the (WB) definition and can be applied provided sequent S does not contain any formula in the antecedent and only a singular formula in the succedent. Moreover, to control the application of the rules, we introduced markers to label whole sequents: $\stackrel{C}{\Rightarrow}$ and $\stackrel{I}{\Rightarrow}$. When we apply rules from premiss to conclusion, label $\stackrel{C}{\Rightarrow}$ allows the application of classical, structural and rule $R^{B}_{\equiv}$; label $\stackrel{I}{\Rightarrow}$ allows the application of classical, structural and left identity-dedicated rules. When we consider applications of rules from conclusion to premiss, we can say that conclusion's label $\stackrel{I}{\Rightarrow}$ allows the application of all rules, whereas conclusion's label $\stackrel{C}{\Rightarrow}$ allows the application of classical and structural rules only. We therefore consider the following structure

$$\Gamma \stackrel{X}{\Rightarrow} \Delta,$$

where $\stackrel{X}{\Rightarrow} \in \{\stackrel{I}{\Rightarrow}, \stackrel{C}{\Rightarrow}\}$. $\Gamma \stackrel{X}{\Rightarrow} \Delta$ is called a WB-sequent.

It is worth commenting on the fact that in this particular system we do not utilize Nergi's strategy of turning axioms into rules of sequent calculus, as was done in the case of $\mathsf{G3_{SCI}}$. The reason for this is rooted in simplicity; WB and two other extensions are obtained through addition of axioms, therefore the set of rules would gradually extend by addition of left-sided identity rules. If we were to apply this strategy in WB, we would obtain six rules, which allow us to introduce into the antecedent of the premiss of the rule a formula $(\phi \equiv \chi)$ which falls under the schema of one of the Boolean axioms, that is:

$$\frac{\phi \equiv \chi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ L^{\mathsf{B}}_{\equiv}$$

From the automatic reasoning perspective, it might be quite tricky to decide what exactly formula $\phi \equiv \chi$ should specifically be. We therefore have decided to formalize a different approach and examine the consequences of such a choice. On the surface the proof-searching process can be easier with $R^{B}_{\equiv}$, but we will analyse certain struggles that this entails. We also note that a system with rule $L^{\mathsf{B}}_{\equiv}$ in it has been examined in [59] and can be compared to the system we present here, as both constitute sound and complete sequent systems for WB (however with several structural differences, particularly with regard to the structural rules' admissibility). Nonetheless, in this thesis we will define $\mathsf{G3_{WB}}$ as $\{L_{\wedge}, R_{\wedge}, L_{\vee}, R_{\vee}, L_{\rightarrow}, R_{\rightarrow}, L_{\leftrightarrow}, R_{\leftrightarrow}, L_{\neg}, R_{\neg}, L^{1}_{\equiv}, L^{2}_{\equiv}, L^{3}_{\equiv}, L^{4}_{\equiv}, R^{B}_{\equiv}, cut\}$.[1]

[1]We will consider two variants, the other one, that is $\{L_{\wedge}, R_{\wedge}, L_{\vee}, R_{\vee}, L_{\rightarrow}, R_{\rightarrow}, L_{\leftrightarrow}, R_{\leftrightarrow}, L_{\neg}, R_{\neg}, L^{1}_{\equiv}, L^{2}_{\equiv}, L^{3}_{\equiv}, L^{4}_{\equiv}, R^{B}_{\equiv}, L_{wk}, R_{wk}\}$, differing in the use of structural rules, which we will comment on later on.

TABLE 4.1: Rules of $\mathsf{G3_{WB}}$: classical rules

$$\frac{\Gamma \overset{X}{\Rightarrow} \Delta, \phi}{\neg\phi, \Gamma \overset{X}{\Rightarrow} \Delta}\, L_\neg \qquad \phi, \Gamma \overset{X}{\Rightarrow} \Delta, \phi \qquad \frac{\phi, \Gamma \overset{X}{\Rightarrow} \Delta}{\Gamma \overset{X}{\Rightarrow} \Delta, \neg\phi}\, R_\neg$$

$$\frac{\phi, \chi, \Gamma \overset{X}{\Rightarrow} \Delta}{\phi \wedge \chi, \Gamma \overset{X}{\Rightarrow} \Delta}\, L_\wedge \qquad \frac{\Gamma \overset{X}{\Rightarrow} \Delta, \phi \quad \Gamma \overset{X}{\Rightarrow} \Delta, \chi}{\Gamma \overset{X}{\Rightarrow} \Delta, \phi \wedge \chi}\, R_\wedge \qquad \frac{\phi, \Gamma \overset{X}{\Rightarrow} \Delta \quad \chi, \Gamma \overset{X}{\Rightarrow} \Delta}{\phi \vee \chi, \Gamma, \overset{X}{\Rightarrow} \Delta}\, L_\vee$$

$$\frac{\Gamma \overset{X}{\Rightarrow} \Delta, \phi, \chi}{\Gamma \overset{X}{\Rightarrow} \Delta, \phi \vee \chi}\, R_\vee \qquad \frac{\Gamma \overset{X}{\Rightarrow} \Delta, \phi \quad \chi, \Gamma \overset{X}{\Rightarrow} \Delta}{\phi \to \chi, \Gamma \overset{X}{\Rightarrow} \Delta}\, L_\to \qquad \frac{\phi, \Gamma \overset{X}{\Rightarrow} \Delta, \chi}{\Gamma \overset{X}{\Rightarrow} \Delta, \phi \to \chi}\, R_\to$$

$$\frac{\phi, \chi, \Gamma \overset{X}{\Rightarrow} \Delta \quad \Gamma \overset{X}{\Rightarrow} \Delta, \phi, \chi}{\phi \leftrightarrow \chi, \Gamma \overset{X}{\Rightarrow} \Delta}\, L_\leftrightarrow \qquad \frac{\phi, \Gamma \overset{X}{\Rightarrow} \chi, \Delta \quad \chi, \Gamma \overset{X}{\Rightarrow} \phi, \Delta}{\Gamma \overset{X}{\Rightarrow} \Delta, \phi \leftrightarrow \chi}\, R_\leftrightarrow$$

where $\overset{X}{\Rightarrow} \in \{\overset{I}{\Rightarrow}, \overset{C}{\Rightarrow}\}$

We can (and will) consider the two following derived axiom schemata

$$\Gamma \overset{X}{\Rightarrow} \Delta, \top \qquad\qquad \bot, \Gamma \overset{X}{\Rightarrow} \Delta$$

as we can obtain them from the rules in Tables 4.1—4.3, for instance (we know that $\top =_{df} \neg\bot =_{df} \neg(p_1 \wedge \neg p_1)$):

$$\cfrac{\cfrac{\cfrac{p_1, \Gamma \overset{X}{\Rightarrow} \Delta, p_1}{p_1, \neg p_1 \Gamma, \overset{X}{\Rightarrow} \Delta}\, L_\neg}{p_1 \wedge \neg p_1, \Gamma \overset{X}{\Rightarrow} \Delta}\, L_\wedge}{\Gamma \overset{X}{\Rightarrow} \Delta, \neg(p_1 \wedge \neg p_1)}\, R_\neg$$

The second sequent axiom schema can be obtained analogously. We use these shortcuts to keep derivations more concise.

Definition 73 (Derivation of WB-sequent $\Gamma \overset{X}{\Rightarrow} \Delta$ in $\mathsf{G3_{WB}}$). *Derivation of WB-sequent $\Gamma \overset{X}{\Rightarrow} \Delta$ in $\mathsf{G3_{WB}}$ is a finite labelled tree with a single root labelled with $\Gamma \overset{X}{\Rightarrow} \Delta$ and each node-label connected with the labels of the (immediate) successor nodes (if any) according to one of the rules of $\mathsf{G3_{WB}}$.*

Definition 74 (Proof of WB-sequent $\Gamma \overset{X}{\Rightarrow} \Delta$ in $\mathsf{G3_{WB}}$). *Proof of WB-sequent $\Gamma \overset{X}{\Rightarrow} \Delta$ in $\mathsf{G3_{WB}}$ is a derivation of $\Gamma \overset{X}{\Rightarrow} \Delta$ in $\mathsf{G3_{WB}}$ with axioms at all of the leaves.*

TABLE 4.2: Rules of G3$_{\mathsf{WB}}$: structural rules

$$\frac{\Gamma \overset{X}{\Rightarrow} \Delta}{\chi, \Gamma \overset{X}{\Rightarrow} \Delta} L_{wk} \qquad \frac{\chi, \chi, \Gamma \overset{X}{\Rightarrow} \Delta}{\chi, \Gamma \overset{X}{\Rightarrow} \Delta} L_{ctr}$$

$$\frac{\Gamma \overset{X}{\Rightarrow} \Delta}{\Gamma \overset{X}{\Rightarrow} \Delta, \chi} R_{wk} \qquad \frac{\Gamma \overset{X}{\Rightarrow} \Delta, \chi, \chi}{\Gamma \overset{X}{\Rightarrow} \Delta, \chi} R_{ctr}$$

$$\frac{\Gamma \overset{X}{\Rightarrow} \Sigma, \phi \quad \phi, \Pi \overset{X}{\Rightarrow} \Delta}{\Gamma, \Pi \overset{X}{\Rightarrow} \Sigma, \Delta} cut$$

where $\overset{X}{\Rightarrow} \in \{\overset{I}{\Rightarrow}, \overset{C}{\Rightarrow}\}$

Let us consider the following proof of $(\phi \vee (\phi \to \bot)) \equiv (\chi \vee (\chi \to \bot))$:

$$\dfrac{\dfrac{\dfrac{\dfrac{(\phi \vee (\phi \to \bot)), \chi \overset{C}{\Rightarrow} \chi, \bot}{(\phi \vee (\phi \to \bot)) \overset{C}{\Rightarrow} \chi, (\chi \to \bot)} R_{\to}}{(\phi \vee (\phi \to \bot)) \overset{C}{\Rightarrow} (\chi \vee (\chi \to \bot))} R_{\vee} \qquad \dfrac{\dfrac{(\chi \vee (\chi \to \bot)), \phi \overset{C}{\Rightarrow} \phi, \bot}{(\chi \vee (\chi \to \bot)) \overset{C}{\Rightarrow} \phi, (\phi \to \bot)} R_{\to}}{(\chi \vee (\chi \to \bot)) \overset{C}{\Rightarrow} (\phi \vee (\phi \to \bot))} R_{\vee}}{\overset{C}{\Rightarrow} (\phi \vee (\phi \to \bot)) \leftrightarrow (\chi \vee (\chi \to \bot))} R_{\leftrightarrow}}{\overset{I}{\Rightarrow} (\phi \vee (\phi \to \bot)) \equiv (\chi \vee (\chi \to \bot))} R^{B}_{\equiv}$$

When searching for a proof, we advise beginning with the root of the derivation tree with label $\overset{I}{\Rightarrow}$ to enable the application of identity-dedicated rules. Of course, if we were to construct a proof for sequent $\Rightarrow \phi$ (assuming there exists one) in G3cp, we could construct a similar proof of $\overset{X}{\Rightarrow} \phi$ in WB. The root of such sequent tree could be labelled either with $\overset{C}{\Rightarrow} \phi$ or $\overset{I}{\Rightarrow} \phi$ as rules from Table 4.1 work on both labels. We can therefore say that labels in the case of classical and structural rules are, in a way, translucent; applying them will not change the label of the WB-sequent.

Fact 1. *Derivations of a* WB*-sequent $\overset{C}{\Rightarrow} \phi$ beginning with leaves labelled with $\overset{C}{\Rightarrow}$ can also be constructed ending with $\overset{I}{\Rightarrow} \phi$ and beginning with leaves labelled with $\overset{I}{\Rightarrow}$.*

The reason for restrictions applied on the form of identity rules, especially $R^{B}_{\equiv}$, lies in controlling the application of rules. Suppose we want to apply rule $L^{1}_{\equiv}$ to an arbitrary sequent S. If we were to start the root-first construction with the root labelled with $\overset{C}{\Rightarrow}$ we would naturally not be able to apply it at all.

Moreover, an examination of the rules, particularly rule $R^{B}_{\equiv}$, shows that we can consider derivations with $\overset{I}{\Rightarrow}$ labelling the root of the tree and $\overset{C}{\Rightarrow}$ labelling

Table 4.3: Rules of $\mathsf{G3_{WB}}$: identity rules

$$\frac{\phi \equiv \phi, \Gamma \overset{I}{\Rightarrow} \Delta}{\Gamma \overset{I}{\Rightarrow} \Delta}\; L^1_{\equiv} \qquad \frac{\chi \leftrightarrow \phi, \chi \equiv \phi, \Gamma \overset{I}{\Rightarrow} \Delta}{\chi \equiv \phi, \Gamma \overset{I}{\Rightarrow} \Delta}\; L^3_{\equiv}$$

$$\frac{\neg\phi \equiv \neg\chi, \Gamma \overset{I}{\Rightarrow} \Delta}{\phi \equiv \chi, \Gamma \overset{I}{\Rightarrow} \Delta}\; L^2_{\equiv} \qquad \frac{(\phi \otimes \chi) \equiv (\psi \otimes \omega), \phi \equiv \psi, \chi \equiv \omega, \Gamma \overset{I}{\Rightarrow} \Delta}{\phi \equiv \psi, \chi \equiv \omega, \Gamma \overset{I}{\Rightarrow} \Delta}\; L^4_{\equiv}$$

$$\frac{\overset{C}{\Rightarrow} \phi \leftrightarrow \chi}{\overset{I}{\Rightarrow} \phi \equiv \chi}\; R^B_{\equiv}$$

the leaves, while the opposite is impossible—we are forbidden to go from a premiss(-es) labelled with $\overset{I}{\Rightarrow}$ to a conclusion labelled with $\overset{C}{\Rightarrow}$. Similarly, if we were to consider *cut* with its conclusion labelled with $\overset{I}{\Rightarrow}$, and premisses labelled with $\overset{I}{\Rightarrow}$ and $\overset{C}{\Rightarrow}$, that is

$$\frac{\Gamma \overset{I}{\Rightarrow} \Delta, \phi \quad \phi, \Pi \overset{C}{\Rightarrow} \Psi}{\Gamma, \Pi \overset{I}{\Rightarrow} \Delta, \Psi}\; cut$$

we could potentially accept its application, but we do not recommend the version of *cut* where at least one of the premisses is labelled with $\overset{I}{\Rightarrow}$ and the conclusion is labelled with $\overset{C}{\Rightarrow}$, that is

$$\frac{\Gamma \overset{I}{\Rightarrow} \Delta, \phi \quad \phi, \Pi \overset{C}{\Rightarrow} \Psi}{\Gamma, \Pi \overset{C}{\Rightarrow} \Delta, \Psi}\; cut$$

as we would be re-allowing further use of identity-dedicated rules[2] in a less controlled way than $R^B_{\equiv}$.

As rule $R^B_{\equiv}$ is not applicable to a sequent with label $\overset{I}{\Rightarrow}$, we have to make a few comments regarding the status of equations appearing in WB-sequents labelled with $\overset{C}{\Rightarrow}$. If we were to consider switching to language $\mathcal{L}'$ without an identity connective, we could use the inverse function of a translation, μ^{-1}, going from $\mathcal{L}$ to $\mathcal{L}'$ (see Definition 64 in the previous chapter): for any $\phi \equiv \chi$ within a formula ψ we can substitute any variable p_i for equation $\phi \equiv \chi$, provided the variable does not appear in ψ. In the case of sequents labelled with C, we can treat equations similarly as in the case of the mentioned translation: as propositional

[2]However, we do note that we could profit from its use in proofs ending with $\overset{C}{\Rightarrow}$ which would at some point require us to utilize identity-dedicated rules; a specific example will be shown in Chapter 7.

variables or, simply, as atomic expressions. This way we can notice that, in fact, the satisfiability (truth and validity) of WB-sequents of the form $\Gamma \overset{C}{\Rightarrow} \Delta$ can be defined on the ground of CPC. Ergo, the notion of ultrafilter will be brought back to the Fregean-realm and depend on the semantic equivalence of two formulae, rather than their semantic non-Fregean identity. This will be visible particularly in the proof of Lemma 14.

Below we once again consider utilization of translation μ.

Lemma 11. *If a* WB*-sequent* $\overset{C}{\Rightarrow} \phi$ *has a proof in* $\mathsf{G3_{WB}}$*, then there is a formula* χ *of language* $\mathcal{L}_{\mathsf{CPC}}$ *such that* χ *is valid in* CPC *and there is a translation* μ *from language* $\mathcal{L}_{\mathsf{CPC}}$ *to* $\mathcal{L}$ *such that* $\mu(\chi) = \phi$*.*

Proof. If we were to build proof for WB-sequent $\overset{C}{\Rightarrow} \phi$, we know that no identity-dedicated rules will be utilized. As in SCI in Theorem 17 we can then use function μ^{-1} to obtain $\mu^{-1}(\phi)$ and build an analogous proof of $\Rightarrow \mu^{-1}(\phi)$ in G3cp, since only classical and/or structural rules are required. Then $\mu^{-1}(\phi)$ through Theorem 5 can be shown to be valid in CPC. □

Lemma 12. *If a* WB*-sequent* $\overset{C}{\Rightarrow} \phi$ *has a proof in* $\mathsf{G3_{WB}}$*, then there is a proof of* $\overset{C}{\Rightarrow} \phi$ *in* $\mathsf{G3_{WB}}$ *with no structural rules utilized.*

Proof. Here we refer to the cut admissibility in G3cp. If we were to build a proof for WB-sequent $\overset{C}{\Rightarrow} \phi$, then per Lemma 11 we may as well build an analogous derivation of sequent $\Rightarrow \mu^{-1}(\phi)$ in G3cp. We know that this particular sequent calculus is cut-free. Therefore, provided $\Rightarrow \mu^{-1}(\phi)$ has a proof in G3cp with no structural rules utilized, we know (considering the same construction as in the case of Theorem 5) there is a way to build the same proof for $\overset{C}{\Rightarrow} \phi$. □

4.3.1 Completeness of $\mathsf{G3_{WB}}$

We prove completeness of the $\mathsf{G3_{WB}}$ indirectly, by means of interpretation of axiomatic system H_{WB} within sequent calculus $\mathsf{G3_{WB}}$.

Theorem 20 (Interpretation of H_{WB} within $\mathsf{G3_{WB}}$)**.** *If formula* ϕ *is provable in axiomatic system* H_{WB}*, then* WB*-sequent* $\overset{I}{\Rightarrow} \phi$ *is provable in* $\mathsf{G3_{WB}}$*.*

Proof. We show completeness of the sequent calculus in the following steps:

1. We show that for every axiom ϕ, a sequent $\overset{I}{\Rightarrow} \phi$ has a proof in $\mathsf{G3_{WB}}$.
2. Based on the proof of a given formula in H_{WB}, we will show that WB-sequent $\overset{I}{\Rightarrow} \phi$ has a proof in $\mathsf{G3_{WB}}$ through simulation of *modus ponens* in sequent calculus.

Here we present proofs of axioms of WB:

($\equiv_5$) $((\phi \wedge \chi) \vee \psi) \equiv ((\chi \vee \psi) \wedge (\phi \vee \psi))$

$$\dfrac{\dfrac{\begin{matrix} D_1 & D_2 \\ \vdots & \vdots \end{matrix}}{\overset{C}{\Rightarrow} ((\phi \wedge \chi) \vee \psi) \leftrightarrow ((\chi \vee \psi) \wedge (\phi \vee \psi))}\, R_{\leftrightarrow}}{\overset{I}{\Rightarrow} ((\phi \wedge \chi) \vee \psi) \equiv ((\chi \vee \psi) \wedge (\phi \vee \psi))}\, R^B_{\equiv}$$

where D_1 is the following derivation:

$$\dfrac{\dfrac{\dfrac{\dfrac{\phi, \chi \overset{C}{\Rightarrow} \chi, \psi}{\phi, \chi \overset{C}{\Rightarrow} \chi \vee \psi}\, R_{\vee} \quad \dfrac{\phi, \chi \overset{C}{\Rightarrow} \phi, \psi}{\phi, \chi \overset{C}{\Rightarrow} \phi \vee \psi}\, R_{\vee}}{\phi, \chi \overset{C}{\Rightarrow} (\chi \vee \psi) \wedge (\phi \vee \psi)}\, R_{\wedge}}{\phi \wedge \chi \overset{C}{\Rightarrow} (\chi \vee \psi) \wedge (\phi \vee \psi)}\, L_{\wedge} \quad \dfrac{\dfrac{\psi \overset{C}{\Rightarrow} \chi, \psi}{\psi \overset{C}{\Rightarrow} \chi \vee \psi}\, R_{\vee} \quad \dfrac{\psi \overset{C}{\Rightarrow} \phi, \psi}{\psi \overset{C}{\Rightarrow} \phi \vee \psi}\, R_{\vee}}{\psi \overset{C}{\Rightarrow} (\chi \vee \psi) \wedge (\phi \vee \psi)}\, R_{\wedge}}{(\phi \wedge \chi) \vee \psi \overset{C}{\Rightarrow} (\chi \vee \psi) \wedge (\phi \vee \psi)}\, L_{\vee}$$

and D_2:

$$\dfrac{\dfrac{\dfrac{\dfrac{\chi, \phi \overset{C}{\Rightarrow} \phi, \psi \quad \chi, \phi \overset{C}{\Rightarrow} \chi, \psi}{\chi, \phi \overset{C}{\Rightarrow} \phi \wedge \chi, \psi}\, R_{\wedge} \quad \chi, \psi \overset{C}{\Rightarrow} \phi \wedge \chi, \psi}{\chi, \phi \vee \psi \overset{C}{\Rightarrow} \phi \wedge \chi, \psi}\, L_{\vee} \quad \psi, \phi \vee \psi \overset{C}{\Rightarrow} \phi \wedge \chi, \psi}{\chi \vee \psi, \phi \vee \psi \overset{C}{\Rightarrow} \phi \wedge \chi, \psi}\, L_{\vee}}{(\chi \vee \psi) \wedge (\phi \vee \psi) \overset{C}{\Rightarrow} (\phi \wedge \chi) \vee \psi}\, L_{\wedge}$$

($\equiv_6$) $((\phi \vee \chi) \wedge \psi) \equiv ((\chi \wedge \psi) \vee (\phi \wedge \psi))$

$$\dfrac{\dfrac{\dfrac{\begin{matrix} D_1 \\ \vdots \end{matrix}}{(\phi \vee \chi) \wedge \psi \overset{C}{\Rightarrow} (\chi \wedge \psi) \vee (\phi \wedge \psi)}\, L_{\wedge} \quad \dfrac{\begin{matrix} D_2 \\ \vdots \end{matrix}}{(\chi \wedge \psi) \vee (\phi \wedge \psi) \overset{C}{\Rightarrow} (\phi \vee \chi) \wedge \psi}\, L_{\vee}}{\overset{C}{\Rightarrow} ((\phi \vee \chi) \wedge \psi) \leftrightarrow ((\chi \wedge \psi) \vee (\phi \wedge \psi))}\, R_{\leftrightarrow}}{\overset{I}{\Rightarrow} ((\phi \vee \chi) \wedge \psi) \equiv ((\chi \wedge \psi) \vee (\phi \wedge \psi))}\, R^B_{\equiv}$$

where D_1 is the following derivation:

$$\dfrac{\dfrac{\dfrac{\phi, \psi \overset{C}{\Rightarrow} \chi \wedge \psi, \phi \quad \phi, \psi \overset{C}{\Rightarrow} \chi \wedge \psi, \psi}{\phi, \psi \overset{C}{\Rightarrow} \chi \wedge \psi, \phi \wedge \psi}\, R_{\wedge} \quad \dfrac{\chi, \psi \overset{C}{\Rightarrow} \chi, \phi \wedge \psi \quad \chi, \psi \overset{C}{\Rightarrow} \psi, \phi \wedge \psi}{\chi, \psi \overset{C}{\Rightarrow} \chi \wedge \psi, \phi \wedge \psi}\, R_{\wedge}}{\phi \vee \chi, \psi \overset{C}{\Rightarrow} \chi \wedge \psi, \phi \wedge \psi}\, L_{\vee}}{\phi \vee \chi, \psi \overset{C}{\Rightarrow} (\chi \wedge \psi) \vee (\phi \wedge \psi)}\, R_{\vee}$$

D_2:

$$\dfrac{\dfrac{\dfrac{\dfrac{\chi,\psi \overset{C}{\Rightarrow} \phi,\chi}{\chi,\psi \overset{C}{\Rightarrow} \phi\vee\chi}\,R_\vee \qquad \chi,\psi \overset{C}{\Rightarrow} \psi}{\chi,\psi \overset{C}{\Rightarrow} (\phi\vee\chi)\wedge\psi}\,R_\wedge}{\chi\wedge\psi \overset{C}{\Rightarrow} (\phi\vee\chi)\wedge\psi}\,L_\wedge \qquad \dfrac{\dfrac{\dfrac{\phi,\psi \overset{C}{\Rightarrow} \phi,\chi}{\phi,\psi \overset{C}{\Rightarrow} \phi\vee\chi}\,R_\vee \qquad \phi,\psi \overset{C}{\Rightarrow} \psi}{\phi,\psi \overset{C}{\Rightarrow} (\phi\vee\chi)\wedge\psi}\,R_\wedge}{\phi\wedge\psi \overset{C}{\Rightarrow} (\phi\vee\chi)\wedge\psi}\,L_\wedge}{(\chi\wedge\psi)\vee(\phi\wedge\psi) \overset{C}{\Rightarrow} (\phi\vee\chi)\wedge\psi}\,L_\vee$$

$(\equiv_7)$ $(\phi\vee(\chi\wedge\neg\chi))\equiv\phi$

$$\dfrac{\dfrac{\dfrac{\phi \overset{C}{\Rightarrow} \phi \qquad \dfrac{\dfrac{\chi \overset{C}{\Rightarrow} \chi,\phi}{\chi,\neg\chi \overset{C}{\Rightarrow} \phi}\,L_\neg}{\chi\wedge\neg\chi \overset{C}{\Rightarrow} \phi}\,L_\wedge}{\phi\vee(\chi\wedge\neg\chi) \overset{C}{\Rightarrow} \phi}\,L_\vee \qquad \dfrac{\phi \overset{C}{\Rightarrow} \phi,\chi\wedge\neg\chi}{\phi \overset{C}{\Rightarrow} \phi\vee(\chi\wedge\neg\chi)}\,R_\vee}{\overset{C}{\Rightarrow} (\phi\vee(\chi\wedge\neg\chi))\leftrightarrow\phi}\,R_\leftrightarrow}{\overset{I}{\Rightarrow} (\phi\vee(\chi\wedge\neg\chi))\equiv\phi}\,R^B_\equiv$$

$(\equiv_8)$ $(\phi\wedge(\chi\vee\neg\chi))\equiv\phi$

$$\dfrac{\dfrac{\dfrac{\phi,\chi\vee\neg\chi \overset{C}{\Rightarrow} \phi}{\phi\wedge(\chi\vee\neg\chi) \overset{C}{\Rightarrow} \phi}\,L_\wedge \qquad \dfrac{\phi \overset{C}{\Rightarrow} \phi \qquad \dfrac{\dfrac{\phi,\chi \overset{C}{\Rightarrow} \chi}{\phi \overset{C}{\Rightarrow} \chi,\neg\chi}\,R_\neg}{\phi \overset{C}{\Rightarrow} \chi\vee\neg\chi}\,R_\vee}{\phi \overset{C}{\Rightarrow} \phi\wedge(\chi\vee\neg\chi)}\,R_\wedge}{\overset{C}{\Rightarrow} (\phi\wedge(\chi\vee\neg\chi))\leftrightarrow\phi}\,R_\leftrightarrow}{\overset{I}{\Rightarrow} (\phi\wedge(\chi\vee\neg\chi))\equiv\phi}\,R^B_\equiv$$

$(\equiv_9)$ $(\phi\to\chi)\equiv(\neg\phi\vee\chi)$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\phi \overset{C}{\Rightarrow} \chi,\phi \qquad \chi,\phi \overset{C}{\Rightarrow} \chi}{\phi\to\chi,\phi \overset{C}{\Rightarrow} \chi}\,L_\to}{\phi\to\chi \overset{C}{\Rightarrow} \neg\phi,\chi}\,R_\neg}{\phi\to\chi \overset{C}{\Rightarrow} \neg\phi\vee\chi}\,R_\vee \qquad \dfrac{\dfrac{\dfrac{\phi \overset{C}{\Rightarrow} \phi,\chi}{\neg\phi,\phi \overset{C}{\Rightarrow} \chi}\,L_\neg \qquad \chi,\phi \overset{C}{\Rightarrow} \chi}{\neg\phi\vee\chi,\phi \overset{C}{\Rightarrow} \chi}\,L_\vee}{\neg\phi\vee\chi \overset{C}{\Rightarrow} \phi\to\chi}\,R_\to}{\overset{C}{\Rightarrow} (\phi\to\chi)\leftrightarrow(\neg\phi\vee\chi)}\,R_\leftrightarrow}{\overset{I}{\Rightarrow} (\phi\to\chi)\equiv(\neg\phi\vee\chi)}\,R^B_\equiv$$

($\equiv_{10}$) $(\phi \leftrightarrow \chi) \equiv ((\phi \to \chi) \wedge (\chi \to \phi))$

$$\dfrac{\dfrac{\begin{matrix}D_1 & D_2\\ \vdots & \vdots\end{matrix}}{\overset{C}{\Rightarrow} (\phi \leftrightarrow \chi) \leftrightarrow ((\phi \to \chi) \wedge (\chi \to \phi))}\,R_{\leftrightarrow}}{\overset{I}{\Rightarrow} (\phi \leftrightarrow \chi) \equiv ((\phi \to \chi) \wedge (\chi \to \phi))}\,R^{B}_{\equiv}$$

where D_1 is the following derivation:

$$\dfrac{\dfrac{\dfrac{\phi, \chi, \phi \overset{C}{\Rightarrow} \chi \quad \phi \overset{C}{\Rightarrow} \chi, \phi, \chi}{\phi \leftrightarrow \chi, \phi \overset{C}{\Rightarrow} \chi}\,L_{\leftrightarrow}}{\phi \leftrightarrow \chi \overset{C}{\Rightarrow} \phi \to \chi}\,R_{\to} \qquad \dfrac{\dfrac{\phi, \chi, \chi \overset{C}{\Rightarrow} \phi \quad \chi \overset{C}{\Rightarrow} \phi, \phi, \chi}{\phi \leftrightarrow \chi, \chi \overset{C}{\Rightarrow} \phi}\,L_{\leftrightarrow}}{\phi \leftrightarrow \chi \overset{C}{\Rightarrow} \chi \to \phi}\,R_{\to}}{\phi \leftrightarrow \chi \overset{C}{\Rightarrow} (\phi \to \chi) \wedge (\chi \to \phi)}\,R_{\wedge}$$

and D_2:

$$\dfrac{\dfrac{\dfrac{\chi \to \phi, \phi \overset{C}{\Rightarrow} \chi, \phi \quad \chi \to \phi, \phi, \chi \overset{C}{\Rightarrow} \chi}{\phi \to \chi, \chi \to \phi, \phi \overset{C}{\Rightarrow} \chi}\,L_{\to} \qquad \dfrac{\phi \to \chi, \chi \overset{C}{\Rightarrow} \phi, \chi \quad \phi \to \chi, \chi, \phi \overset{C}{\Rightarrow} \phi}{\phi \to \chi, \chi \to \phi, \chi \overset{C}{\Rightarrow} \phi}\,L_{\to}}{\phi \to \chi, \chi \to \phi \overset{C}{\Rightarrow} \phi \leftrightarrow \chi}\,R_{\leftrightarrow}}{(\phi \to \chi) \wedge (\chi \to \phi) \overset{C}{\Rightarrow} \phi \leftrightarrow \chi}\,L_{\wedge}$$

Modus ponens is obtainable in G3$_{\mathsf{WB}}$ through utilization of *cut*:

$$\dfrac{\begin{matrix}D_1\\ \vdots\\ \overset{I}{\Rightarrow} \chi\end{matrix} \qquad \dfrac{\begin{matrix}D_2\\ \vdots\\ \overset{I}{\Rightarrow} \chi \to \phi\end{matrix} \quad \dfrac{\chi \overset{I}{\Rightarrow} \chi \quad \phi \overset{I}{\Rightarrow} \phi}{\chi, \chi \to \phi \overset{I}{\Rightarrow} \phi}\,L_{\to}}{\chi \overset{I}{\Rightarrow} \phi}\,cut}{\overset{I}{\Rightarrow} \phi}\,cut$$

where D_1 and D_2 are proofs of $\overset{I}{\Rightarrow} \chi$, $\overset{I}{\Rightarrow} \chi \to \phi$, respectively. The remaining reasoning is analogous to that presented in Theorem 11. □

Proofs of axioms ($\equiv_5$)—($\equiv_{10}$) and the shape of the rules utilized suggest that if ϕ is a theorem characteristic of WB, the proof of WB-sequent $\overset{I}{\Rightarrow} \phi$ will at some point require us to utilize rules other than those from G3$_{\mathsf{SCI}}$. In G3$_{\mathsf{WB}}$ we add rule $R^{B}_{\equiv}$, which allows us to build a proof for formulae characteristic of WB. We know that some labels of WB-sequents will be changed from $\overset{C}{\Rightarrow}$ to $\overset{I}{\Rightarrow}$. Suppose we have a proof of $\overset{I}{\Rightarrow} \phi$ beginning with axioms labelled only with $\overset{I}{\Rightarrow}$. This formula is not characteristic of WB (but is in CPC and/or SCI) as $R^{B}_{\equiv}$ has not been used. This fact will be recalled in later sections, particularly regarding a failed cut elimination procedure. These observations lead us to the following lemma:

Lemma 13. *If* $\phi \in \mathsf{WB} \setminus \mathsf{SCI}$*, then every proof of* WB*-sequent* $\overset{I}{\Rightarrow} \phi$ *starts with at least one axiom labelled with* $\overset{C}{\Rightarrow}$*.*

Proof. We refer to the fact that the only rule allowing a change of labels is $R^B_{\equiv}$. □

If $R^B_{\equiv}$ was not used, labels would not change: proof of $\overset{C}{\Rightarrow} \phi$ starting with all leaves labelled with $\overset{C}{\Rightarrow}$ means that ϕ is a theorem of CPC (expressed in language $\mathcal{L}_{\mathsf{SCI}}$) and proof of $\overset{I}{\Rightarrow} \phi$ starting with all leaves labelled with $\overset{I}{\Rightarrow}$ means that ϕ is a theorem of SCI (some of which will be also theorems of CPC).

4.3.2 Soundness of $\mathsf{G3_{WB}}$

We now define semantic notions with regard to WB-sequents (analogous to those in the previous chapter):

Definition 75 (Satisfiability of a WB-sequent)**.** *Let* $\mathcal{M} = \langle \mathcal{A}, F \rangle$ *be an arbitrary but fixed* B*-model and let* $h \in Hom(\mathcal{L}, \mathcal{A})$*.* WB*-sequent* $\Gamma \overset{X}{\Rightarrow} \Delta$ *is* <u>*satisfied in* $\mathcal{M}$ *under h*</u> *provided if all formulae from* Γ *are satisfied in* $\mathcal{M}$ *under h that is* $h(\chi) \in F$ *(for all* $\chi \in \Gamma$*), then at least one formula in* Δ *is satisfied in* $\mathcal{M}$ *under h as well.*

Definition 76 (Truth of a WB-sequent)**.** *Let* $\mathcal{M} = \langle \mathcal{A}, F \rangle$ *be an arbitrary* B*-model.* WB*-sequent* $\Gamma \overset{X}{\Rightarrow} \Delta$ *is* <u>*true in* $\mathcal{M}$</u> *provided that for each* $h \in Hom(\mathcal{L}, \mathcal{A})$*,* WB*-sequent* $\Gamma \overset{X}{\Rightarrow} \Delta$ *is satisfied in* $\mathcal{M}$ *under h.*

Definition 77 (Validity of a WB-sequent)**.** WB*-sequent* $\Gamma \overset{X}{\Rightarrow} \Delta$ *is* <u>*valid in* WB</u>*, if it is true in each* B*-model.*

Yet again we consider two properties of rules: preservability of satisfiability/validity of WB-sequents (correctness) and invertibility. Similarly as it was for $\mathsf{G3_{SCI}}$, the logical rule $R_\neg, L_\neg, R_\vee, L_\vee, R_\wedge, L_\wedge, R_\to, L_\to, R_\leftrightarrow$, and $L_\leftrightarrow$ have both properties, and we will not include proofs showing that. We will also omit proof of correctness and invertibility of left identity-dedicated rules, which can also be found in the previous chapter. The definitions below are analogous to ones from Chapter 3 and concern transfer of satisfiability. In the case of rule $R^B_{\equiv}$ we will consider transfer of validity between premiss and conclusion.

Definition 78 (Correctness of rule)**.** *Rule R is* <u>*correct in* WB</u> *provided that for each* B*-model* $\mathcal{M}$ *and for every valuation h in* $\mathcal{M}$*, if the premiss(-es) of R is (are) satisfied in* $\mathcal{M}$ *under h, then so is its (their) conclusion.*

Definition 79 (Invertibility of rule)**.** *Rule R is* <u>*invertible in* WB</u> *provided that for each* B*-model* $\mathcal{M}$ *and for every valuation h in* $\mathcal{M}$*, if the conclusion of R is satisfied in* $\mathcal{M}$ *under h, then so is (are) its premisse(s).*

For the following lemmas, $\mathcal{L}$ denotes the language algebra and $\mathcal{A}$ denotes a B-algebra.

$R^B_{\equiv}$ is not correct in the sense of Definition 78. The counterexample is an arbitrary equivalence satisfied in some model $\mathcal{M}$ under some homomorphism h and equation of the same formulae not satisfied in $\mathcal{M}$ under h. For the next lemma we will consider another property of the rule: preservation of validity. To prove it we will refer to Theorem 4. We will start, however, with the following corollary:

Corollary 2. *If a* WB*-sequent* $\overset{C}{\Rightarrow} \phi$ *has a proof in* $\mathsf{G3_{WB}}$*, sequent* $\overset{C}{\Rightarrow} \phi$ *is valid in* WB*.*

Proof. Here we depend on the well documented semantic correctness of both classical and structural sets of rules for classical connectives and the fact that axioms of $\mathsf{G3_{WB}}$ are valid in CPC. □

Lemma 14. *If* WB*-sequent* $\overset{I}{\Rightarrow} \phi \equiv \chi$ *has a proof in* $\mathsf{G3_{WB}}$*, a root of which is a conclusion of rule* $R^B_{\equiv}$*, then* $\overset{I}{\Rightarrow} \phi \equiv \chi$ *is valid in* WB*.*

Proof. Let us examine a proof of a WB-sequent $\overset{I}{\Rightarrow} \phi \equiv \chi$, where the last utilized rule is $R^B_{\equiv}$. Let us have the following:

(1) $\overset{C}{\Rightarrow} \phi \leftrightarrow \chi$ has a proof in $\mathsf{G3_{WB}}$,

(2) $\overset{I}{\Rightarrow} \phi \equiv \chi$ is not valid in WB.

From (2) we know that $\phi \equiv \chi$ is not satisfied in some B-model $\mathcal{M} = \langle \mathcal{A}, F \rangle$ under some h, hence $h(\phi) \doteqdot h(\chi) \notin F$. F is normal, therefore $h(\phi) \neq h(\chi)$, so we have $h(\chi) \not\leq h(\phi)$ or $h(\phi) \not\leq h(\chi)$. From Theorem 4 we know that there is a prime filter F^* such that

(*) $h(\phi) \notin F^*$ and $h(\chi) \in F^*$

(or the other way round, the reasoning is the same in both cases). As a result we have $h(\phi) \dot{\leftrightarrow} h(\chi) \notin F^*$, that is,

(**) $h(\phi \leftrightarrow \chi) \notin F^*$.

Theorem 3 guarantees that F^* is an ultrafilter, but $\langle \mathcal{A}, F^* \rangle$ does not particularly have to be a B-model, as F^* can potentially not be normal. Therefore we need to use the fact that sequent $\overset{C}{\Rightarrow} \phi \leftrightarrow \chi$ has been proved without the application of identity-dedicated rules.

From (1) we know that WB-sequent $\overset{C}{\Rightarrow} \phi \leftrightarrow \chi$ has a proof in $\mathsf{G3_{WB}}$. From Lemma 11 we have a formula ψ of language $\mathcal{L}_{\mathsf{CPC}}$ such that ψ is valid in Boolean algebras and there exists a translation μ from language $\mathcal{L}_{\mathsf{CPC}}$ to $\mathcal{L}$ such that $\mu(\psi) = \phi \leftrightarrow \chi$. If we consider algebra $\mathcal{A}$ without $\doteqdot$, that is $\mathcal{A}^* = \langle A^*, \dot{\neg}, \dot{\cap}, \dot{\cup}, \dot{\rightarrow}, \dot{\leftrightarrow} \rangle$, then it is a Boolean algebra of similarity type $\langle 1, 2, 2, 2, 2 \rangle$, therefore ψ must be true in the CPC-model $\langle \mathcal{A}^*, F^* \rangle$. However, the composition $h \circ \mu$ of function μ and h is a homomorphism from $\mathcal{L}_{\mathsf{CPC}}$ to $\mathcal{A}^*$ and:

$$h \circ \mu(\psi) = h(\mu(\psi)) = h(\phi \leftrightarrow \chi) \notin F^*$$

that is, the homomorphism sends ψ outside F^*. A contradiction. $\square$

We now move to invertibility of the right identity-dedicated rule. Here, for all rules we will stay with the same property: transmission of the satisfiability of a WB-sequent from the conclusion to the premiss.

Lemma 15. *$R^B_\equiv$ is invertible in* WB.

Proof. Let $\mathcal{M} = \langle \mathcal{A}, F \rangle$ be an arbitrary, but fixed B-model and h be a homomorphism from $\mathcal{L}$ into $\mathcal{A}$. Suppose that the following conditions hold:

(1) $\overset{I}{\Rightarrow} \phi \equiv \chi$ is satisfied in $\mathcal{M}$ under h;

(2) $\overset{C}{\Rightarrow} \phi \leftrightarrow \chi$ is not satisfied in $\mathcal{M}$ under h.

From (2) we know that $\phi \leftrightarrow \chi$ is not satisfied in $\mathcal{M}$ under h. This means that it is not the case that $h(\phi) \dot{\leftrightarrow} h(\chi) \in F$, by which we know that either $h(\phi) \in F$ and $h(\chi) \notin F$ or $h(\phi) \notin F$ and $h(\chi) \in F$. Therefore $h(\phi) \neq h(\chi)$, which means $h(\phi \equiv \chi) \notin F$. $\square$

Theorem 21 (Completeness)**.** *If a sequent $\overset{X}{\Rightarrow} \phi$ is valid in* WB*, it is provable in* $\mathsf{G3_{WB}}$.

Proof. By completeness of H_{WB} and Theorem 20. $\square$

Theorem 22 (Soundness)**.** *If a sequent $\Gamma \overset{X}{\Rightarrow} \Delta$ is provable in* $\mathsf{G3_{WB}}$*, it is valid in* WB.

Proof. The proof is analogous to the one of Theorem 13 in Chapter 3. $\square$

As a result from Theorems 21 and 22 we have

Theorem 23 (Adequacy)**.** *Sequent $\overset{X}{\Rightarrow} \phi$ is provable in* $\mathsf{G3_{WB}}$ *iff $\overset{X}{\Rightarrow} \phi$ is valid in* WB.

Chapter 5

WT logic and sequent calculus $\mathsf{G3}_{\mathsf{WT}}$

The following extension of SCI is a formal interpretation of proposition

> 5.141 If p follows from q and q from p then they are one and the same proposition.

from Wittgenstein's *Tractatus*, which we can interpret as the fact that two logically equivalent sentences constitute different variants of the same proposition. WT corresponds to modal system S4 and both of them are finitely axiomatized [47]. We can easily propose a translation from WT to S4, as $\Box$ can be interpreted through means of identity connective $\equiv$. However, Suszko underlines that we cannot identify non-Fregean logics with modal logics, particularly regarding NFLs extensionality and two-valuedness [51, p. 204]

WT is a set of sentences valid in topological Boolean algebras. Moreover, we can interpret equations $\phi \equiv \chi$ as $\Box(\phi \leftrightarrow \chi)$, where $\Box$ could be interpreted as an interior operator on the Boolean algebra of situations [51, p. 200], which will be further discussed with regard to the semantics.

5.1 Hilbert system for WT

We yet again utilize the same base language algebra as in the case of WB language, that is an algebra of the similarity type $\langle 1, 2, 2, 2, 2, 2\rangle$:

$$\mathcal{L} = \langle L, \neg, \wedge, \vee, \rightarrow, \leftrightarrow, \equiv\rangle.$$

In [38; 51] the language of WT (as well as the language of WH) is extended by the addition of a binary connective $\preceq$ and a unary connective $\Box$. A formula of the form $\phi \preceq \chi$ is read as: situation χ is included in situation ϕ, or: situation ϕ involves situation χ, or: situation χ occurs in situation ϕ.[1] We will stay faithful to the language $\mathcal{L}$ and use the following abbreviations (knowing that we can also define $\bot =_{df} p_1 \wedge \neg p_1$ and $\top =_{df} \neg\bot$):

$$(\phi \preceq \chi) =_{df} ((\phi \rightarrow \chi) \equiv \top)$$

[1] In the literature symbol "$\leq$" appears in place of "$\preceq$"; we use "$\preceq$" as "$\leq$" holds a different meaning (ordering relation) in semantics utilized in this dissertation.

as well as use the fact that $\Box$ and $\equiv$ are interdefinable in WT [51]:

$$\Box\phi =_{df} \phi \equiv \top$$

Also, we know that $\phi \equiv \chi$ can be written as $(\phi \leftrightarrow \chi) \equiv \top$.

An axiomatic system for WT [51] is obtained by adding the following axiom schemata (WTA) to the axiomatic system for WB (on the left side we present the original version of the axiom, and on the right side—the axiom after using the abbreviations omitting $\Box$ and $\leq$, but utilizing $\top$ for simpler notation):

$(\equiv_{11})$	$\Box(\phi \leftrightarrow \chi) \equiv (\phi \equiv \chi)$	$((\phi \leftrightarrow \chi) \equiv \top) \equiv (\phi \equiv \chi)$
$(\equiv_{12})$	$\Box\phi \preceq \phi$	$((\phi \equiv \top) \rightarrow \phi) \equiv \top$
$(\equiv_{13})$	$\Box(\phi \wedge \chi) \equiv (\Box\phi \wedge \Box\chi)$	$((\phi \wedge \chi) \equiv \top) \equiv ((\phi \equiv \top) \wedge (\chi \equiv \top))$
$(\equiv_{14})$	$\Box\Box\phi \equiv \Box\phi$	$((\phi \equiv \top) \equiv \top) \equiv (\phi \equiv \top)$

Below we present examples of WT theorems (which are not in WB), similarly as above, in two shapes:

1.	$\Box\top \equiv \top$	$(\top \equiv \top) \equiv \top$
2.	$(\phi \equiv \chi) \equiv \Box(\phi \equiv \chi)$	$(\phi \equiv \chi) \equiv ((\phi \equiv \chi) \equiv \top)$
3.	$(\phi \equiv \chi) \preceq (\phi \leftrightarrow \chi)$	$((\phi \equiv \chi) \rightarrow (\phi \leftrightarrow \chi)) \equiv \top$
4.	$(\phi \equiv \chi) \equiv (\chi \equiv \phi)$	$(\phi \equiv \chi) \equiv (\chi \equiv \phi)$
5.	$((\phi \equiv \chi) \wedge (\chi \equiv \psi)) \preceq (\phi \equiv \psi)$	$(((\phi \equiv \chi) \wedge (\chi \equiv \psi)) \rightarrow (\phi \equiv \psi))$ $\equiv \top$
6.	$(\phi \preceq \chi) \equiv ((\phi \wedge \chi) \equiv \phi)$	$((\phi \rightarrow \chi) \equiv \top) \equiv ((\phi \wedge \chi) \equiv \phi)$
7.	$(\Box\phi \equiv \neg\Box\neg\phi) \rightarrow ((\phi \equiv \top) \vee (\phi \equiv \bot))$	$((\phi \equiv \top) \equiv \neg(\neg\phi \equiv \top))$ $\rightarrow ((\phi \equiv \top) \vee (\phi \equiv \bot))$

The definition of formal proof of a given formula ϕ in the axiomatic system for WT is analogous to the earlier iterations of it in the non-Fregean systems introduced earlier.

Definition 80 (Derivation, formal proof)**.** *Let* Φ *stand for a set of formulae of* $\mathcal{L}$*. A finite sequence* $\phi_1, ..., \phi_n$ *of formulae of* $\mathcal{L}$ *is a* <u>*derivation*</u> *of* ϕ *from* Φ *provided* $\phi_n = \phi$ *and formula* $\phi_i, i \leq n$*, is either from* Φ *or has been derived from some* ϕ_{i_1}, ϕ_{i_2}*,* $(i_1, i_2 < i)$ *through an application of modus ponens. If* $\Phi = \mathsf{TFA} \cup \mathsf{IDA} \cup \mathsf{WBA} \cup \mathsf{WTA}$*, then* $\phi_1, ..., \phi_n$ *is a* <u>*proof of* ϕ *in the axiomatic system for* WT</u>.

We gradually add more axioms to successive non-Fregean systems, therefore yet again consequence operation C_{WT} is defined by a set $\mathsf{TFA} \cup \mathsf{IDA} \cup \mathsf{WBA} \cup \mathsf{WTA}$ of axioms and a singular inference rule, *modus ponens*, which we define analogously as was done for WB. Elements of $C_{\mathsf{WT}}(\emptyset)$ are called logical theorems of WT.

In WT we can prove equations of the form $\phi \equiv \chi$, provided $\phi \leftrightarrow \chi$ can be proved using axioms from the set $\mathsf{TFA} \cup \mathsf{IDA}$, which is depicted in the below definition of WT [38, p. 106]:

$$\mathsf{WT} = C(\{\alpha \equiv \beta : (\alpha \leftrightarrow \beta) \in C(\emptyset)\}).$$

WT is closed under both the Gödel rule (all three versions below are equivalent)

$$\frac{\phi, \chi}{\phi \equiv \chi} \qquad \frac{\phi}{\phi \equiv \top} \qquad \frac{\phi}{\Box\phi}$$

(if $\phi, \chi \in \mathsf{WT}$, then $\phi \equiv \chi \in \mathsf{WT}$) and the *quasi-Fregean* rule, that is

$$\frac{\phi \leftrightarrow \chi}{\phi \equiv \chi} \qquad \frac{\phi \rightarrow \chi, \chi \rightarrow \phi}{\phi \equiv \chi}$$

Theories closed under the *quasi*-Fregan rule are called *quasi*-Fregean. In subsequent sections we will utilize a version of the quasi-Fregean rule in sequent calculus (right identity-dedicated rule). This particular approach to formalization of WT can be compared to systems proposed by Cresswell [8] and Greniewski [20]. Cresswell introduced a Calculus of Functions of Propositions, where a version of the quasi-Fregean rule is added in order to obtain a system corresponding to modal logic S4. Greniewski adds a similar rule to his system and obtains a logic corresponding to S4 as well. WT's correspondence to S4 means we can translate formulae from one logic to the other and state that for given formulae $\phi, \chi \in \mathcal{L}$ [38, p. 108]:

$$\phi \equiv \chi \in \mathsf{WT} \text{ iff } \Box(\phi \leftrightarrow \chi) \in \mathsf{S4}$$

Once again, in keeping with Suszko we introduce the logic WT as a deductive system H_{WT}.

Definition 81. $H_{\mathsf{WT}} = \langle \mathcal{L}, C_{\mathsf{WT}} \rangle$.

The length of a formal proof of ϕ in H_{WT} is defined in the same way as in H_{WB}. *Modus ponens* is the only inference rule, however we can also consider the secondary inference rules presented above, all variants of the Gödel and *quasi-Fregean* rules.

Theorem 24 ([56, p. 173])**.** *If ϕ is a* CPC *theorem, then $\phi \equiv \top$ is a theorem of* WT*.*

Proof. We show it is the case by means of applying the Gödel rule to a given CPC theorem ϕ. □

A similar observation can be made with regard to the possibilities of *quasi-Fregean* rule application, so we can state that

Theorem 25. *If $\phi \leftrightarrow \chi$ is a* WB *theorem, then $\phi \equiv \chi$ is a theorem of* WT*.*

Proof. Similarly as above, but this time we show it by means of utilizing the quasi-Fregean rule. □

Lemma 16. *If ϕ is a theorem of* SCI *then $\phi \equiv \top$ is a theorem of* WT.

Proof. We show it as above, through the application of the Gödel rule. □

Omyła [38, p. 107] points out that a given binary connective $\otimes$ is called equivalence in some theory Φ provided the following holds for its axiomatic system $H_X = \langle \mathcal{L}^*, C_X \rangle$, $\phi, \chi \in L^*$:

$$\phi \otimes \chi \in \Phi \text{ iff } C_X(\Phi \cup \{\phi\}) = C_X(\Phi \cup \{\chi\}).$$

Then, if Φ is quasi-Fregean, we will have the following ($\phi, \chi \in L^*$):

$$\phi \equiv \chi \in \Phi \text{ iff } C(\Phi \cup \{\phi\}) = C(\Phi \cup \{\chi\}).$$

However, as Omyła points out, the identity connective in SCI still cannot be identified with equivalence, as for any formula ϕ we have the following [38]:

$$(\phi \equiv \neg\neg\phi) \notin C(\emptyset), \text{ even though we have } C(\{\phi\}) = C(\{\neg\neg\phi\})$$

On the other hand, in quasi-Fregean theories an equivalence connective is the same as the identity connective.

As we have correspondence with S4 we can also state that WT is quasi-complete/Halldén-complete.[2]

Definition 82 (Quasi-completeness/Halldén-completeness [13, p. 65])**.** *Let Φ be an invariant* SCI*-theory in language $\mathcal{L}^*$. Φ is quasi-complete/Halldén-complete provided there are no formulae $\phi, \chi \in L^*$ satisfying all of the following conditions:*

- *ϕ, χ do not share propositional variables;*
- *$\phi \vee \chi \in \Phi$;*
- *$\phi \notin \Phi$ and $\chi \notin \Phi$.*

5.2 Semantics of WT

A standard definition of topological Boolean algebra can be found in [43, p. 93]

Definition 83. *By a topological Boolean algebra we understand a Boolean algebra $\mathcal{A} = \langle A, \dot{\neg}, \dot{\cap}, \dot{\cup}, \dot{\rightarrow}, \dot{\leftrightarrow} \rangle$ with an operation I which, to every element $a \in A$, associates an element $Ia \in A$ in such a way that the following axioms are satisfied:*

(I_1) $I(a \mathbin{\dot{\cap}} b) = Ia \mathbin{\dot{\cap}} Ib$,

(I_2) $Ia \leq a$,

(I_3) $IIa = Ia$,

[2] It is noteworthy that WB is not Halldén complete [70].

(I_4) $I1 = 1$.

Every operation I (in a Boolean algebra) satisfying I_1–I_4 is called an interior operation.

We will also utilize an alternative definition of a topological Boolean algebra proposed by Kagan [32].

Definition 84. *Let $\mathcal{B} = \langle B, \dot{\neg}, \dot{\cap}, \dot{\cup}, \dot{\rightarrow}, \dot{\leftrightarrow} \rangle$ be a Boolean algebra, let $\doteqdot$ be a binary operator on $B \times B$ into B and let $\mathcal{A} = \langle \mathcal{B}, \doteqdot \rangle$. Then $\mathcal{A}$ is a topological Boolean algebra, iff $\doteqdot$ satisfies the following conditions for all $a, b, c, d \in B$:*

1. $a \doteqdot a = 1$,
2. $(a \doteqdot b) \leq (a \dot{\leftrightarrow} b)$,
3. $(a \doteqdot c) \dot{\cap} (b \doteqdot c) \leq ((a \otimes b) \doteqdot (c \otimes d))$,

where $\otimes \in \{\dot{\cup}, \dot{\cap}, \dot{\rightarrow}, \dot{\leftrightarrow}, \doteqdot\}$[3].

Instead of conditions $1-3$ Suszko [47] proposes the following conditions:

1*. $a \doteqdot a = 1$,

2*. $(a \doteqdot b) \dot{\rightarrow} (a \dot{\leftrightarrow} b) = 1$,

3*. $((a \dot{\leftrightarrow} b) \doteqdot 1) \dot{\rightarrow} ((a \doteqdot b) \doteqdot 1) = 1$,

4*. $(a \doteqdot c) \dot{\cap} (b \doteqdot c) = ((a \dot{\cup} b \dot{\cup} c \dot{\rightarrow} a \dot{\cap} b \dot{\cap} c) \doteqdot 1)$.

however, in future considerations we will utilize Kagan's definition (Definition 84).

Theorem 26 ([32], p. 103)**.** *If $\mathcal{A} = \langle \mathcal{B}, \doteqdot \rangle$ is a topological Boolean algebra with interior operator defined by $Ia = a \doteqdot 1$, then $\mathcal{A}$ is a topological Boolean algebra $\mathcal{B}$ with interior operator I satisfying conditions (I_1) to (I_4) from Definition 83.*

Theorem 27 ([32], p. 104)**.** *If $\langle \mathcal{B}, I \rangle$ is a topological Boolean algebra with interior operator I, then a binary operator $\doteqdot$ defined by $a \doteqdot b = I(a \dot{\leftrightarrow} b)$ will satisfy conditions 1–3 from Definition 84.*

Theorem 28 ([32], p. 104)**.** *In every topological Boolean algebra $\mathcal{A}$ the following equations hold for all $a, b \in A$:*

1. $a \doteqdot b = b \doteqdot a$,
2. *If* $a \leq b$, *then* $a \doteqdot 1 \leq b \doteqdot 1$,
3. $(a \doteqdot b) \doteqdot 1 = a \doteqdot b$,
4. $(a \dot{\leftrightarrow} b) \doteqdot 1 = a \doteqdot b$,

[3] In Kagan's definition operations $\dot{\rightarrow}, \dot{\leftrightarrow}$ are not listed. Moreover, Kagan's consideration are concerned with closure algebras, where closure operator is utilized instead of interior operator. As these two formalizations of topological space – by closure operators and by interior operators – are equivalent, we utilize algebras with an interior operator.

5. $\dot{\neg}a \doteqdot \dot{\neg}b = a \doteqdot b$.

Let us also note that in a topological Boolean algebra we have [47, p. 23]:

$$a \doteqdot b = I(a \dot{\leftrightarrow} b) \text{ implies } a = b \text{ iff } a \dot{\leftrightarrow} b = 1.$$

Element a of a topological Boolean algebra is called open if $Ia = a$ (or $a \doteqdot 1 = a$). Similarly as for topological spaces we observe that, if b is open, then, for every a : $b \leq a$ iff $b \leq Ia$. Therefore Ia is the greatest open element contained in a.

In the case of WT I can be interpreted as a semantic correlate of $\Box$. $\Box$ and $\equiv$ are interdefinable, and so is I with $\doteqdot$. Therefore, as $\doteqdot$ may be used instead of I as a primitive operation of topological Boolean algebras, we will use the following definitional equations $Ia = (a \doteqdot 1), a \doteqdot b = I(a \dot{\leftrightarrow} b)$, per Theorems 26 and 27. Instead of "topological Boolean algebra" we will use "TB-algebra", for short.

If h is a homomorphism from a TB-algebra $\mathcal{A}$ into a TB-algebra $\mathcal{B}$, it preserves the Boolean operations and additionally:

- $h(Ia) = Ih(a)$ or $h(a \doteqdot 1) = h(a) \doteqdot 1$

for all $a \in A$.

The following definition is not utilized in the dissertation, however it expresses the necessary condition for the existence of a normal ultrafilter in a TB-algebra.

Definition 85. *Any* TB-*algebra* $\mathcal{A} = \langle A, \dot{\neg}, \dot{\cup}, \dot{\cap}, \dot{\rightarrow}, \dot{\leftrightarrow}, \doteqdot \rangle$ *is* *well-connected* *[38, p. 50] iff for any* $a, b, c, d \in A$ *the following condition holds:*

$$\textit{if } (a \doteqdot b) \ \dot{\cup} \ (c \doteqdot d) = 1, \ \textit{then } (a = b) \textit{ or } (c = d).$$

Theorem 29. *In any* TB-*algebra* $\mathcal{A} = \langle A, \dot{\neg}, \dot{\cup}, \dot{\cap}, \dot{\rightarrow}, \dot{\leftrightarrow}, \doteqdot \rangle$ *there is a normal ultrafilter iff* $\mathcal{A}$ *is well-connected [50].*

Definition 86 (TB-model)**.** *Pair* $\langle \mathcal{A}, F \rangle$ *is called a* TB-*model if and only if* $\mathcal{A}$ *is a* TB-*algebra and* F *is a normal ultrafilter in* TB-*algebra.*

Definition 87 (Satisfiability of a formula in a model)**.** *Let* $\mathcal{M} = \langle \mathcal{A}, F \rangle$ *be an arbitrary but fixed* TB-*model and* h *be an arbitrary homomorphism from* $\mathcal{L}$ *in* $\mathcal{A}$*. Formula* ϕ *is satisfied in* $\mathcal{M}$ *under* h *if and only if* $h(\phi) \in F$*.*

Definition 88 (Truth of a formula in a model)**.** *Let* $\mathcal{M} = \langle \mathcal{A}, F \rangle$ *be an arbitrary but fixed* TB-*model. Formula* ϕ *is true in* $\mathcal{M}$ *if and only if: for all* $h \in Hom(\mathcal{L}, \mathcal{A})$ ϕ *is satisfied in* $\mathcal{M}$ *under* h*.*

Definition 89 (Validity of a formula)**.** *Formula* ϕ *is valid in* WT *iff* ϕ *is true in all* TB-*models* $\mathcal{M}$*.*

We then can show that WT is sound and complete with respect to the presented algebraic semantics [38, p. 109].

Theorem 30. WT *is the set of all and only formulae true in every* TB-*model.*

Proof can be found in [38, p. 109].

5.3 Sequent Calculus $\mathsf{G3_{WT}}$

As WT is an axiomatic extension of WB, our goal is to utilize the same base for each system and introduce certain modifications that will underline the differences regarding the identity connective properties. In the case of WB we utilized labels to have control over the possible application of identity-dedicated rules. In the case of WT we modify the right identity rule in a way that would still allow the previous use of left-sided identity rules. As a result we will consider the following right identity rule

$$\frac{\Gamma^{\equiv} \Rightarrow \phi \leftrightarrow \chi}{\Gamma^{\equiv} \Rightarrow \phi \equiv \chi}\ R^{T}_{\equiv}$$

where $\Gamma^{\equiv}$ consists of equations only. Why do we restrict the antecedent of rule $R^{T}_{\equiv}$ to consist of equations only? Its main aim is to simply prevent the unwelcome possibility of proving the Fregean Axiom:

$$\frac{\begin{array}{c}\vdots\\ \phi \leftrightarrow \chi \Rightarrow \phi \equiv \chi\end{array}}{\Rightarrow (\phi \leftrightarrow \chi) \to (\phi \equiv \chi)}\ R_{\to}$$

In the above fragment of derivation we have non-identity in the antecedent of a $R_{\to}$ premiss, therefore the sequent cannot be a conclusion of $R^{T}_{\equiv}$, thus we are safe from proving the Fregean Axiom; identity is still separate from equivalence.

Here, similarly as was done for $\mathsf{G3_{WB}}$, we decided on formalizing as rule the definition of WT through a consequence operation instead of applying Negri's strategy. Axioms characterizing WT are all equations, which means that if we were to apply the abovementioned strategy, we would obtain a rule similar to $L^{B}_{\equiv}$, where we would additionally be able to consider four axioms of WT. This rule would allow us to introduce into derivation formulae of the shape of axioms $(\equiv_{11})$–$(\equiv_{14})$:

$$\frac{\phi, \Gamma \overset{I}{\Rightarrow} \Delta}{\Gamma \overset{I}{\Rightarrow} \Delta}\ L^{T}_{\equiv}$$

where ϕ is one of WT axioms. This calculus has not been examined as of yet, but it is interesting whether it would allow us (as in the case of calculus with $L^{B}_{\equiv}$) to show that structural rules are admissible. Since we would consider only left-sided rules, it appears plausible that cut and other structural rules would be admissible (since we would not need to consider the case with cut-formula of the shape of equation being principal in both premisses of cut; this topic will be elaborated on in the next chapter).

$\mathsf{G3_{WT}}$ will be understood as the set of the rules $\{L_{\wedge}, R_{\wedge}, L_{\vee}, R_{\vee}, L_{\to}, R_{\to}, L_{\leftrightarrow}, R_{\leftrightarrow}, L_{\neg}, R_{\neg}, L^{1}_{\equiv}, L^{2}_{\equiv}, L^{3}_{\equiv}, L^{4}_{\equiv}, R^{T}_{\equiv}, \mathit{cut}\}$, see Tables 5.1, 5.3, 5.2.

Definition 90 (Derivation of sequent $\Gamma \Rightarrow \Delta$ in $\mathsf{G3_{WT}}$)**.** *Derivation of* $\Gamma \Rightarrow \Delta$ *in* $\mathsf{G3_{WT}}$ *is a labelled finite tree with a single root labelled with* $\Gamma \Rightarrow \Delta$ *and each*

TABLE 5.1: Rules of $\mathsf{G3_{WT}}$: classical rules

$$\phi, \Gamma \Rightarrow \Delta, \phi$$

$$\frac{\Gamma \Rightarrow \Delta, \phi}{\neg\phi, \Gamma \Rightarrow \Delta}\ L_{\neg} \qquad \frac{\phi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg\phi}\ R_{\neg}$$

$$\frac{\phi, \chi, \Gamma \Rightarrow \Delta}{\phi \wedge \chi, \Gamma \Rightarrow \Delta}\ L_{\wedge} \qquad \frac{\Gamma \Rightarrow \Delta, \phi \quad \Gamma \Rightarrow \Delta, \chi}{\Gamma \Rightarrow \Delta, \phi \wedge \chi}\ R_{\wedge}$$

$$\frac{\phi, \Gamma \Rightarrow \Delta \quad \chi, \Gamma \Rightarrow \Delta}{\phi \vee \chi, \Gamma, \Rightarrow \Delta}\ L_{\vee} \qquad \frac{\Gamma \Rightarrow \Delta, \phi, \chi}{\Gamma \Rightarrow \Delta, \phi \vee \chi}\ R_{\vee}$$

$$\frac{\Gamma \Rightarrow \Delta, \phi \quad \chi, \Gamma \Rightarrow \Delta}{\phi \rightarrow \chi, \Gamma \Rightarrow \Delta}\ L_{\rightarrow} \qquad \frac{\phi, \Gamma \Rightarrow \Delta, \chi}{\Gamma \Rightarrow \Delta, \phi \rightarrow \chi}\ R_{\rightarrow}$$

$$\frac{\phi, \chi, \Gamma \Rightarrow \Delta \quad \Gamma \Rightarrow \Delta, \phi, \chi}{\phi \leftrightarrow \chi, \Gamma \Rightarrow \Delta}\ L_{\leftrightarrow} \qquad \frac{\phi, \Gamma \Rightarrow \chi, \Delta \quad \chi, \Gamma \Rightarrow \phi, \Delta}{\Gamma \Rightarrow \Delta, \phi \leftrightarrow \chi}\ R_{\leftrightarrow}$$

node-label connected with the labels of the (immediate) successor nodes (if any) according to one of the rules.

Definition 91 (Proof of $\Gamma \Rightarrow \Delta$ in $\mathsf{G3_{WT}}$). *Proof of sequent $\Gamma \Rightarrow \Delta$ in $\mathsf{G3_{WT}}$ is a derivation of $\Gamma \Rightarrow \Delta$ with axioms at all of its top nodes.*

Similarly as in the case of $\mathsf{G3_{WB}}$, we can consider two additional axiom schemata utilizing constants $\bot$ and $\top$, that is

$$\Gamma \Rightarrow \Delta, \top \qquad\qquad \bot, \Gamma \Rightarrow \Delta$$

proofs of which are analogous to the ones presented in Chapter 4; the only difference being the lack of labels.

When it comes to the rule-set of sequent calculus $\mathsf{G3_{WT}}$, we will consider a set of standard classical and structural rules, similarly as it was in the case of $\mathsf{G3_{WB}}$, but we will omit the labels, as they are unnecessary in this sequent calculus. As for the identity-dedicated rules, we will consider rules in Table 5.3, mostly to keep derivations in $\mathsf{G3_{WT}}$ as concise as possible, although we can consider other possibilities of defining a sequent system for WT, depending on our goal. There are two rules $L^3_{\equiv}$ and $R^T_{\equiv}$ that will be sufficient to prove both axioms characterizing WB and the added axioms characterizing WT. Moreover, both rules additionally work as a formalization of the quasi-Fregean rule mentioned above. We know that, naturally, axiom ($\equiv_3$) can be proved using the above rules. Below we present derivations for axioms ($\equiv_1$) and ($\equiv_2$).

TABLE 5.2: Rules of $\mathsf{G3_{WT}}$: structural rules

$$\frac{\Gamma \Rightarrow \Delta}{\chi, \Gamma \Rightarrow \Delta}\ L_{wk} \qquad \frac{\chi, \chi, \Gamma \Rightarrow \Delta}{\chi, \Gamma \Rightarrow \Delta}\ L_{ctr}$$

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \chi}\ R_{wk} \qquad \frac{\Gamma \Rightarrow \Delta, \chi, \chi}{\Gamma \Rightarrow \Delta, \chi}\ R_{ctr}$$

$$\frac{\Gamma \Rightarrow \Sigma, \phi \quad \phi, \Pi \Rightarrow \Delta}{\Gamma, \Pi \Rightarrow \Sigma, \Delta}\ cut$$

TABLE 5.3: $\mathsf{G3_{WT}}$: identity rules

$$\frac{\phi \equiv \phi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ L^1_{\equiv} \qquad \frac{\chi \leftrightarrow \phi, \chi \equiv \phi, \Gamma \Rightarrow \Delta}{\chi \equiv \phi, \Gamma \Rightarrow \Delta}\ L^3_{\equiv}$$

$$\frac{\neg\phi \equiv \neg\chi, \phi \equiv \chi, \Gamma \Rightarrow \Delta}{\phi \equiv \chi, \Gamma \Rightarrow \Delta}\ L^2_{\equiv} \qquad \frac{(\phi \otimes \chi) \equiv (\psi \otimes \omega), \phi \equiv \psi, \chi \equiv \omega, \Gamma \Rightarrow \Delta}{\phi \equiv \psi, \chi \equiv \omega, \Gamma \Rightarrow \Delta}\ L^4_{\equiv}$$

$$\frac{\Gamma^{\equiv} \Rightarrow \phi \leftrightarrow \chi}{\Gamma^{\equiv} \Rightarrow \phi \equiv \chi}\ R^T_{\equiv}$$

($\equiv_1$):

$$\dfrac{\dfrac{\phi \Rightarrow \phi \quad \phi \Rightarrow \phi}{\Rightarrow \phi \leftrightarrow \phi}\ R_{\leftrightarrow}}{\Rightarrow \phi \equiv \phi}\ R^T_{\equiv}$$

($\equiv_2$):

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\delta, \phi, \chi, \phi \Rightarrow \chi \quad \delta, \phi \Rightarrow \phi, \chi, \chi}{\delta, \phi \leftrightarrow \chi, \phi \Rightarrow \chi}\ L_{\leftrightarrow}}{\delta, \phi \leftrightarrow \chi, \neg\chi \Rightarrow \neg\phi}\ R_{\neg}, L_{\neg} \quad \dfrac{\dfrac{\delta, \phi, \chi, \chi \Rightarrow \phi \quad \delta, \chi \Rightarrow \phi, \phi, \chi}{\delta, \phi \leftrightarrow \chi, \chi \Rightarrow \phi}\ L_{\leftrightarrow}}{\delta, \phi \leftrightarrow \chi, \neg\phi \Rightarrow \neg\chi}\ R_{\neg}, L_{\neg}}{\delta, \phi \leftrightarrow \chi \Rightarrow \neg\phi \leftrightarrow \neg\chi}\ R_{\leftrightarrow}}{\delta \Rightarrow \neg\phi \leftrightarrow \neg\chi}\ L^3_{\equiv}}{\delta \Rightarrow \neg\phi \equiv \neg\chi}\ R^T_{\equiv}}{\Rightarrow (\phi \equiv \chi) \rightarrow (\neg\phi \equiv \neg\chi)}\ R_{\rightarrow}$$

where $\delta = \phi \equiv \chi$. Let us also consider the following fragment of the proof of an instance of axiom ($\equiv_4$), where $\Gamma^{\equiv} = \{\phi \equiv \psi, \chi \equiv \omega, \phi \equiv \chi\}$:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{\phi \leftrightarrow \psi, \chi \leftrightarrow \omega, \phi \leftrightarrow \chi, \Gamma^{\equiv} \Rightarrow \psi \leftrightarrow \omega}}{\phi \equiv \psi, \chi \equiv \omega, \phi \equiv \chi \Rightarrow \psi \leftrightarrow \omega} L^3_{\equiv} \times 3}{\phi \equiv \psi, \chi \equiv \omega, \phi \equiv \chi \Rightarrow \psi \equiv \omega} R^T_{\equiv} \quad \dfrac{\vdots}{\phi \equiv \psi, \chi \equiv \omega, \psi \equiv \omega \Rightarrow \phi \equiv \chi}}{\phi \equiv \psi, \chi \equiv \omega \Rightarrow (\phi \equiv \chi) \leftrightarrow (\psi \equiv \omega)} R_{\leftrightarrow}}{\phi \equiv \psi, \chi \equiv \omega \Rightarrow (\phi \equiv \chi) \equiv (\psi \equiv \omega)} R^T_{\equiv}}{(\phi \equiv \psi) \wedge (\chi \equiv \omega) \Rightarrow (\phi \equiv \chi) \equiv (\psi \equiv \omega)} L_{\wedge}}{\Rightarrow ((\phi \equiv \psi) \wedge (\chi \equiv \omega)) \rightarrow ((\phi \equiv \chi) \equiv (\psi \equiv \omega))} R_{\rightarrow}
$$

We can easily prove it through the use of $L^3_{\equiv}$ and $R^T_{\equiv}$ (and other classical rules), we just have to keep in mind the context in which right identity can be applied. We therefore could potentially choose the wrong order of application of certain rules (in proof-search through root-first search) and close our way to constructing the proof.

Also, to shorten the proofs we will consider two additional rules, which are derivable with the use of $R^T_{\equiv}$, $L^3_{\equiv}$ and cut or with the use of $R^T_{\equiv}$, $L^3_{\equiv}$ and L_{wk}:

$$
\dfrac{\Gamma, \phi \equiv \top, \phi \Rightarrow \Delta}{\Gamma, \phi \equiv \top \Rightarrow \Delta} L_{\equiv\top} \qquad\qquad \dfrac{\Gamma^{\equiv} \Rightarrow \phi}{\Gamma^{\equiv} \Rightarrow \phi \equiv \top} R_{\equiv\top}
$$

where $\Gamma^{\equiv}$ consists of equations only. These two rules are derivable in $\mathsf{G3_{WT}} \cup \{L_{wk}\}$:

- $L_{\equiv\top}$:

$$
\dfrac{\dfrac{\dfrac{\Gamma, \phi \equiv \top, \phi \Rightarrow \Delta}{\Gamma, \phi \equiv \top, \phi, \top \Rightarrow \Delta} L_{wk} \quad \Gamma, \phi \equiv \top \Rightarrow \phi, \top, \Delta}{\Gamma, \phi \equiv \top, \phi \leftrightarrow \top \Rightarrow \Delta} L_{\leftrightarrow}}{\Gamma, \phi \equiv \top \Rightarrow \Delta} L^3_{\equiv}
$$

- $R_{\equiv\top}$

$$
\dfrac{\dfrac{\Gamma^{\equiv}, \phi \Rightarrow \top \quad \dfrac{\Gamma^{\equiv} \Rightarrow \phi}{\Gamma^{\equiv}, \top \Rightarrow \phi} L_{wk}}{\Gamma^{\equiv} \Rightarrow \phi \leftrightarrow \top} R_{\leftrightarrow}}{\Gamma^{\equiv} \Rightarrow \phi \equiv \top} R^T_{\equiv}
$$

or, instead of weakening, using cut, that is, they are derivable in $\mathsf{G3_{WT}}$:

- $L_{\equiv\top}$:

$$
\dfrac{\dfrac{\dfrac{\phi, \top \Rightarrow \phi \quad \phi, \Gamma, \phi \equiv \top \Rightarrow \Delta}{\Gamma, \phi \equiv \top, \phi, \top \Rightarrow \Delta} cut \quad \Gamma, \phi \equiv \top \Rightarrow \phi, \top, \Delta}{\Gamma, \phi \equiv \top, \phi \leftrightarrow \top \Rightarrow \Delta} L_{\leftrightarrow}}{\Gamma, \phi \equiv \top \Rightarrow \Delta} L^3_{\equiv}
$$

- $R_{\equiv\top}$

$$\dfrac{\Gamma^{\equiv}, \phi \Rightarrow \top \qquad \dfrac{\Gamma^{\equiv} \Rightarrow \phi \quad \phi, \top \Rightarrow \phi}{\Gamma^{\equiv}, \top \Rightarrow \phi}\ cut}{\dfrac{\Gamma^{\equiv} \Rightarrow \phi \leftrightarrow \top}{\Gamma^{\equiv} \Rightarrow \phi \equiv \top}\ R^{T}_{\equiv}}\ R_{\leftrightarrow}$$

We can also consider introducing them as primary rules, in addition to $L^3_{\equiv}$ and $R^T_{\equiv}$.

We mentioned that WT corresponds to S4 and the identity connective $\equiv$ can be interpreted as a version of the necessity connective $\Box$. We may therefore examine how rules for $\Box$ in sequent calculus for S4 can be translated to rules characterizing $\equiv$ in WT. In order to obtain sequent calculus for S4 we add to G3cp the following modal rules [24]:

$$\frac{\phi, \Gamma \Rightarrow \Delta}{\Box\phi, \Gamma \Rightarrow \Delta}\ L_{\Box} \qquad \frac{\Gamma \Rightarrow \Delta, \phi}{\Gamma \Rightarrow \Delta, \Diamond\phi}\ R_{\Diamond}$$

$$\frac{\Box\Gamma \Rightarrow \Diamond\Delta, \phi}{\Box\Gamma \Rightarrow \Diamond\Delta, \Box\phi}\ R_{\Box} \qquad \frac{\phi, \Box\Gamma \Rightarrow \Diamond\Delta}{\Diamond\phi, \Box\Gamma \Rightarrow \Diamond\Delta}\ L_{\Diamond}$$

We can use the fact of correspondence between the two logics and consider a pair of rules for the identity connective which are based on the modal rules. We translate formulae preceded by $\Box$ to language of WT and obtain rules $L_{\equiv\top}$ and $R_{\equiv\top}$ (which, as we showed before, can be obtain through utilisation of $L^3_{\equiv}$ and $R^T_{\equiv}$ and cut or weakening):

$$\frac{\phi, \Gamma \Rightarrow \Delta}{\Box\phi, \Gamma \Rightarrow \Delta}\ L_{\Box} \qquad \rightsquigarrow \qquad \frac{\phi, \Gamma \Rightarrow \Delta}{\phi \equiv \top, \Gamma \Rightarrow \Delta}$$

$$\frac{\Box\Gamma \Rightarrow \Diamond\Delta, \phi}{\Box\Gamma \Rightarrow \Diamond\Delta, \Box\phi}\ R_{\Box} \qquad \rightsquigarrow \qquad \frac{\Gamma^{\equiv\top} \Rightarrow \Delta^{\neg(\equiv\top)}, \phi}{\Gamma^{\equiv\top} \Rightarrow \Delta^{\neg(\equiv\top)}, \phi \equiv \top}$$

where $\Gamma^{\equiv\top}$ consists of formulae ϕ_i in Γ, which are equal with $\top$, i.e. for each formula $\phi_i \in \Gamma$, $\phi_i \equiv \top \in \Gamma^{\equiv}$. Similarly for each $\chi_i \in \Delta$, $\neg(\chi_i \equiv \top) \in \Delta^{\neg(\equiv\top)}$.

To sum up, we can consider three different approaches regarding specifying the set of identity-dedicated rules:

(a) the set of two primary rules: $\{L^3_{\equiv}, R^T_{\equiv}\}$,

(b) the set of four primary rules: $\{L^3_{\equiv}, R^T_{\equiv}, L_{\equiv\top}, R_{\equiv\top}\}$,

(c) the set of five primary rules: $\{L^1_{\equiv}, L^2_{\equiv}, L^3_{\equiv}, L^4_{\equiv}, R^T_{\equiv}\}$.

In order to minimize the size of derivations we will opt for version (c) with two additional, derivable rules $\{L_{\equiv\top}, R_{\equiv\top}\}$, thus defining G3$_{\mathsf{WT}}$ as $\{L_{\wedge}, R_{\wedge}, L_{\vee}, R_{\vee}, L_{\rightarrow}, R_{\rightarrow}, L_{\leftrightarrow}, R_{\leftrightarrow}, L_{\neg}, R_{\neg}, L^1_{\equiv}, L^2_{\equiv}, L^3_{\equiv}, L^4_{\equiv}, R^T_{\equiv}, cut\}$.

For WT and WH we consider slightly modified definition of the satisfiability of a sequent.

Definition 92 (Satisfiability of a sequent)**.** *Let $\mathcal{M} = \langle \mathcal{A}, F\rangle$ be an arbitrary but fixed* TB*-model and let $h \in Hom(\mathcal{L}, \mathcal{A})$. Sequent $\Gamma \Rightarrow \Delta$ is satisfied in $\mathcal{M}$ under h provided $h(\phi_1)\dot{\cap}...\dot{\cap}h(\phi_n) \leq h(\chi_1)\dot{\cup}...\dot{\cup}h(\chi_k)$, where $\Gamma = \phi_1, ..., \phi_n$ and $\Delta = \chi_1, ..., \chi_k$.*

Definition 93 (Truth of a sequent)**.** *Let $\mathcal{M} = \langle \mathcal{A}, F\rangle$ be an arbitrary* TB*-model. Sequent $\Gamma \Rightarrow \Delta$ is true in $\mathcal{M}$ provided that for each $h \in Hom(\mathcal{L}, \mathcal{A})$, sequent $\Gamma \Rightarrow \Delta$ is satisfied in $\mathcal{M}$ under h.*

Definition 94 (Validity of a sequent)**.** *Sequent $\Gamma \Rightarrow \Delta$ is valid in* WT*, if it is true in each* TB*-model.*

We can now examine the following example of a derivation (of WT theorem $((\phi \equiv \chi) \to (\phi \leftrightarrow \chi)) \equiv \top$):

$$\dfrac{\dfrac{(\phi \equiv \chi) \to (\phi \leftrightarrow \chi) \Rightarrow \top \qquad \dfrac{\dfrac{\top, \phi \leftrightarrow \chi, \phi \equiv \chi \Rightarrow \phi \leftrightarrow \chi}{\top, \phi \equiv \chi \Rightarrow \phi \leftrightarrow \chi}\, L^3_{\equiv}}{\top \Rightarrow (\phi \equiv \chi) \to (\phi \leftrightarrow \chi)}\, R_{\to}}{\Rightarrow ((\phi \equiv \chi) \to (\phi \leftrightarrow \chi)) \leftrightarrow \top}\, R_{\leftrightarrow}}{\Rightarrow ((\phi \equiv \chi) \to (\phi \leftrightarrow \chi)) \equiv \top}\, R^T_{\equiv}$$

5.3.1 Completeness of $\mathsf{G3}_{\mathsf{WT}}$

We use a similar approach as was in the case of $\mathsf{G3}_{\mathsf{SCI}}$ and $\mathsf{G3}_{\mathsf{WB}}$. We will show that axiomatic system H_{WT} can be simulated within $\mathsf{G3}_{\mathsf{WT}}$.

Theorem 31 (Interpretation of H_{WT} in $\mathsf{G3}_{\mathsf{WT}}$)**.** *If formula ϕ is provable in axiomatic system H_{WT}, then sequent $\Rightarrow \phi$ is provable in* $\mathsf{G3}_{\mathsf{WT}}$*.*

Proof. We use the same construction as for $\mathsf{G3}_{\mathsf{WB}}$. We know that *modus ponens* is derivable in $\mathsf{G3}_{\mathsf{WT}}$ with the use of *cut*, so we omit this part and show that for all axioms ϕ of WT, sequents $\Rightarrow \phi$ have proofs in $\mathsf{G3}_{\mathsf{WT}}$:

($\equiv_{11}$) $\Box(\phi \leftrightarrow \chi) \equiv (\phi \equiv \chi) =_{df} ((\phi \leftrightarrow \chi) \equiv \top) \equiv (\phi \equiv \chi)$

$$\dfrac{\dfrac{\dfrac{\dfrac{(\phi \leftrightarrow \chi) \equiv \top, \phi \leftrightarrow \chi \Rightarrow \phi \leftrightarrow \chi}{(\phi \leftrightarrow \chi) \equiv \top \Rightarrow \phi \leftrightarrow \chi}\, L_{\equiv\top}}{(\phi \leftrightarrow \chi) \equiv \top \Rightarrow \phi \equiv \chi}\, R^T_{\equiv} \qquad \dfrac{\dfrac{\phi \leftrightarrow \chi, \phi \equiv \chi \Rightarrow \phi \leftrightarrow \chi}{\phi \equiv \chi \Rightarrow \phi \leftrightarrow \chi}\, L^3_{\equiv}}{\phi \equiv \chi \Rightarrow (\phi \leftrightarrow \chi) \equiv \top}\, R_{\equiv\top}}{\Rightarrow ((\phi \leftrightarrow \chi) \equiv \top) \leftrightarrow (\phi \equiv \chi)}\, R_{\leftrightarrow}}{\Rightarrow ((\phi \leftrightarrow \chi) \equiv \top) \equiv (\phi \equiv \chi)}\, R^T_{\equiv}$$

($\equiv_{12}$) $\Box\phi \leq \phi =_{df} (((\phi \equiv \top) \to \phi) \equiv \top)$

$$\dfrac{\dfrac{\dfrac{\phi \equiv \top, \phi \Rightarrow \phi}{\phi \equiv \top \Rightarrow \phi}\, L_{\equiv\top}}{\Rightarrow (\phi \equiv \top) \to \phi}\, R_{\to}}{\Rightarrow ((\phi \equiv \top) \to \phi) \equiv \top}\, R_{\equiv\top}$$

$(\equiv_{13})$ $\Box(\phi \wedge \chi) \equiv (\Box\phi \wedge \Box q) =_{df} ((\phi \wedge \chi) \equiv \top) \equiv ((\phi \equiv \top) \wedge (\chi \equiv \top))$

$$
\dfrac{
\dfrac{
\dfrac{\dfrac{\dfrac{\dfrac{(\phi\wedge\chi)\equiv\top,\phi,\chi\Rightarrow\phi}{(\phi\wedge\chi)\equiv\top,\phi\wedge\chi\Rightarrow\phi}\,L_\wedge}{(\phi\wedge\chi)\equiv\top\Rightarrow\phi}\,L_{\equiv\top}}{(\phi\wedge\chi)\equiv\top\Rightarrow\phi\equiv\top}\,R_{\equiv\top}
\quad
\dfrac{\dfrac{\dfrac{(\phi\wedge\chi)\equiv\top,\phi,\chi\Rightarrow\chi}{(\phi\wedge\chi)\equiv\top,\phi\wedge\chi\Rightarrow\chi}\,L_\wedge}{(\phi\wedge\chi)\equiv\top\Rightarrow\chi}\,L_{\equiv\top}}{(\phi\wedge\chi)\equiv\top\Rightarrow\chi\equiv\top}\,R_{\equiv\top}}{(\phi\wedge\chi)\equiv\top\Rightarrow(\phi\equiv\top)\wedge(\chi\equiv\top)}\,R_\wedge
\quad
\begin{array}{c}D_1\\ \vdots\end{array}
}{\Rightarrow((\phi\wedge\chi)\equiv\top)\leftrightarrow((\phi\equiv\top)\wedge(\chi\equiv\top))}\,R_\leftrightarrow}{\Rightarrow((\phi\wedge\chi)\equiv\top)\equiv((\phi\equiv\top)\wedge(\chi\equiv\top))}\,R^T_\equiv
$$

where D_1:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\phi\equiv\top,\chi\equiv\top,\phi,\chi\Rightarrow\phi \quad \phi\equiv\top,\chi\equiv\top,\phi,\chi\Rightarrow\chi}{\phi\equiv\top,\chi\equiv\top,\phi,\chi\Rightarrow(\phi\wedge\chi)}\,R_\wedge}{\phi\equiv\top,\chi\equiv\top\Rightarrow(\phi\wedge\chi)}\,2\times L_{\equiv\top}}{\phi\equiv\top,\chi\equiv\top\Rightarrow(\phi\wedge\chi)\equiv\top}\,R_{\equiv\top}}{(\phi\equiv\top)\wedge(\chi\equiv\top)\Rightarrow((\phi\wedge\chi)\equiv\top)}\,L_\wedge
$$

$(\equiv_{14})$ $\Box\Box\phi \equiv \Box\phi =_{\mathrm{df}} ((\phi \equiv \top) \equiv \top) \equiv (\phi \equiv \top)$

$$
\dfrac{\dfrac{\dfrac{\phi\equiv\top,(\phi\equiv\top)\equiv\top\Rightarrow\phi\equiv\top}{(\phi\equiv\top)\equiv\top\Rightarrow\phi\equiv\top}\,L_{\equiv\top}
\quad
\dfrac{\phi\equiv\top\Rightarrow\phi\equiv\top}{\phi\equiv\top\Rightarrow(\phi\equiv\top)\equiv\top}\,R_{\equiv\top}}{\Rightarrow((\phi\equiv\top)\equiv\top)\leftrightarrow(\phi\equiv\top)}\,R_\leftrightarrow}{\Rightarrow((\phi\equiv\top)\equiv\top)\equiv(\phi\equiv\top)}\,R^T_\equiv
$$

The remaining reasoning is analogous to that in the proof of Theorem 11. □

Theorem 32 (Completeness)**.** *If a sequent $\Rightarrow \phi$ is valid in* WT*, it is provable in* $\mathsf{G3_{WT}}$*.*

Proof. Analogous to that of Theorem 12 presented in Chapter 3. □

5.3.2 Soundness of $\mathsf{G3_{WT}}$

Since the correctness of rules uses different approach to the satisfiability of sequent in some model $\mathcal{M}$ under homomorphism h, below we present correctness of both classical and identity-dedicated rules.

Lemma 17. *$L_\neg$ is correct in* WT*.*

Proof. Let $\mathcal{M} = \langle \mathcal{A}, F\rangle$ be an arbitrary, but fixed TB-model and h be a homomorphism of $\mathcal{L}$ into $\mathcal{A}$. Suppose that the following condition holds: $\Gamma \Rightarrow \Delta, \phi$ is satisfied in $\mathcal{M}$ under h. It means that $\dot{\bigcap} h(\Gamma) \leq \dot{\bigcup} h(\Delta) \mathbin{\dot\cup} h(\phi)$. We will show that $\neg\phi, \Gamma \Rightarrow \Delta$ is also satisfied in $\mathcal{M}$ under h, that is $\dot\neg h(\phi) \mathbin{\dot\cap} \dot{\bigcap} h(\Gamma) \leq \dot{\bigcup} h(\Delta)$. We begin with multiplying both sides of $\dot{\bigcap} h(\Gamma) \leq \dot{\bigcup} h(\Delta) \mathbin{\dot\cup} h(\phi)$ with $\dot\neg h(\phi)$, thus obtaining $\dot\neg h(\phi) \mathbin{\dot\cap} \dot{\bigcap} h(\Gamma) \leq \dot{\bigcup}(h(\Delta) \mathbin{\dot\cup} h(\phi)) \mathbin{\dot\cap} \dot\neg h(\phi)$. Through distributivity we have $\dot\neg h(\phi) \mathbin{\dot\cap} \dot{\bigcap} h(\Gamma) \leq \dot{\bigcup}(h(\Delta) \mathbin{\dot\cap} \dot\neg h(\phi)) \mathbin{\dot\cup} (h(\phi) \mathbin{\dot\cap} \dot\neg h(\phi))$. Since $h(\phi) \mathbin{\dot\cap} \dot\neg h(\phi)$ equals 0, we have $\dot\neg h(\phi) \mathbin{\dot\cap} \dot{\bigcap} h(\Gamma) \leq (\dot{\bigcup} h(\Delta) \mathbin{\dot\cap} \dot\neg h(\phi)) \mathbin{\dot\cup} 0$.

Then, as $a \mathbin{\dot\cup} 0 = a$, we have $\dot\neg h(\phi) \mathbin{\dot\cap} \dot\bigcap h(\Gamma) \leq \dot\bigcup h(\Delta) \mathbin{\dot\cap} \dot\neg h(\phi)$. Through properties of $\leq$ we have both $\dot\neg h(\phi) \mathbin{\dot\cap} \dot\bigcap h(\Gamma) \leq \dot\bigcup h(\Delta)$ and $\dot\neg h(\phi) \mathbin{\dot\cap} \dot\bigcap h(\Gamma) \leq \dot\neg h(\phi)$. The former can be brought back to the sequent form $\neg\phi, \Gamma \Rightarrow \Delta$ thus showing that $L_\neg$ is correct in WT. □

Lemma 18. *$R_\neg$ is correct in* WT.

Proof. Let $\mathcal{M} = \langle \mathcal{A}, F \rangle$ be an arbitrary, but fixed TB-model and h be a homomorphism of $\mathcal{L}$ into $\mathcal{A}$. Suppose that the following condition holds: $\phi, \Gamma \Rightarrow \Delta$ is satisfied in $\mathcal{M}$ under h. It means that $h(\phi) \mathbin{\dot\cap} \dot\bigcap h(\Gamma) \leq \dot\bigcup h(\Delta)$. We will show that $\Gamma \Rightarrow \Delta, \neg\phi$ is also satisfied in $\mathcal{M}$ under h, that is $\dot\bigcap h(\Gamma) \leq \dot\bigcup h(\Delta) \mathbin{\dot\cup} \dot\neg h(\phi)$. We begin with adding to both sides of $h(\phi) \mathbin{\dot\cap} \dot\bigcap h(\Gamma) \leq \dot\bigcup h(\Delta)$ expression $\dot\neg h(\phi)$, thus obtaining $\dot\neg h(\phi) \mathbin{\dot\cup} (h(\phi) \mathbin{\dot\cap} \dot\bigcap h(\Gamma)) \leq \dot\bigcup h(\Delta) \mathbin{\dot\cup} \dot\neg h(\phi)$. Through distributivity we have $(\dot\neg h(\phi) \mathbin{\dot\cup} h(\phi)) \mathbin{\dot\cap} (\dot\neg h(\phi) \mathbin{\dot\cup} \dot\bigcap h(\Gamma)) \leq \dot\bigcup h(\Delta) \mathbin{\dot\cup} \dot\neg h(\phi)$. Since $\dot\neg h(\phi) \mathbin{\dot\cup} h(\phi)$ equals 1, we have $1 \mathbin{\dot\cap} (\dot\neg h(\phi) \mathbin{\dot\cup} \dot\bigcap h(\Gamma)) \leq \dot\bigcup h(\Delta) \mathbin{\dot\cup} \dot\neg h(\phi)$. Then, as $a \mathbin{\dot\cap} 1 = a$, we have $\dot\neg h(\phi) \mathbin{\dot\cup} \dot\bigcap h(\Gamma) \leq \dot\bigcup h(\Delta) \mathbin{\dot\cup} \dot\neg h(\phi)$. Through properties of $\leq$ we have both $\dot\neg h(\phi) \leq \dot\bigcup h(\Delta) \mathbin{\dot\cup} \dot\neg h(\phi)$ and $\dot\bigcap h(\Gamma) \leq \dot\bigcup h(\Delta) \mathbin{\dot\cup} \dot\neg h(\phi)$. The latter can be brought back to the sequent form $\Gamma \Rightarrow \Delta, \neg\phi$ thus showing that $R_\neg$ is correct in WT. □

Lemma 19. *$L_\wedge$ is correct in* WT.

Proof. Let $\mathcal{M} = \langle \mathcal{A}, F \rangle$ be an arbitrary, but fixed TB-model and h be a homomorphism of $\mathcal{L}$ into $\mathcal{A}$. Suppose that the following condition holds: $\phi, \chi, \Gamma \Rightarrow \Delta$ is satisfied in $\mathcal{M}$ under h. That means that $h(\phi) \mathbin{\dot\cap} h(\chi) \mathbin{\dot\cap} \dot\bigcap h(\Gamma) \leq \dot\bigcup h(\Delta)$. Then also $h(\phi \wedge \chi) \mathbin{\dot\cap} \dot\bigcap h(\Gamma) \leq \dot\bigcup h(\Delta)$. □

Lemma 20. *$R_\wedge$ is correct in* WT.

Proof. Let $\mathcal{M} = \langle \mathcal{A}, F \rangle$ be an arbitrary, but fixed TB-model and h be a homomorphism of $\mathcal{L}$ into $\mathcal{A}$. Suppose that the following condition holds: $\Gamma \Rightarrow \Delta, \phi$ and $\Gamma \Rightarrow \Delta, \chi$ are satisfied in $\mathcal{M}$ under h. That means that the following hold: $\dot\bigcap h(\Gamma) \leq \dot\bigcup h(\Delta) \mathbin{\dot\cup} h(\phi)$ and $\dot\bigcap h(\Gamma) \leq \dot\bigcup h(\Delta) \mathbin{\dot\cup} h(\chi)$. We will show that $\Gamma \Rightarrow \Delta, \phi \wedge \chi$ is also satisfied in $\mathcal{M}$ under h, that is $\dot\bigcap h(\Gamma) \leq \dot\bigcup h(\Delta), h(\phi \wedge \chi)$. From $\dot\bigcap h(\Gamma) \leq \dot\bigcup h(\Delta) \mathbin{\dot\cup} h(\phi)$ and $\dot\bigcap h(\Gamma) \leq \dot\bigcup h(\Delta) \mathbin{\dot\cup} h(\chi)$ we have $\dot\bigcap h(\Gamma) \mathbin{\dot\cap} \dot\bigcap h(\Gamma) \leq (\dot\bigcup h(\Delta) \mathbin{\dot\cup} h(\phi)) \mathbin{\dot\cap} (\dot\bigcup h(\Delta) \mathbin{\dot\cup} h(\chi))$ which then can be turned into $\dot\bigcap h(\Gamma) \leq (\dot\bigcup h(\Delta) \mathbin{\dot\cup} h(\phi)) \mathbin{\dot\cap} (\dot\bigcup h(\Delta) \mathbin{\dot\cup} h(\chi))$. Through distributivity, $(\dot\bigcap \Gamma) \leq \dot\bigcup h(\Delta) \mathbin{\dot\cup} (h(\phi) \mathbin{\dot\cap} h(\chi))$. □

Lemma 21. *$L_\vee$ is correct in* WT.

Proof. Let $\mathcal{M} = \langle \mathcal{A}, F \rangle$ be an arbitrary, but fixed TB-model and h be a homomorphism of $\mathcal{L}$ into $\mathcal{A}$. Suppose that the following condition holds: $\phi, \Gamma \Rightarrow \Delta$ and $\chi, \Gamma \Rightarrow \Delta$ are satisfied in $\mathcal{M}$ under h. That means that the following hold: $h(\phi) \mathbin{\dot\cap} \dot\bigcap h(\Gamma) \leq \dot\bigcup h(\Delta)$ and $h(\chi) \mathbin{\dot\cap} \dot\bigcap h(\Gamma) \leq \dot\bigcup h(\Delta)$. We will show that $\phi \vee \chi, \Gamma \Rightarrow \Delta$ is also satisfied in $\mathcal{M}$ under h, that is $h(\phi \vee \chi) \mathbin{\dot\cap} h(\dot\bigcap \Gamma) \leq \dot\bigcup h(\Delta)$. From $h(\phi) \mathbin{\dot\cap} \dot\bigcap h(\Gamma) \leq \dot\bigcup h(\Delta)$ and $h(\chi) \mathbin{\dot\cap} \dot\bigcap h(\Gamma) \leq \dot\bigcup h(\Delta)$ we have $(\dot\bigcap h(\Gamma) \mathbin{\dot\cap} h(\phi)) \mathbin{\dot\cup} (\dot\bigcap h(\Gamma) \mathbin{\dot\cap} h(\chi)) \leq \dot\bigcup h(\Delta)$ and then, through distributivity, $(h(\phi) \mathbin{\dot\cup} h(\chi)) \mathbin{\dot\cap} \dot\bigcap h(\Gamma) \leq \dot\bigcup h(\Delta)$. □

Lemma 22. *$R_\vee$ is correct in* WT.

Proof. Let $\mathcal{M} = \langle \mathcal{A}, F \rangle$ be an arbitrary, but fixed TB-model and h be a homomorphism of $\mathcal{L}$ into $\mathcal{A}$. Suppose that the following condition holds: $\Gamma \Rightarrow \Delta, \phi, \chi$ is satisfied in $\mathcal{M}$ under h. That means that $\dot{\bigcap} h(\Gamma) \leq \dot{\bigcup} h(\Delta) \mathbin{\dot{\cup}} h(\phi) \mathbin{\dot{\cup}} h(\chi)$. From Definition 51 we have $\dot{\bigcap} h(\Gamma) \leq \dot{\bigcup} h(\Delta) \mathbin{\dot{\cup}} h(\phi \vee \chi)$. $\square$

Lemma 23. *$L_{\rightarrow}$ is correct in* WT.

Proof. Let $\mathcal{M} = \langle \mathcal{A}, F \rangle$ be an arbitrary, but fixed TB-model and h be a homomorphism of $\mathcal{L}$ into $\mathcal{A}$. Suppose that the following condition holds: $\Gamma \Rightarrow \Delta, \phi$ and $\chi, \Gamma \Rightarrow \Delta$ are satisfied in $\mathcal{M}$ under h. That means that the following hold: $\dot{\bigcap} h(\Gamma) \leq \dot{\bigcup} h(\Delta) \mathbin{\dot{\cup}} h(\phi)$ and $h(\chi) \mathbin{\dot{\cap}} \dot{\bigcap} h(\Gamma) \leq \dot{\bigcup} h(\Delta)$. We will show that $\phi \rightarrow \chi, \Gamma \Rightarrow \Delta$ is also satisfied in $\mathcal{M}$ under h, that is $h(\phi \rightarrow \chi) \mathbin{\dot{\cap}} h(\dot{\bigcap} \Gamma) \leq \dot{\bigcup} h(\Delta)$. We apply Lemma 17 to $\dot{\bigcap} h(\Gamma) \leq \dot{\bigcup} h(\Delta) \mathbin{\dot{\cup}} h(\phi)$ thus obtaining $\dot{\neg} h(\phi) \mathbin{\dot{\cap}} \dot{\bigcap} h(\Gamma) \leq \dot{\bigcup} h(\Delta)$. By Lemma 21 from $\dot{\neg} h(\phi) \mathbin{\dot{\cap}} \dot{\bigcap} h(\Gamma) \leq \dot{\bigcup} h(\Delta)$ and $h(\chi) \mathbin{\dot{\cap}} \dot{\bigcap} h(\Gamma) \leq \dot{\bigcup} h(\Delta)$ we have $(\dot{\neg} h(\phi) \mathbin{\dot{\cup}} h(\chi)) \mathbin{\dot{\cap}} \dot{\bigcap} h(\Gamma) \leq \dot{\bigcup} h(\Delta)$, then $h(\neg\phi \vee \chi) \mathbin{\dot{\cap}} \dot{\bigcap} h(\Gamma) \leq \dot{\bigcup} h(\Delta)$ which then through Definition 51 gives us $h(\neg\phi \rightarrow \chi) \mathbin{\dot{\cap}} \dot{\bigcap} h(\Gamma) \leq \dot{\bigcup} h(\Delta)$. $\square$

Lemma 24. *$R_{\rightarrow}$ is correct in* WT.

Proof. Let $\mathcal{M} = \langle \mathcal{A}, F \rangle$ be an arbitrary, but fixed TB-model and h be a homomorphism of $\mathcal{L}$ into $\mathcal{A}$. Suppose that the following condition holds: $\phi, \Gamma \Rightarrow \Delta, \chi$ is satisfied in $\mathcal{M}$ under h. That means that $h(\phi) \mathbin{\dot{\cap}} \dot{\bigcap} h(\Gamma) \leq \dot{\bigcup} h(\Delta) \mathbin{\dot{\cup}} h(\chi)$. We will show that $\Gamma \Rightarrow \Delta, \phi \rightarrow \chi$ is also satisfied in $\mathcal{M}$ under h, that is $\dot{\bigcap} h(\Gamma) \leq \dot{\bigcup} h(\Delta) \mathbin{\dot{\cup}} h(\phi \rightarrow \chi)$. We apply Lemma 18 to $h(\phi) \mathbin{\dot{\cap}} \dot{\bigcap} h(\Gamma) \leq \dot{\bigcup} h(\Delta) \mathbin{\dot{\cup}} h(\chi)$ thus obtaining $\dot{\bigcap} h(\Gamma) \leq \dot{\bigcup} h(\Delta) \mathbin{\dot{\cup}} h(\chi) \mathbin{\dot{\cup}} \dot{\neg} h(\phi)$. By Lemma 22 we have $\dot{\bigcap} h(\Gamma) \leq \dot{\bigcup} h(\Delta) \mathbin{\dot{\cup}} h(\neg\phi \vee \chi)$ which then through Definition 51 gives us $\dot{\bigcap} h(\Gamma) \leq \dot{\bigcup} h(\Delta) \mathbin{\dot{\cup}} h(\phi \rightarrow \chi)$. $\square$

Lemma 25. *$L_{\leftrightarrow}$ is correct in* WT.

Proof. Let $\mathcal{M} = \langle \mathcal{A}, F \rangle$ be an arbitrary, but fixed TB-model and h be a homomorphism of $\mathcal{L}$ into $\mathcal{A}$. Suppose that the following condition holds: $\phi, \chi, \Gamma \Rightarrow \Delta$ and $\Gamma \Rightarrow \Delta, \phi, \chi$ are satisfied in $\mathcal{M}$ under h. Through Lemmas 19 and 23 and condition (6) from Definition 17 we conclude that $\phi \leftrightarrow \chi, \Gamma \Rightarrow \Delta$ is also satisfied in $\mathcal{M}$ under h. $\square$

Lemma 26. *$R_{\leftrightarrow}$ is correct in* WT.

Proof. Let $\mathcal{M} = \langle \mathcal{A}, F \rangle$ be an arbitrary, but fixed TB-model and h be a homomorphism of $\mathcal{L}$ into $\mathcal{A}$. Suppose that the following condition holds: $\phi, \Gamma \Rightarrow \Delta, \chi$ and $\chi, \Gamma \Rightarrow \Delta, \phi$ are satisfied in $\mathcal{M}$ under h. Through Lemmas 20 and 24 and condition (6) from Definition 17 we conclude that $\phi \leftrightarrow \chi, \Gamma \Rightarrow \Delta$ is also satisfied in $\mathcal{M}$ under h. $\square$

Lemma 27. *$L^1_{\equiv}$ is correct in* WT.

Proof. Let $\mathcal{M} = \langle \mathcal{A}, F \rangle$ be an arbitrary, but fixed TB-model and h be a homomorphism of $\mathcal{L}$ into $\mathcal{A}$. Suppose that the following condition holds: $\phi \equiv \phi, \Gamma \Rightarrow \Delta$ is satisfied in $\mathcal{M}$ under h, that is $h(\phi \equiv \phi) \mathbin{\dot{\cap}} \dot{\bigcap} h(\Gamma) \leq \dot{\bigcup} h(\Delta)$. From Definition 51 we have $(h(\phi) \doteq h(\phi)) \mathbin{\dot{\cap}} \dot{\bigcap} h(\Gamma) \leq \dot{\bigcup} h(\Delta)$. By condition 1 from Definition 84 we have $h(\phi) \doteq h(\phi)$ being equal to 1, therefore we conclude $1 \mathbin{\dot{\cap}} \dot{\bigcap} h(\Gamma) \leq \dot{\bigcup} h(\Delta)$ which by $a \mathbin{\dot{\cap}} 1 = a$ entails $\dot{\bigcap} h(\Gamma) \leq \dot{\bigcup} h(\Delta)$. $\square$

Lemma 28. *$L^2_\equiv$ is correct in* WT.

Proof. Let $\mathcal{M} = \langle \mathcal{A}, F \rangle$ be an arbitrary, but fixed TB-model and h be a homomorphism of $\mathcal{L}$ into $\mathcal{A}$. Suppose that the following condition does not hold: $\phi \equiv \chi, \Gamma \Rightarrow \Delta$ is not satisfied in $\mathcal{M}$ under h, that is $h(\phi \equiv \chi) \dot\cap \dot\bigcap h(\Gamma) \leq \dot\bigcup h(\Delta)$ does not hold. We know that $(\phi \equiv \chi) \to (\neg\phi \equiv \neg\chi)$ is a tautology of WT. We can therefore add $\neg\phi \equiv \neg\chi$ to the antecedent of a sequent knowing the algebraic value of the antecedent will not change. This means that $h(\neg\phi \equiv \neg\chi) \dot\cap h(\phi \equiv \chi) \dot\cap \dot\bigcap h(\Gamma) \not\leq \dot\bigcup h(\Delta)$, therefore $\neg\phi \equiv \neg\chi, \phi \equiv \chi, \Gamma \Rightarrow \Delta$ is not satisfied in $\mathcal{M}$ under h. □

Lemma 29. *$L^3_\equiv$ is correct in* WT.

Proof. Let $\mathcal{M} = \langle \mathcal{A}, F \rangle$ be an arbitrary, but fixed TB-model and h be a homomorphism of $\mathcal{L}$ into $\mathcal{A}$. Suppose that the following condition does not hold: $\phi \equiv \chi, \Gamma \Rightarrow \Delta$ is not satisfied in $\mathcal{M}$ under h, that is $h(\phi \equiv \chi) \dot\cap \dot\bigcap h(\Gamma) \leq \dot\bigcup h(\Delta)$ does not hold. We know that $(\phi \equiv \chi) \to (\phi \leftrightarrow \chi)$ is a tautology of WT. We can therefore add $\phi \leftrightarrow \chi$ to the antecedent of a sequent knowing the algebraic value of the antecedent will not change. This means that $h(\phi \leftrightarrow \chi) \dot\cap h(\phi \equiv \chi) \dot\cap \dot\bigcap h(\Gamma) \not\leq \dot\bigcup h(\Delta)$, therefore $\phi \leftrightarrow \chi, \phi \equiv \chi, \Gamma \Rightarrow \Delta$ is not satisfied in $\mathcal{M}$ under h. □

Lemma 30. *$L^4_\equiv$ is correct in* WT.

Proof. Let $\mathcal{M} = \langle \mathcal{A}, F \rangle$ be an arbitrary, but fixed TB-model and h be a homomorphism of $\mathcal{L}$ into $\mathcal{A}$. Suppose that the following condition does not hold: $\phi \equiv \psi, \chi \equiv \omega, \Gamma \Rightarrow \Delta$ is not satisfied in $\mathcal{M}$ under h, that is $h(\phi \equiv \psi) \dot\cap h(\chi \equiv \omega) \dot\cap \dot\bigcap h(\Gamma) \leq \dot\bigcup h(\Delta)$ does not hold. We know that $((\phi \equiv \psi) \wedge (\chi \equiv \omega)) \to ((\phi \otimes \chi) \equiv (\psi \otimes \omega))$ is a tautology of WT. We can therefore add $(\phi \otimes \chi) \equiv (\psi \otimes \omega)$ to the antecedent of a sequent knowing the algebraic value of the antecedent will not change. This means that $h((\phi \otimes \chi) \equiv (\psi \otimes \omega)) \dot\cap h(\phi \equiv \psi) \dot\cap h(\chi \equiv \omega) \dot\cap \dot\bigcap h(\Gamma) \not\leq \dot\bigcup h(\Delta)$, therefore sequent $(\phi \otimes \chi) \equiv (\psi \otimes \omega), \phi \equiv \psi, \chi \equiv \omega, \Gamma \Rightarrow \Delta$ is not satisfied in $\mathcal{M}$ under h. □

In the case of WT we can show that $R^T_\equiv$ preserves a weaker property than we examined for $R^B_\equiv$ (which, in turn, provides a stronger rule property), the satisfiability of a sequent.

Lemma 31. *$R^T_\equiv$ is correct in* WT.

Proof. Let $\mathcal{M} = \langle \mathcal{A}, F \rangle$ be an arbitrary, but fixed TB-model and h be a homomorphism of $\mathcal{L}$ into $\mathcal{A}$. Suppose that the following condition holds: $\Gamma^\equiv \Rightarrow \phi \leftrightarrow \chi$ is satisfied in $\mathcal{M}$ under h. Now, let $\Gamma^\equiv = \{\phi_1 \equiv \chi_1, ..., \phi_n \equiv \chi_n\}$ (n might be 0). We know that the following objects are identical:

(a) $h(\phi_i \equiv \chi_i) = (h(\phi_i) \doteq h(\chi_i)) \doteq 1$ (per condition 3 from Theorem 28)

(b) $h((\phi_1 \equiv \chi_1) \wedge ... \wedge (\phi_n \equiv \chi_n)) \doteq 1 =$

$$((h(\phi_1) \doteq h(\chi_1)) \dot\cap ... \dot\cap (h(\phi_n) \doteq h(\chi_n))) \doteq 1$$

By assumption, $h((\phi_1 \equiv \chi_1) \wedge ... \wedge (\phi_n \equiv \chi_n)) \leq h(\phi \leftrightarrow \chi)$. By condition 2 from Theorem 28,

$$h((\phi_1 \equiv \chi_1) \wedge ... \wedge (\phi_n \equiv \chi_n)) \doteqdot 1 \leq h(\phi \leftrightarrow \chi) \doteqdot 1.$$

By (b)

$$((h(\phi_1) \doteqdot h(\chi_1)) \dot{\cap} ... \dot{\cap} (h(\phi_n) \doteqdot h(\chi_n))) \doteqdot 1 \leq h(\phi \leftrightarrow \chi) \doteqdot 1.$$

By the definition of $\doteqdot$ in terms of I and clause (I_1) of Definition 83:

$$((h(\phi_1) \doteqdot h(\chi_1)) \doteqdot 1) \dot{\cap} ... \dot{\cap} ((h(\phi_n) \doteqdot h(\chi_n)) \doteqdot 1) \leq h(\phi \leftrightarrow \chi) \doteqdot 1$$

Then by (a),

$$h(\phi_1 \equiv \chi_1) \dot{\cap} ... \dot{\cap} h(\phi_n \equiv \chi_n) \leq h(\phi \leftrightarrow \chi) \doteqdot 1.$$

The left side is just

$$h((\phi_1 \equiv \chi_1) \wedge ... \wedge (\phi_n \equiv \chi_n))$$

whereas $h(\phi \leftrightarrow \chi) \doteqdot 1$ is $(h(\phi) \dot{\leftrightarrow} h(\chi)) \doteqdot 1$ and by point 4 of Theorem 28 it is equal to $h(\phi) \doteqdot h(\chi)$. Hence finally

$$h((\phi_1 \equiv \chi_1) \wedge ... \wedge (\phi_n \equiv \chi_n)) \leq h(\phi \equiv \chi).$$

Therefore the conclusion is also satisfied in $\mathcal{M}$ under h. □

Lemma 32. *$R^T_{\equiv}$ is invertible in* WT.

Proof. Through clause 2 of Definition 84 we notice that the satisfiability of a conclusion entails the satisfiability of the premiss of the rule. □

Theorem 33 (Soundness)**.** *If a sequent is provable in* $\mathsf{G3_{WT}}$*, it is valid in* WT.

Proof. The proof is analogous to that of Theorem 13 in Chapter 3. □

As a result from Theorems 32 and 33 we have

Theorem 34 (Adequacy)**.** *Sequent* $\Rightarrow \phi$ *is provable in* $\mathsf{G3_{WT}}$ *iff* $\Rightarrow \phi$ *is valid in* WT.

Chapter 6

WH logic and sequent calculus $\mathsf{G3}_{\mathsf{WH}}$

WH, the third extension of SCI discussed by Suszko, is a formalization of the following proposition from Wittgenstein's Tractatus:

> 5.5303 Roughly speaking: to say of **two** things that they are identical is nonsense, and to say of **one** thing that it is identical with itself is to say nothing.

which is often quoted to emphasize aversion towards the sign of identity. Suszko formalized this proposition through WH, which is a set of sentences interpreted within Henle algebra. In WH we can state that a given situation is either necessary or impossible. This notion brings us yet again closer to modal logic. WH corresponds to modal logic S5, where $\Box$ can be interpreted as interior operator "I".

6.1 Hilbert system for WH

We utilize the same language as in the previous extensions, that is an algebra of the similarity type $\langle 1, 2, 2, 2, 2, 2\rangle$

$$\mathcal{L} = \langle L, \neg, \wedge, \vee, \rightarrow, \leftrightarrow, \equiv\rangle$$

in which, yet again, we define $\bot =_{df} p_1 \wedge \neg p_1$, $\top =_{df} \neg\bot$, $(\phi \leq \chi) =_{\mathrm{df}} \big((\phi \rightarrow \chi) \equiv \top\big)$ and $\Box\phi =_{df} \phi \equiv \top$. Suszko and Omyła utilize different language, that is algebra of the similarity type $\langle 1, 2, 2, 2, 2, 2, 1\rangle$

$$\mathcal{L}^{\Box} = \langle L, \neg, \wedge, \vee, \rightarrow, \leftrightarrow, \equiv, \Box\rangle$$

which allows us to simplify the shape of formulae. Identity, as was the case for WT, can be thought of as a version of necessity. We will, however, remain with the formerly defined language for continuity and cohesion of all three extensions. Depending on the language we choose to work with, an axiomatic system for WH can be obtained in three different ways. If we had $\Box$ in our language, an axiomatic system for WH could be obtained by means of the addition of formulae falling under $(\equiv^*_{15})$ to the axiomatic system of WT:

$(\equiv^*_{15})$ $(\Box\phi \equiv \top) \vee (\Box\phi \equiv \bot)$

or, for the language algebra $\mathcal{L}$ which we will utilize, it is obtained by the addition of the following axiom scheme to the axiomatic system of WT:

$(\equiv_{15})$ $((\phi \equiv \chi) \equiv \top) \vee ((\phi \equiv \chi) \equiv \bot)$

which can be described as the non-Fregean law of excluded middle. For any equation $\phi \equiv \chi$ we can say that it is either impossible or necessary. The formula $(\equiv_{15})$ was also introduced by Greniewski in [20] where he axiomatized S5 through adding $(\equiv_{15})$ to CPC with identity connective (in the original article introduced through symbol $\dot{\equiv}$) together with the following quasi-Fregean rule:

$$\frac{\phi \leftrightarrow \chi}{\phi \equiv \chi}$$

Formula $(\equiv_{15})$, if we were to be faithful to the original notation, would be written as $((p \;\dot{\equiv}\; q) \;\dot{\equiv}\; r^0) \vee ((p \;\dot{\equiv}\; q) \;\dot{\equiv}\; \bar{r^0})$. Greniewski referred to the mentioned identity connective as another variant of equivalence, which is separate from material equivalence. Additionally Greniewski considered a third option, the so-called *Hamilton's equivalence* $\ddot{\equiv}$, which can be defined as $\phi \;\ddot{\equiv}\; \chi =_{df} (\phi \equiv \chi) \wedge \bar{\phi^2}$ (where $\bar{\phi^2}$ can be read as "logic does not tell us whether ϕ occurs"[20]). A similar approach was undertaken by Cresswell [8]. He formalized Calculus of Functions of Propositions (FC) with non-truth-functional variables. In the variant of the logic corresponding to modal logic S5, identity is characterized by the quasi-Fregean rule, axiom $((\phi \equiv \chi) \to \bot) \to ((\phi \equiv \chi) \equiv \bot)$, reflexivity axiom and standard replacement rule.

The third axiomatization for WH comes from [56]. It can be formulated as Boole algebra axioms and the following set:

$(\equiv^{\#}_{15})$ $\top \equiv (\phi \vee \neg\phi)$,

$(\equiv^{\#}_{16})$ $\bot \equiv (\phi \wedge \neg\phi)$,

$(\equiv^{\#}_{17})$ $(\phi \equiv \chi) \equiv ((\phi \equiv \chi) \equiv \top)$,

$(\equiv^{\#}_{18})$ $\neg(\phi \equiv \chi) \equiv ((\phi \equiv \chi) \equiv \bot)$.

Axiom $(\equiv_{15})$ can be obtained from the axioms $(\equiv^{\#}_{17})$ and $(\equiv^{\#}_{18})$ from the third axiomatization variant for WH [56]. In further considerations we will refer to set WHA, which will consist of all instances of axiom scheme $(\equiv_{15})$.

We define formal proof in axiomatic system in the standard way, following the second variant of axiomatization:

Definition 95 (Derivation, formal proof)**.** *Let Φ stand for a set of formulae of $\mathcal{L}$. A finite sequence $\phi_1, ..., \phi_n$ of formulae of $\mathcal{L}$ is a derivation of ϕ from Φ provided $\phi_n = \phi$ and formula $\phi_i, i \leq n$, is either from Φ or has been derived from some ϕ_{i_1}, ϕ_{i_2}, $(i_1, i_2 < i)$ through an application of modus ponens. If $\Phi = \mathsf{TFA} \cup \mathsf{IDA} \cup \mathsf{WBA} \cup \mathsf{WTA} \cup \mathsf{WHA}$, then $\phi_1, ..., \phi_n$ is a proof of ϕ in the axiomatic system for* WH.

If $\phi_1, ..., \phi_n$ is a proof of ϕ from $\mathsf{TFA} \cup \mathsf{IDA} \cup \mathsf{WBA} \cup \mathsf{WTA} \cup \mathsf{WHA}$, then the length of this proof is n. We undertake the same approach (as it was in the previous chapters) to the consequence operation. This time we extend axiom set by the addition of the set WHA and, in connection with this, consequence operation C_{WH} is defined by the set $\mathsf{TFA} \cup \mathsf{IDA} \cup \mathsf{WBA} \cup \mathsf{WTA} \cup \mathsf{WHA}$ of axioms and a singular inference rule, *modus ponens*. Elements of $C_{\mathsf{WH}}(\emptyset)$ are called logical theorems of WH.

As previously, in keeping with Suszko we introduce the logic WH as a deductive system H_{WH}.

Definition 96. *Pair $H_{\mathsf{WH}} = \langle \mathcal{L}, C_{\mathsf{WH}} \rangle$ is a deductive system H_{WH}.*

6.2 Semantics of WH

In the previous section we presented three approaches to defining axiomatic system for WH. We can add four axioms to the axiomatic system of WB or one axiom to the axiomatic system for WT ($(\equiv^*_{15})$ or $(\equiv_{15})$, depending on language). Following that, we present two equivalent approaches to defining Henle algebra.

Definition 97. *Algebra*

$$\mathcal{A} = \langle A, \dot{\neg}, \dot{\cup}, \dot{\cap}, \dot{\rightarrow}, \dot{\leftrightarrow}, \dot{\equiv} \rangle$$

is called a Henle algebra or H*-algebra if and only if $\langle A, \dot{\neg}, \dot{\cup}, \dot{\cap}, \dot{\rightarrow}, \dot{\leftrightarrow} \rangle$ is a Boole algebra and for any $a, b \in A$ operation $\dot{\equiv}$ satisfies the following:*

$$a \dot{\equiv} b = \begin{cases} 1, & \text{if } a = b \\ 0, & \text{if } a \neq b \end{cases} \tag{6.1}$$

Moreover, every H-algebra is also a well-connected TB-algebra [38, p. 52].

Omyła in [38, p. 110] notes that in B-models equations which are theorems of WB do not have to be assigned to 1 in the algebra, but to other elements of the filter. In models of WH logic, every equation satisfied in Henle model is assigned 1, whereas equations not satisfied in a model are assigned 0.

Definition 98. *A given* TB*-algebra $\mathcal{A} = \langle A, \dot{\neg}, \dot{\cap}, \dot{\cup}, \dot{\rightarrow}, \dot{\leftrightarrow}, \dot{\equiv} \rangle$ is a Henle algebra provided for any $a, b \in A$ the following condition is satisfied: $\dot{\neg}(a \dot{\equiv} b) = ((a \dot{\equiv} b) \dot{\equiv} 0)$.*

Definition 99 (Henle model)**.** *By Henle model we understand any* SCI*-model $\mathcal{M} = \langle \mathcal{A}, F \rangle$ such that $\mathcal{A}$ is a Henle algebra [38, p. 114].*

As modal logic S_5 corresponds to WH we can translate formulae between them using the following definitions:

- $\Box \phi =_{df} \phi \equiv (\phi \vee \neg \phi)$
- $\phi \equiv \chi =_{df} \Box(\phi \leftrightarrow \chi)$

Definition 100 (Satisfiability of a formula in a model). *Let $\mathcal{M} = \langle \mathcal{A}, F \rangle$ be an arbitrary but fixed Henle model, h be an arbitrary homomorphism from $\mathcal{L}$ to $\mathcal{A}$. Formula ϕ is satisfied in $\mathcal{M}$ under h if and only if $h(\phi) \in F$.*

Definition 101 (Truth of a formula in a model). *Let $\mathcal{M} = \langle \mathcal{A}, F \rangle$ be an arbitrary but fixed Henle model. Formula ϕ is true in $\mathcal{M}$ if and only if: for all $h \in Hom(\mathcal{L}, \mathcal{A})$ ϕ is satisfied in $\mathcal{M}$ under h.*

Definition 102 (Validity of a formula). *Formula ϕ is valid in WH iff ϕ is true in all Henle models.*

Corollary 3. $((\phi \equiv \chi) \equiv \top) \vee ((\phi \equiv \chi) \equiv \bot)$ *is valid in* WH.

Proof. Suppose it is not the case. We then have one Henle model $\mathcal{M} = \langle \mathcal{A}, F \rangle$ and homomorphism h such that $h(((\phi \equiv \chi) \equiv \top) \vee ((\phi \equiv \chi) \equiv \bot)) \notin F$, which in turn means that, consecutively $h((\phi \equiv \chi) \equiv \top)) \notin F$ and $h((\phi \equiv \chi) \equiv \bot) \notin F$, and further $((h(\phi) \dot{\equiv} h(\chi)) \dot{\equiv} 1)) \notin F$ and $((h(\phi) \dot{\equiv} v(\chi)) \dot{\equiv} 0) \notin F$. We also know that since $\mathcal{M}$ is a Henle model, for any two elements $a, b \in A$ we have $(a \dot{\equiv} b) = ((a \dot{\equiv} b) \dot{\equiv} 1)$ and $\dot{\neg}(a \dot{\equiv} b) = ((a \dot{\equiv} b) \dot{\equiv} 0)$. But, since $((h(\phi) \dot{\equiv} h(\chi)) \dot{\equiv} 0) \notin F$, then so is $\dot{\neg}(h(\phi) \dot{\equiv} h(\chi)) \notin F$. We arrive at $h(\phi) \dot{\equiv} v(\chi) \notin F$ and $\dot{\neg}(h(\phi) \dot{\equiv} v(\chi)) \notin F$. But, condition for semantical negation shows us that $\dot{\neg} a \in F$ iff $a \in F$. A contradiction. $\square$

Theorem 35. WH *is the set of all and only formulae true in every Henle model.*

The proof can be found in [38, p. 114].

6.3 Sequent Calculus G3$_{\mathsf{WH}}$

Sequent Calculus for G3$_{\mathsf{WH}}$ is obtained through the continuation of extending rule set of previously presented systems. WH can be obtained through an addition of axioms to either WB or WT. For continuity, as G3$_{\mathsf{WT}}$ has been obtained through modification of G3$_{\mathsf{WB}}$, we will expand rule set of G3$_{\mathsf{WT}}$ to obtain G3$_{\mathsf{WH}}$.

We here pivot to the way ℓG3$_{\mathsf{SCI}}$ was obtained and we yet again utilize Negri's strategy of turning axioms into sequent calculus rules. This way we obtain the following rule from axiom $(\equiv_{15})$:

$$\frac{\Gamma, (\phi \equiv \chi) \equiv \top \Rightarrow \Delta \quad \Gamma, (\phi \equiv \chi) \equiv \bot \Rightarrow \Delta}{\Gamma \Rightarrow \Delta} L^5_{\equiv}$$

which is added to the set of rules characterizing G3$_{\mathsf{WT}}$ and will constitute G3$_{\mathsf{WH}}$. In $L^5_{\equiv}$ we have two active formulae and no principal formula. In a way its use is similar to *cut*; (looking bottom-up) we introduce two possible scenarios in our derivation—a given equation $\phi \equiv \chi$ can be either necessary or impossible. However, in contrast to *cut*, $L^5_{\equiv}$ is a shared-context rule. Naturally, if we were to turn axiom $(\equiv^*_{15})$ into sequent rule using the same strategy, we would meet the analogous outcome.

TABLE 6.1: $\mathsf{G3_{WH}}$: identity rules

$$\dfrac{\phi \equiv \phi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ L^1_{\equiv} \qquad \dfrac{\chi \leftrightarrow \phi, \chi \equiv \phi, \Gamma \Rightarrow \Delta}{\chi \equiv \phi, \Gamma \Rightarrow \Delta}\ L^3_{\equiv}$$

$$\dfrac{\neg\phi \equiv \neg\chi, \phi \equiv \chi, \Gamma \Rightarrow \Delta}{\phi \equiv \chi, \Gamma \Rightarrow \Delta}\ L^2_{\equiv} \qquad \dfrac{\Gamma^{\equiv} \Rightarrow \phi \leftrightarrow \chi}{\Gamma^{\equiv} \Rightarrow \phi \equiv \chi}\ R^T_{\equiv}$$

$$\dfrac{(\phi \otimes \chi) \equiv (\psi \otimes \omega), \phi \equiv \psi, \chi \equiv \omega, \Gamma \Rightarrow \Delta}{\phi \equiv \psi, \chi \equiv \omega, \Gamma \Rightarrow \Delta}\ L^4_{\equiv}$$

$$\dfrac{(\phi \equiv \chi) \equiv \top, \Gamma \Rightarrow \Delta \quad (\phi \equiv \chi) \equiv \bot, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ L^5_{\equiv}$$

where $\Gamma^{\equiv}$ consists of equations only.

For the full sequent system we consider the standard set of classical and structural rules (Tables 5.1 and 5.2) and the following set of identity-dedicated rules:

As a result $\mathsf{G3_{WB}}$ will be defined by the following set of rules $\{L_\wedge$, $R_\wedge$, $L_\vee$, $R_\vee$, $L_\rightarrow$, $R_\rightarrow$, $L_\leftrightarrow$, $R_\leftrightarrow$, $L_\neg$, $R_\neg$, $L^1_\equiv$, $L^2_\equiv$, $L^3_\equiv$, $L^4_\equiv$, $L^5_\equiv$, $R^T_\equiv$, *cut*$\}$.

Definition 103 (Derivation of $\Gamma \Rightarrow \Delta$ in $\mathsf{G3_{WH}}$)**.** *Derivation of $\Gamma \Rightarrow \Delta$ in $\mathsf{G3_{WH}}$ is a labelled finite tree with a single root and each node-label connected with the labels of the (immediate) successor nodes (if any) according to one of the rules.*

Definition 104 (Proof of $\Gamma \Rightarrow \Delta$ in $\mathsf{G3_{WH}}$)**.** *A proof of $\Gamma \Rightarrow \Delta$ in $\mathsf{G3_{WH}}$ is a derivation of $\Gamma \Rightarrow \Delta$ with axioms at all of the top nodes.*

We shall now examine several examples of derivations in $\mathsf{G3_{WH}}$. $(\phi \equiv \chi) \equiv ((\phi \equiv \chi) \equiv \top)$ is an example of WT theorem, therefore we do not need rule $L^5_\equiv$ to prove it in $\mathsf{G3_{WH}}$. Similarly to previously examined systems, we shall utilize additional, derivable axioms with $\top$ and $\bot$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\phi \equiv \chi, \phi \equiv \chi \Rightarrow \top \quad \top, \phi \equiv \chi \Rightarrow \phi \equiv \chi}{\phi \equiv \chi \Rightarrow ((\phi \equiv \chi) \leftrightarrow \top)}\ R_\leftrightarrow}{\phi \equiv \chi \Rightarrow ((\phi \equiv \chi) \equiv \top)}\ R^T_\equiv \qquad \begin{matrix} D_1 \\ \vdots \end{matrix}}{\Rightarrow (\phi \equiv \chi) \leftrightarrow ((\phi \equiv \chi) \equiv \top)}\ R_\leftrightarrow}{\Rightarrow (\phi \equiv \chi) \equiv ((\phi \equiv \chi) \equiv \top)}\ R^T_\equiv$$

where D_1 is the following:

$$\cfrac{\cfrac{\phi\equiv\chi,\top,(\phi\equiv\chi)\equiv\top\Rightarrow\phi\equiv\chi \quad (\phi\equiv\chi)\equiv\top\Rightarrow\phi\equiv\chi,\phi\equiv\chi,\top}{(\phi\equiv\chi)\leftrightarrow\top,(\phi\equiv\chi)\equiv\top\Rightarrow\phi\equiv\chi}\,L_{\leftrightarrow}}{(\phi\equiv\chi)\equiv\top\Rightarrow\phi\equiv\chi}\,L^3_{\equiv}$$

However, the rule is necessary to prove $\neg(\phi\equiv\chi)\equiv((\phi\equiv\chi)\equiv\bot)$.

$$\cfrac{\cfrac{\cfrac{\begin{matrix}D_1\\ \vdots\end{matrix}\quad \neg(\phi\equiv\chi),(\phi\equiv\chi)\equiv\bot\Rightarrow(\phi\equiv\chi)\equiv\bot}{\neg(\phi\equiv\chi)\Rightarrow((\phi\equiv\chi)\equiv\bot)}\,L^5_{\equiv} \quad \begin{matrix}D_2\\ \vdots\end{matrix}}{\Rightarrow\neg(\phi\equiv\chi)\leftrightarrow((\phi\equiv\chi)\equiv\bot)}\,R_{\leftrightarrow}}{\Rightarrow\neg(\phi\equiv\chi)\equiv((\phi\equiv\chi)\equiv\bot)}\,R^T_{\equiv}$$

where D_1 is the following derivation:

$$\cfrac{\cfrac{\cfrac{\phi\equiv\chi,\top,(\phi\equiv\chi)\equiv\top\Rightarrow\phi\equiv\chi,(\phi\equiv\chi)\equiv\bot \quad S_1}{(\phi\equiv\chi)\leftrightarrow\top,(\phi\equiv\chi)\equiv\top\Rightarrow(\phi\equiv\chi)\equiv\bot,\phi\equiv\chi}\,L_{\leftrightarrow}}{(\phi\equiv\chi)\equiv\top\Rightarrow(\phi\equiv\chi)\equiv\bot,\phi\equiv\chi}\,L^3_{\equiv}}{\neg(\phi\equiv\chi),(\phi\equiv\chi)\equiv\top\Rightarrow(\phi\equiv\chi)\equiv\bot}\,L_{\neg}$$

where S_1 stands for sequent $(\phi\equiv\chi)\equiv\top\Rightarrow\top,(\phi\equiv\chi)\equiv\top,\phi\equiv\chi$ and D_2 is the following derivation:

$$\cfrac{\cfrac{\cfrac{\phi\equiv\chi,\phi\equiv\chi,\bot,(\phi\equiv\chi)\equiv\bot\Rightarrow \quad \phi\equiv\chi,(\phi\equiv\chi)\equiv\bot\Rightarrow\phi\equiv\chi,\bot}{(\phi\equiv\chi)\leftrightarrow\bot,\phi\equiv\chi,(\phi\equiv\chi)\equiv\bot\Rightarrow}\,L_{\leftrightarrow}}{(\phi\equiv\chi)\equiv\bot,\phi\equiv\chi\Rightarrow}\,L^3_{\equiv}}{(\phi\equiv\chi)\equiv\bot\Rightarrow\neg(\phi\equiv\chi)}\,R_{\neg}$$

We follow with definitions analogous to those for $\mathsf{G3_{WT}}$.

Definition 105 (Satisfiability of a sequent)**.** *Let $\mathcal{M}=\langle\mathcal{A},F\rangle$ be an arbitrary but fixed Henle model and let $h\in Hom(\mathcal{L},\mathcal{A})$. Sequent $\Gamma\Rightarrow\Delta$ is satisfied in $\mathcal{M}$ under h provided $h(\phi_1)\dot\cap...\dot\cap h(\phi_n)\leq h(\chi_1)\dot\cup...\dot\cup h(\chi_k)$, where $\Gamma=\phi_1,...,\phi_n$ and $\Delta=\chi_1,...,\chi_k$.*

Definition 106 (Truth of a sequent)**.** *Let $\mathcal{M}=\langle\mathcal{A},F\rangle$ be an arbitrary Henle model. Sequent $\Gamma\Rightarrow\Delta$ is true in $\mathcal{M}$ provided that for each $h\in Hom(\mathcal{L},\mathcal{A})$, sequent $\Gamma\Rightarrow\Delta$ is satisfied in $\mathcal{M}$ under h.*

Definition 107 (Validity of a sequent)**.** *Sequent $\Gamma\Rightarrow\Delta$ is valid in* WH*, if it is true in each Henle model.*

6.3.1 Completeness of $\mathsf{G3_{WH}}$

Theorem 36 (Interpretation of H_{WH} within $\mathsf{G3_{WH}}$)**.** *If formula ϕ is provable in deductive system for H_{WH}, then sequent $\Rightarrow\phi$ is provable in $\mathsf{G3_{WH}}$.*

Proof. *Modus ponens* can be obtained as in previous systems, through *cut*. Naturally, axiom ($\equiv_{15}$) can be easily proved by:

$$\dfrac{\dfrac{(\phi \equiv \chi) \equiv \top \Rightarrow (\phi \equiv \chi) \equiv \top, (\phi \equiv \chi) \equiv \bot \quad S_1}{\Rightarrow (\phi \equiv \chi) \equiv \top, (\phi \equiv \chi) \equiv \bot}\, L^5_{\equiv}}{\Rightarrow ((\phi \equiv \chi) \equiv \top) \vee ((\phi \equiv \chi) \equiv \bot)}\, R_{\vee}$$

where S_1 is sequent $(\phi \equiv \chi) \equiv \bot \Rightarrow (\phi \equiv \chi) \equiv \top, (\phi \equiv \chi) \equiv \bot$. Similarly ($\equiv^*_{15}$) can be proven in similar manner (with $\Box\phi =_{df} \phi \equiv \top$).

The rest of the reasoning is analogous to those in previous sections. □

As a result, $\mathsf{G3_{WH}}$ is complete.

Theorem 37 (Completeness)**.** *If a sequent $\Rightarrow \phi$ is valid in* WH*, it is provable in* $\mathsf{G3_{WH}}$*.*

Proof. Analogous to the one of Theorem 12 presented in Chapter 3. □

6.3.2 Soundness of $\mathsf{G3_{WH}}$

In the case of correctness and invertibility of rules of $\mathsf{G3_{WH}}$ we refer back to the Definitions 62 and 63.

Lemma 33. *Rule $L^5_{\equiv}$ is correct in* WH*.*

Proof. Suppose it is not the case, that is, we have the following conditions:

(1) $\Gamma, (\phi \equiv \chi) \equiv \top \Rightarrow \Delta$ is satisfied in $\mathcal{M}$ under h and

(2) $\Gamma, (\phi \equiv \chi) \equiv \bot \Rightarrow \Delta$ is satisfied in $\mathcal{M}$ under h.

From (1) and (2) we have, respectively, $\dot{\bigcap} h(\Gamma) \mathbin{\dot\cap} h((\phi \equiv \chi) \equiv \top) \leq \dot{\bigcup} h(\Delta)$ and $\dot{\bigcap} h(\Gamma) \mathbin{\dot\cap} h((\phi \equiv \chi) \equiv \bot) \leq \dot{\bigcup} h(\Delta)$. Then, through the properties of $\leq$, we have $\dot{\bigcap} h(\Gamma) \mathbin{\dot\cap} (h((\phi \equiv \chi) \equiv \top) \mathbin{\dot\cup} h((\phi \equiv \chi) \equiv \bot)) \leq \dot{\bigcup} h(\Delta)$. Since valid equations in Henle algebras are assigned with 1, expression $\dot{\bigcap} h(\Gamma) \mathbin{\dot\cap} (h((\phi \equiv \chi) \equiv \top) \mathbin{\dot\cup} h((\phi \equiv \chi) \equiv \bot)) \leq \dot{\bigcup} h(\Delta)$ being satisfied in $\mathcal{M}$ under h means that so is expression $\dot{\bigcap} h(\Gamma) \leq \dot{\bigcup} h(\Delta)$. □

Lemma 34. *Rule $L^5_{\equiv}$ is invertible in* WH*.*

Proof. Suppose it is not the case, that is, we have the following conditions for $\mathcal{M}$ and h:

1. $\Gamma, (\phi \equiv \chi) \equiv \top \Rightarrow \Delta$ is not satisfied in $\mathcal{M}$ under h or

2. $\Gamma, (\phi \equiv \chi) \equiv \bot \Rightarrow \Delta$ is not satisfied in $\mathcal{M}$ under h and

3. $\Gamma \Rightarrow \Delta$ is satisfied in $\mathcal{M}$ under h.

We notice that 1 and 2 are contradictory with regards to 3 as an addition of formulae $(\phi \equiv \chi) \equiv \top$ and $(\phi \equiv \chi) \equiv \bot \Rightarrow \Delta$ to the premiss satisfied in $\mathcal{M}$ under h will not alter the said property. □

Theorem 38 (Soundness)**.** *If a sequent* $\Gamma \Rightarrow \Delta$ *is provable in* $\mathsf{G3_{WH}}$*,* $\Gamma \Rightarrow \Delta$ *is valid in* WH*.*

Proof. The proof is analogous to one of Theorem 13 in Chapter 3. □

As a result from Theorems 37 and 38 we have

Theorem 39 (Adequacy)**.** *Sequent* $\Rightarrow \phi$ *is provable in* $\mathsf{G3_{WH}}$ *iff* $\Rightarrow \phi$ *is valid in* WH*.*

Chapter 7

The case of cut

In this chapter we shall examine certain issues regarding cut elimination procedure across sequent calculi for all three axiomatic extensions of SCI. The reason for treating this as a separate matter is the fact that the outcome of the following investigations is not definitive. Our goal is not to present a formal and indisputable reason for the impossibility of constructing a cut elimination proof. Our goal is rather to discuss certain limitations of the proposed sequent formalizations and propose ways in which we could potentially overcome them. We also treat this topic as a separate chapter in this work, since the problems we will encounter are consecutively inherited, which, of course, comes from our approach of building three sequent calculi as extensions of ones for weaker non-Fregean logics. Ergo, some of the properties (or lack of them) from $\mathsf{G3_{WB}}$ will partly appear in $\mathsf{G3_{WT}}$ and those from $\mathsf{G3_{WT}}$ will be visible in $\mathsf{G3_{WH}}$. We also purposely focus on the strategy of proving cut elimination that was developed by Dragalin. Of course, we may consider other strategies, but the mentioned method has been used for the sequent base $\ell\mathsf{G3_{SCI}}$. We shall show how changes we introduce to the system lead to the failure of this particular method and we point to the sources of such an outcome.

7.1 Cut elimination for $\mathsf{G3_{SCI}}$

We shall begin with a full proof of cut elimination for $\mathsf{G3_{SCI}}$, which differs from the shortened proof presented in [4] due to several changes to the set of classical and identity-dedicated rules. We shall begin with the weakening and contraction, even though these rules are not a part of the examined sequent system. The reason for this strategy relates to the cut elimination procedure, which requires the presence of these structural rules. Therefore we start with proof of the admissibility of weakening and contraction.

Definition 108 (Weight of a formula [36])**.** *The weight $w(\phi)$ of a formula ϕ of language $\mathcal{L}$ is defined inductively by*

1. $w(p_i) = 1$,
2. $w(\neg\phi) = w(\phi) + 1$,
3. $w(\phi \otimes \chi) = w(\phi) + w(\chi) + 1$, *where* $\otimes \in \{\wedge, \vee, \to, \leftrightarrow, \equiv\}$.

By $\vdash_n \Gamma \Rightarrow \Delta$ we mean there exists a derivation of $\Gamma \Rightarrow \Delta$ of height at most n, where height of a derivation is defined as follows.

Definition 109 (Height of derivation)**.** *A sequent of the form: $\phi, \Gamma \Rightarrow \Delta, \phi$ has a derivation of height 0. If a sequent ϕ has a derivation D of height n, then the following derivation:*

$$\dfrac{\begin{array}{c} D \\ \vdots \\ \phi \end{array}}{\psi}\ R$$

has height $n+1$, where R is a one-premiss rule. If sequents ψ and ϕ have derivations D', D'' of heights n and m respectively, then the following derivation:

$$\dfrac{\begin{array}{c} D' \\ \vdots \\ \phi' \end{array} \quad \begin{array}{c} D'' \\ \vdots \\ \phi'' \end{array}}{\psi}\ R'$$

has height $max(m, n) + 1$, where R' is a two-premiss rule.

Theorem 40 (Admissibility of L_{wk} and R_{wk})**.** *If $\vdash_n \Gamma \Rightarrow \Delta$, then $\vdash_n \phi, \Gamma \Rightarrow \Delta$ and $\vdash_n \Gamma \Rightarrow \Delta, \phi$.*

Proof. The proof is based on the fact that any derivation of $\Gamma \Rightarrow \Delta$ can be transformed into derivation of $\phi, \Gamma \Rightarrow \Delta$ (or $\Gamma \Rightarrow \phi, \Delta$) through addition of formula ϕ to the antecedent (succedent) of each sequent of the derivation. □

Strictly speaking, calculus $\mathsf{G3}_{\mathsf{SCI}}$ <u>does not</u> have the property of height-preserving invertibility of rules. Example:

$$\neg p \Rightarrow \neg p$$

has a proof in $\mathsf{G3}_{\mathsf{SCI}}$ of height 0, since it is an axiom of $\mathsf{G3}_{\mathsf{SCI}}$. But sequent $\Rightarrow p, \neg p$ does not have a proof in $\mathsf{G3}_{\mathsf{SCI}}$ of height 0. Hence the existence of a proof of a conclusion of some R (a rule of $\mathsf{G3}_{\mathsf{SCI}}$) which is of height n does not guarantee the existence of a proof of the premise of R that has height at most n. We therefore introduce system $a\mathsf{G3}_{\mathsf{SCI}}$ to overcome this particular issue.[1] Let $a\mathsf{G3}_{\mathsf{SCI}}$ stand for sequent calculus that differs from $\mathsf{G3}_{\mathsf{SCI}}$ only in one respect: $\phi, \Gamma \Rightarrow \Delta, \phi$ is an axiom of $a\mathsf{G3}_{\mathsf{SCI}}$ iff ϕ is a propositional variable or an equation.

Lemma 35. *If a sequent has a proof in $\mathsf{G3}_{\mathsf{SCI}}$, then it also has a proof in $a\mathsf{G3}_{\mathsf{SCI}}$.*

Proof. It suffices to show that every sequent of the form $\phi, \Gamma \Rightarrow \Delta, \phi$, where ϕ is a complex formula, but not an equation, is provable in $a\mathsf{G3}_{\mathsf{SCI}}$. The proof is by simple induction on weight of ϕ.

Base: for $w(\phi) = 1$ the claim is trivially satisfied, since $w(\phi) > 1$ by assumption.

[1] This solution comes from Dorota Leszczyńska-Jasion in [33].

Inductive part: let $w(\phi) = n + 1$. The reasoning depends on the shape of ϕ. E.g. if it is $\phi = \phi_1 \to \phi_2$, then consider:

$$\dfrac{\dfrac{\Gamma, \phi_1 \Rightarrow \Delta, \phi_2, \phi_1 \quad \phi_2, \Gamma, \phi_1 \Rightarrow \Delta, \phi_2}{\phi_1 \to \phi_2, \Gamma, \phi_1 \Rightarrow \Delta, \phi_2}\, L_\to}{\phi_1 \to \phi_2, \Gamma \Rightarrow \Delta, \phi_1 \to \phi_2}\, R_\to$$

Since $w(\phi_1), w(\phi_2) \leq n$, by inductive hypothesis the top sequents of this derivations have proofs in $a\mathsf{G3_{SCI}}$. Hence there is also a proof of $\phi_1 \to \phi_2, \Gamma \Rightarrow \Delta, \phi_1 \to \phi_2$ in $a\mathsf{G3_{SCI}}$. □

What is more, every proof of a sequent in $a\mathsf{G3_{SCI}}$ is also a proof of the same sequent in $\mathsf{G3_{SCI}}$. Every rule of $a\mathsf{G3_{SCI}}$ is also a rule of $\mathsf{G3_{SCI}}$; the same goes for every axiom of $a\mathsf{G3_{SCI}}$ being one of $\mathsf{G3_{SCI}}$. Contraction is admissible in $a\mathsf{G3_{SCI}}$ (which will be showed later on in the section). Later it will be shown that admissibility of contraction $\mathsf{G3_{SCI}}$ boils down to admissibility of contraction in $a\mathsf{G3_{SCI}}$

Below we will consider height-preserving invertibility of rules, which will be of use in the subsequent proofs for admissibility of contraction in $a\mathsf{G3_{SCI}}$. The proof is standard and can be found in [25].

Lemma 36. *For every rule R_x of $a\mathsf{G3_{SCI}}$, if the conclusion of the R_x has a proof of height n, its premisses have proofs of height $\leq n$.*

Proof. The proof for each rule is inductive. Below we will focus on rules $L_\neg$, $L_\wedge$ and $L^2_\equiv$ the other cases are analogous.

$L_\neg$ We begin with the base, where the height of the derivation of the conclusion $\neg\phi, \Gamma \Rightarrow \Delta$ equals 0, therefore the sequent is an axiom. This means that some formula $\psi \in \Gamma \cap \Delta$. We know that if $\psi \in \Gamma \cap \Delta$, then sequent $\Gamma \Rightarrow \phi, \Delta$ is also an axiom of $a\mathsf{G3_{SCI}}$. Induction hypothesis says that the lemma can be applied for any proof of height n of the conclusion of the rule. We shall show it works for the case in which the height of the derivation of the conclusion equals $n + 1$. Let us consider the proof of height $n + 1$ with the root labelled with sequent $\neg\phi, \Gamma \Rightarrow \Delta$. Two cases are considered:

(a) $\neg\phi$ is principal;

(b) $\neg\phi$ is not principal.

In the case of (a), if the proof of $\neg\phi, \Gamma \Rightarrow \Delta$ is of height $n + 1$, then we know that the proof of $\Gamma \Rightarrow \Delta, \phi$ is of height n. In (b) $\neg\phi$ is a side formula and the above sequent $\neg\phi, \Gamma \Rightarrow \Delta$ results by an application of some other rule R_x. If R_x is a one-premiss rule we have the following:

$$\dfrac{\neg\phi, \Gamma' \Rightarrow \Delta'}{\neg\phi, \Gamma \Rightarrow \Delta}\, R_x$$

As $\neg\phi, \Gamma' \Rightarrow \Delta'$ has a proof of the height n, the induction hypothesis can be used to conclude that sequent $\Gamma' \Rightarrow \phi, \Delta'$ has a proof of height at most

n as well. Through the use of rule R_x we can conclude $\Gamma \Rightarrow \phi, \Delta$ at height at most $n+1$. The reasoning is similar if R_x is a two-premise rule.

$L_\wedge$ We begin with the base, where the height of the derivation of the conclusion $\phi \wedge \chi, \Gamma \Rightarrow \Delta$ equals 0, therefore the sequent is an axiom. This means that some formula $\psi \in \Gamma \cap \Delta$. We know that if $\psi \in \Gamma \cap \Delta$, then sequent $\phi, \chi, \Gamma \Rightarrow \Delta$ is also an axiom of $a\mathsf{G3}_{\mathsf{SCI}}$. Induction hypothesis says that the lemma can be applied for any proof of height n of the conclusion of the rule. We shall show it works for the case in which the height of the derivation of the conclusion equals $n+1$. Let us consider the proof of height $n+1$ with the root labelled with sequent $\phi \wedge \chi, \Gamma \Rightarrow \Delta$. Two cases are considered:

(a) $\phi \wedge \chi$ is principal;

(b) $\phi \wedge \chi$ is not principal.

In the case of (a) if proof of $\phi \wedge \chi, \Gamma \Rightarrow \Delta$ is of height $n+1$, then we know that proof of $\phi, \chi, \Gamma \Rightarrow \Delta$ is of height n. In (b) $\phi \wedge \chi$ is a side formula and the above sequent $\phi \wedge \chi, \Gamma \Rightarrow \Delta$ results by an application of some other rule. In case of two-premiss rule we have the following:

$$\frac{\phi \wedge \chi, \Gamma' \Rightarrow \Delta' \quad \phi \wedge \chi, \Gamma'' \Rightarrow \Delta''}{\phi \wedge \chi, \Gamma \Rightarrow \Delta}\ R_x$$

As $\phi \wedge \chi, \Gamma' \Rightarrow \Delta'$ and $\phi \wedge \chi, \Gamma''$ have proofs of the height n, the induction hypothesis can be used to both of these sequents to conclude that sequents $\phi, \chi, \Gamma' \Rightarrow \Delta'$ and $\phi, \chi, \Gamma'' \Rightarrow \Delta''$ have proofs of height at most n as well. Through the use of rule R_x we can conclude $\phi, \chi, \Gamma \Rightarrow \Delta$ at height at most $n+1$.

$L^2_\equiv$ We begin with the base, where the height of the derivation of the conclusion $\phi \equiv \chi, \Gamma \Rightarrow \Delta$ equals 0, therefore the sequent is an axiom. This means that some formula $\psi \in \Gamma \cap \Delta$ or $\phi \equiv \chi \in \Delta$. We know that if $\psi \in \Gamma \cap \Delta$, then sequent $\neg\phi \equiv \neg\chi, \phi \equiv \chi, \Gamma \Rightarrow \Delta$ is also an axiom of $a\mathsf{G3}_{\mathsf{SCI}}$, the same goes for the situation in which $\phi \equiv \chi \in \Delta$. Induction hypothesis says that the lemma can be applied for any proof of height n of the conclusion of the rule. We shall show it works for the case in which the height of the derivation of the conclusion equals $n+1$. Let us consider the proof of height $n+1$ with the root labelled with sequent $\phi \equiv \chi, \Gamma \Rightarrow \Delta$. Two cases are considered:

(a) $\phi \equiv \chi$ is principal;

(b) $\phi \equiv \chi$ is not principal.

In the case of (a) if proof of $\phi \equiv \chi, \Gamma \Rightarrow \Delta$ is of height $n+1$, then we know that proof of $\neg\phi \equiv \neg\chi, \phi \equiv \chi, \Gamma \Rightarrow \Delta$ is of height n. In (b) $\phi \equiv \chi$ is a side formula and the above sequent $\phi \equiv \chi, \Gamma \Rightarrow \Delta$ results by an application of some other rule. In the case of one-premiss rule we have the following:

$$\frac{\phi \equiv \chi, \Gamma' \Rightarrow \Delta'}{\phi \equiv \chi, \Gamma \Rightarrow \Delta} R_x$$

As $\phi \equiv \chi, \Gamma' \Rightarrow \Delta'$ has a proof of the height n, the induction hypothesis can be used to conclude that sequent $\neg\phi \equiv \neg\chi, \phi \equiv \chi, \Gamma' \Rightarrow \Delta'$ has a proof of height at most n as well. Then, through the use of rule R_x we can conclude $\neg\phi \equiv \neg\chi, \phi \equiv \chi, \Gamma \Rightarrow \Delta$ at height at most $n+1$.

The remaining cases are analogous. □

In order to proceed with the proof of contraction admissibility, the following corollary needs to be addressed:

Corollary 4. *Let D be a derivation of a sequent S in a*$\mathsf{G3}_{\mathsf{SCI}}$ *such that the rules L_{ctr} and $L^4_{\equiv}$ are applied in the following manner*

$$\cfrac{\cfrac{\phi \equiv \psi, \phi \equiv \psi, (\phi \otimes \phi) \equiv (\psi \otimes \psi), \Gamma \Rightarrow \Delta}{\phi \equiv \psi, \phi \equiv \psi, \Gamma \Rightarrow \Delta} L^4_{\equiv}}{\phi \equiv \psi, \Gamma \Rightarrow \Delta} L_{ctr}$$

Then D can be transformed in the derivation of the same sequent S in a$\mathsf{G3}_{\mathsf{SCI}}$ *such that no L_{ctr} as shown above has been applied.*

Proof. The mentioned fragment of the derivation can be replaced with the following derivation, where no contraction rule has been utilized:

$$\cfrac{\cfrac{\cfrac{\cfrac{\chi, \phi \equiv \psi, \phi \equiv \psi, \phi \equiv \psi, \Gamma^* \Rightarrow \Delta}{\phi \equiv \psi, \phi \equiv \psi, \phi \equiv \psi, \Gamma^* \Rightarrow \Delta} L^4_{\equiv} \quad \phi \equiv \psi, \Gamma^* \Rightarrow \Delta, \phi \equiv \psi, \phi \equiv \psi}{(\phi \equiv \psi) \leftrightarrow (\phi \equiv \psi), \delta, \phi \equiv \psi, \Gamma \Rightarrow \Delta} L_{\leftrightarrow}}{\delta, \phi \equiv \psi, \Gamma \Rightarrow \Delta} L^3_{\equiv}}{\phi \equiv \psi, \Gamma \Rightarrow \Delta} L^1_{\equiv}$$

where $\chi = (\phi \otimes \phi) \equiv (\psi \otimes \psi), \delta = (\phi \equiv \psi) \equiv (\phi \equiv \psi)$, $\Gamma^* = \Gamma \cup \{\delta\}$. In the obtained derivation sequent $\chi, \phi \equiv \psi, \phi \equiv \psi, \phi \equiv \psi, \Gamma^* \Rightarrow \Delta$ contains two more occurrences of formulas, than the leaf of the original derivation: $\phi \equiv \psi$ and δ. By Theorem 40, if the leaf of the original derivation has a proof in $\mathsf{G3}_{\mathsf{SCI}}$, then the left leaf of the replacing derivation has a proof as well. □

It is worth underlining that when we apply Corollary 4, we do not maintain the same height of the resulting derivation, moreover, the use of weakening is necessary, but the height of the premiss of the contraction is no longer a subject of consideration since the use of contraction is no longer present in the derivation.

Any application of contraction rule which does not fall under the case described in Corollary 4 will be called a standard contraction application. A non-standard application of contraction is one that is not standard.

Theorem 41 (Admissibility of standard applications of contraction in $a\mathsf{G3}_{\mathsf{SCI}}$).

1. *If* $\vdash_n \Gamma \Rightarrow \Delta, \phi, \phi$ *in* $a\mathsf{G3}_{\mathsf{SCI}}$*, then* $\vdash_n \Gamma \Rightarrow \Delta, \phi$ *in* $a\mathsf{G3}_{\mathsf{SCI}}$*.*
2. *If a sequent* $\phi, \phi, \Gamma \Rightarrow \Delta$ *has a proof* D *in* $a\mathsf{G3}_{\mathsf{SCI}}$ *of height at most* n *and* D *does not end with a non-standard application of contraction, then* $\phi, \Gamma \Rightarrow \Delta$ *has a proof in* $a\mathsf{G3}_{\mathsf{SCI}}$ *of height at most* n*.*

Proof. The proof is by induction on the height of the derivation of the premiss of contraction. We begin with the height of the premiss of the contraction equal to 0; this situation is described by the case in which the premiss of the contraction is an axiom. Then we consider the height of the premiss of the contraction equal to $n+1$, within which two cases are considered: contraction formula ϕ not being a principal formula and contraction formula ϕ being the principal formula. To simplify the reasoning we will refer to these cases in the following way:

1a The premiss of the contraction is an axiom;

2a Contraction formula ϕ is not a principal formula;

2b Contraction formula ϕ is a principal formula.

Below we consider an application of L_{ctr}.

We begin with case 1a. If $\phi, \phi, \Gamma \Rightarrow \Delta$ is an axiom, then we know that so is $\phi, \Gamma \Rightarrow \Delta$.

We now move to induction step.

We move to case 2a. Contraction formula ϕ is not principal:

($L_\neg$) We begin with the original derivation, where the height of the derivation of the premiss of L_{ctr} equals $h+1$, where $+1$ refers to the application of $L_\neg$ and h is the height of the derivation of $\phi, \phi, \Gamma \Rightarrow \Delta, \psi$:

$$\dfrac{\dfrac{\phi, \phi, \Gamma \Rightarrow \Delta, \psi}{\phi, \phi, \neg\psi \Rightarrow \Delta}\, L_\neg}{\phi, \neg\psi, \Gamma \Rightarrow \Delta}\, L_{ctr}$$

which we transform so L_{ctr} is applied at the lesser height of the derivation, that is h:

$$\dfrac{\dfrac{\phi, \phi, \Gamma \Rightarrow \Delta, \psi}{\phi, \Gamma \Rightarrow \Delta, \psi}\, L_{ctr}}{\phi, \neg\psi, \Gamma \Rightarrow \Delta}\, L_\neg$$

($L_\wedge$) In the original derivation the height of the derivation of the premiss of L_{ctr} equals $h+1$

$$\dfrac{\dfrac{\phi, \phi, \psi, \chi, \Gamma \Rightarrow \Delta}{\phi, \phi, \psi \wedge \chi, \Gamma \Rightarrow \Delta}\, L_\wedge}{\phi, \psi \wedge \chi, \Gamma \Rightarrow \Delta}\, L_{ctr}$$

which is modified so the height of the derivation of the premiss of L_{ctr} equals now h.

$$\dfrac{\dfrac{\phi, \phi, \psi, \chi, \Gamma \Rightarrow \Delta}{\phi, \psi, \chi, \Gamma \Rightarrow \Delta}\, L_{ctr}}{\phi, \psi \wedge \chi, \Gamma \Rightarrow \Delta}\, L_{\wedge}$$

($L_{\vee}$) In the original derivation the height of the derivation of the premiss of L_{ctr} equals $max(h_l, h_r) + 1$

$$\dfrac{\dfrac{\phi, \phi, \psi, \Gamma \Rightarrow \Delta \quad \phi, \phi, \chi, \Gamma \Rightarrow \Delta}{\phi, \phi, \psi \vee \chi, \Gamma \Rightarrow \Delta}\, L_{\vee}}{\phi, \psi \vee \chi, \Gamma \Rightarrow \Delta}\, L_{ctr}$$

which is modified so the height of the derivation of the premisses of applications of L_{ctr} equals now, respectively, h_l and h_r.

$$\dfrac{\dfrac{\phi, \phi, \psi, \Gamma \Rightarrow \Delta}{\phi, \psi, \Gamma \Rightarrow \Delta}\, L_{ctr} \quad \dfrac{\phi, \phi, \chi, \Gamma \Rightarrow \Delta}{\phi, \chi, \Gamma \Rightarrow \Delta}\, L_{ctr}}{\phi, \psi \vee \chi, \Gamma \Rightarrow \Delta}\, L_{\vee}$$

The remaining cases for classical connectives and identity-dedicated rules are analogous.

We move to case 2b. Again, we consider all left-sided rules (transformations for right-sided rules are analogous).

($L_{\neg}$) We begin with the original derivation:

$$\dfrac{\dfrac{\neg\phi, \Gamma \Rightarrow \Delta, \phi}{\neg\phi, \neg\phi, \Gamma \Rightarrow \Delta}\, L_{\neg}}{\neg\phi, \Gamma \Rightarrow \Delta}\, L_{ctr}$$

which we modify in the following manner:

$$\dfrac{\dfrac{\Gamma \Rightarrow \Delta, \phi, \phi}{\Gamma \Rightarrow \Delta, \phi}\, R_{ctr}}{\neg\phi, \Gamma \Rightarrow \Delta}\, L_{\neg}$$

The obtained derivation moves contraction application at the lesser height and through height-preserving invertibility of rule $L_{\neg}$ we know that sequent $\Gamma \Rightarrow \Delta, \phi, \phi$ can be proved by derivation of the height at most h.

($L_{\wedge}$) We begin with the original derivation, where the height of the derivation of L_{ctr} premiss is $h + 1$:

$$\dfrac{\dfrac{\psi, \chi, \psi \wedge \chi, \Gamma \Rightarrow \Delta}{\psi \wedge \chi, \psi \wedge \chi, \Gamma \Rightarrow \Delta}\, L_{\wedge}}{\psi \wedge \chi, \Gamma \Rightarrow \Delta}\, L_{ctr}$$

and modify it in the following way:

$$\dfrac{\dfrac{\psi, \psi, \chi, \chi, \Gamma \Rightarrow \Delta}{\psi, \chi, \Gamma \Rightarrow \Delta}\ L_{ctr} \times 2}{\psi \wedge \chi, \Gamma \Rightarrow \Delta}\ L_{\wedge}$$

By Lemma 36, the top sequent of this derivation is at height at most h. The double application of L_{ctr} amounts to double application of the induction hypothesis. By application of $L_{\wedge}$ we obtain the final sequent at height at most $h+1$.

($L_{\vee}$) We begin with the original derivation, where the height of the contraction premiss is $h+1$:

$$\dfrac{\dfrac{\phi, \phi \vee \chi, \Gamma \Rightarrow \Delta \quad \chi, \phi \vee \chi, \Gamma \Rightarrow \Delta}{\phi \vee \chi, \phi \vee \chi, \Gamma \Rightarrow \Delta}\ L_{\vee}}{\phi \vee \chi, \Gamma \Rightarrow \Delta}\ L_{ctr}$$

and modify it in the following way:

$$\dfrac{\dfrac{\phi, \phi, \Gamma \Rightarrow \Delta}{\phi, \Gamma \Rightarrow \Delta}\ L_{ctr} \quad \dfrac{\chi, \chi, \Gamma \Rightarrow \Delta}{\chi, \Gamma \Rightarrow \Delta}\ L_{ctr}}{\phi \vee \chi, \Gamma \Rightarrow \Delta}\ L_{\vee}$$

which results in the derivation where at least one of the premisses of $L_{\vee}$ has a derivation of height at most h. The derivation of the second premiss can be even shorter. Here we again refer to the height-preserving invertibility of rule $L_{\vee}$ (shown in Lemma 36) as the leaves in two derivations are labelled with different sequents.

The remaining cases for classical connectives are analogous. We omit the case for rule $L^1_{\equiv}$ since there is no principal formula in its conclusion. Cases for identity-dedicated rules $L^2_{\equiv}$, $L^3_{\equiv}$, $L^4_{\equiv}$ are trivial since the principal formula is kept in the premiss. Below we consider transformations for $L^2_{\equiv}$ and $L^3_{\equiv}$, the transformation for $L^4_{\equiv}$ is analogous.

($L^2_{\equiv}$) We begin with the height of the derivation of the premiss of L_{ctr} being equal to $h+1$:

$$\dfrac{\dfrac{\neg\phi \equiv \neg\chi, \phi \equiv \chi, \phi \equiv \chi, \Gamma \Rightarrow \Delta}{\phi \equiv \chi, \phi \equiv \chi, \Gamma \Rightarrow \Delta}\ L^2_{\equiv}}{\phi \equiv \chi, \Gamma \Rightarrow \Delta}\ L_{ctr}$$

and modify it in the following way, obtaining lesser height of the premiss of the contraction – h:

$$\dfrac{\dfrac{\neg\phi \equiv \neg\chi, \phi \equiv \chi, \phi \equiv \chi, \Gamma \Rightarrow \Delta}{\neg\phi \equiv \neg\chi, \phi \equiv \chi, \Gamma \Rightarrow \Delta}\ L_{ctr}}{\phi \equiv \chi}\ L^2_{\equiv}$$

($L^3_\equiv$) We begin with the original derivation with the height of the derivation of the premiss of L_{ctr} being equal to $h+1$:

$$\dfrac{\dfrac{\phi \leftrightarrow \chi, \phi \equiv \chi, \phi \equiv \chi, \Gamma \Rightarrow \Delta}{\phi \equiv \chi, \phi \equiv \chi, \Gamma \Rightarrow \Delta}\, L^3_\equiv}{\phi \equiv \chi, \Gamma \Rightarrow \Delta}\, L_{ctr}$$

and modify it in the following way, obtaining lesser height of the derivation of the premiss of $L_{ctr} - h$:

$$\dfrac{\dfrac{\phi \leftrightarrow \chi, \phi \equiv \chi, \phi \equiv \chi, \Gamma \Rightarrow \Delta}{\phi \leftrightarrow \chi, \phi \equiv \chi, \Gamma \Rightarrow \Delta}\, L_{ctr}}{\phi \equiv \chi}\, L^3_\equiv$$

□

Theorem 42 (Admissibility of L_{ctr} and R_{ctr} in $\mathsf{G3_{SCI}}$)**.**

1. *If* $\vdash_n \Gamma \Rightarrow \Delta, \phi, \phi$ *in* $\mathsf{G3_{SCI}}$*, then* $\vdash_n \Gamma \Rightarrow \Delta, \phi$ *in* $\mathsf{G3_{SCI}}$*.*
2. *If* $\vdash_n \phi, \phi, \Gamma \Rightarrow \Delta$ *in* $\mathsf{G3_{SCI}}$*, then* $\vdash_n \phi, \Gamma \Rightarrow \Delta$ *in* $\mathsf{G3_{SCI}}$*.*

Proof. Suppose there is a proof of sequent $\phi, \phi, \Gamma \Rightarrow \Delta$ in $\mathsf{G3_{SCI}}$. By Lemma 35, sequent $\phi, \phi, \Gamma \Rightarrow \Delta$ has also a proof in $a\mathsf{G3_{SCI}}$. Through Theorem 41 $\phi, \Gamma \Rightarrow \Delta$ has a proof in $a\mathsf{G3_{SCI}}$, provided it falls under the assumption of Theorem 41. However, if sequent $\phi, \phi, \Gamma \Rightarrow \Delta$ does not fall under conditions from Theorem 41, i.e. it is obtained through non-standard contraction application, then per Corollary 4 sequent $\phi, \Gamma \Rightarrow \Delta$ has a proof in $a\mathsf{G3_{SCI}}$ in which contraction is not used. This particular proof is a proof of sequent $\phi, \Gamma \Rightarrow \Delta$ in $\mathsf{G3_{SCI}}$. The reasoning for point 1 is analogous. □

Definition 110 (Cut-height)**.** *By height of an application of cut in a derivation we shall understand the sum of heights of derivations of its two premisses.*

Theorem 43 (Admissibility of cut)**.** *The cut rule of the form:*

$$\dfrac{\Gamma \Rightarrow \Delta, \psi \quad \psi, \Theta \Rightarrow \Pi}{\Gamma, \Theta \Rightarrow \Delta, \Pi}$$

is admissible in $\mathsf{G3_{SCI}}$*.*

Proof. The proof is analogous to the one presented in [4], we just have to consider three connectives in language $\mathcal{L}_{\mathsf{SCI}}$ that did not appear in [4] and the rules defining them.

We use the following structure of the proof, which relies on the induction on the height of the cut. We begin with the cut height being equal to 0; under this scenario we consider the possibility of each cut premiss being an axiom. Then we move to induction step, where we consider cut height equal to $n+1$. Under this condition we consider the presence of different rules applied over cut. This strategy has been proposed by Dragalin [9], and then simplified by Negri and von Plato in [36], where the authors consider the following cases:

(1) (cut) application with its left premiss being of an axiom form.

(2) (cut) application with its right premiss being of an axiom form.

(3) (cut) application with neither of the premisses of an axiom form.

   (3.1) Cut-formula not principal in the left premiss.

   (3.2) Cut-formula principal in the left premiss only.

   (3.3) Cut-formula principal in both premisses.

In the proof we show that moving the cut upwards to the leaves will result in its overall elimination from the whole derivation.

Let us start with (1). We can consider the following scenarios:

$$\frac{\Gamma \Rightarrow \Delta, \psi \quad \psi, \Pi \Rightarrow \Theta}{\Gamma, \Pi \Rightarrow \Theta, \Delta}$$

(a) cut formula $\psi \in \Gamma$: we obtain $\Gamma, \Pi \Rightarrow \Theta, \Delta$ through weakening of the second premiss;

(b) Γ and Δ share the same formula δ: $\Gamma, \Pi \Rightarrow \Theta, \Delta$ is then also an axiom.

In the case of (2) we have the following scenarios:

(c) cut formula $\psi \in \Theta$: we obtain $\Gamma, \Pi \Rightarrow \Theta, \Delta$ through weakening of the first premiss;

(d) Θ and Π share the same formula δ: $\Gamma, \Pi \Rightarrow \Theta, \Delta$ is an axiom;

We now move to (3.1). We consider the following cases in which cut-formula is not principal in the left premiss:

$(L_\neg)$ We begin with the original derivation:

$$\dfrac{\dfrac{\Gamma \Rightarrow \phi, \Theta, \psi}{\neg\phi, \Gamma \Rightarrow \Theta, \psi}\,L_\neg \qquad \psi, \Pi \Rightarrow \Delta}{\neg\phi, \Gamma, \Pi \Rightarrow \Theta, \Delta}\,cut$$

Where the height of the cut equals the sum of the heights of its two premisses. In the derivation above (and in the subsequent ones) the height of the left leaf will be denoted by h_l and the height of the second, right leaf will be denoted by h_r. In this case the height of the (cut) application is equal to $h_l + h_r + 1$. After the transformation in which the cut is moved upwards, the value of the cut-height is equal to $h_l + h_r$, as we moved the application of the cut upwards, therefore the objective has been met.

$$\dfrac{\dfrac{\Gamma \Rightarrow \phi, \psi, \Theta \quad \psi, \Pi \Rightarrow \Delta}{\Gamma, \Pi \Rightarrow \phi, \Theta, \Delta}\,cut}{\neg\phi, \Gamma, \Pi \Rightarrow \Theta, \Delta}\,L_\neg$$

($L_\wedge$) In this case the height of the (cut) application is equal to $h_l + h_r + 1$. The original derivation is the following:

$$\dfrac{\dfrac{\phi, \chi, \Gamma \Rightarrow \psi, \Theta}{\phi \wedge \chi, \Gamma \Rightarrow \psi, \Theta}\, L_\wedge \quad \psi, \Pi \Rightarrow \Delta}{\phi \wedge \chi, \Gamma, \Pi \Rightarrow \Theta, \Delta}\, cut$$

After the transformation in which the cut is moved upwards, the value of the cut-height is equal to $h_l + h_r$, therefore the objective has been met.

$$\dfrac{\dfrac{\phi, \chi, \Gamma \Rightarrow \psi, \Theta \quad \psi, \Pi \Rightarrow \Delta}{\phi, \chi, \Gamma, \Pi \Rightarrow \Theta, \Delta}\, cut}{\phi \wedge \chi, \Gamma, \Pi \Rightarrow \Theta, \Delta}\, L_\wedge$$

($L_\vee$) Below by h_l^1 and h_l^2 we mean the heights of, respectively, the leftmost leaf and the right leaf of the left branch in the derivation. Consequently, in this case the height of the (cut) application is equal to $max(h_l^1, h_l^2) + h_r + 1$. The original derivation is the following:

$$\dfrac{\dfrac{\phi, \Gamma \Rightarrow \psi, \Delta \quad \chi, \Gamma \Rightarrow \psi, \Delta}{\phi \vee \chi, \Gamma \Rightarrow \psi, \Delta}\, L_\vee \quad \psi, \Pi \Rightarrow \Theta}{\phi \vee \chi, \Gamma, \Pi \Rightarrow \Theta, \Delta}\, cut$$

After the transformation in which the cut is moved upwards, we obtain two applications of cut. The value of the cut-height on the left branch is equal to $h_l^1 + h_r$, whereas the cut-height on the right branch equals $h_l^2 + h_r$, therefore the objective has been met.

$$\dfrac{\dfrac{\phi, \Gamma \Rightarrow \Delta, \psi \quad \psi, \Pi \Rightarrow \Theta}{\phi, \Gamma, \Pi \Rightarrow \Theta, \Delta}\, cut \quad \dfrac{\chi, \Gamma \Rightarrow \Delta, \psi \quad \psi, \Pi \Rightarrow \Theta}{\chi, \Gamma, \Pi \Rightarrow \Theta, \Delta}\, cut}{\phi \vee \chi, \Gamma, \Pi \Rightarrow \Theta, \Delta}\, L_\vee$$

($L_\rightarrow$) In this case the height of the (cut) application is equal to $max(h_l^1, h_l^2) + h_r + 1$. The original derivation is the following:

$$\dfrac{\dfrac{\Gamma \Rightarrow \phi, \psi, \Theta \quad \chi, \Gamma \Rightarrow \psi, \Theta}{\phi \rightarrow \chi, \Gamma \Rightarrow \psi, \Theta}\, L_\rightarrow \quad \psi, \Pi \Rightarrow \Delta}{\phi \rightarrow \chi, \Gamma, \Pi \Rightarrow \Theta, \Delta}\, cut$$

After the transformation in which the cut is moved upwards, we, similarly as for $L_\vee$, obtain two applications of cut. The value of the cut-height on the left branch is equal to $h_l^1 + h_r$, whereas the cut-height on the right branch equals $h_l^2 + h_r$, therefore the objective has been met.

$$\dfrac{\dfrac{\Gamma \Rightarrow \phi, \Theta, \psi \quad \psi, \Pi \Rightarrow \Delta}{\Gamma, \Pi \Rightarrow \phi, \Theta, \Delta}\, cut \qquad \dfrac{\chi, \Gamma \Rightarrow \Theta, \psi \quad \psi, \Pi \Rightarrow \Delta}{\chi, \Gamma, \Pi \Rightarrow \Theta, \Delta}\, cut}{\phi \to \chi, \Gamma, \Pi \Rightarrow \Theta, \Delta}\, L_{\to}$$

($L_{\leftrightarrow}$) In this case the height of the (cut) application is equal to $max(h_l^1, h_l^2) + h_r + 1$. The original derivation is the following:

$$\dfrac{\dfrac{\phi, \chi, \Gamma \Rightarrow \Delta, \psi \quad \Gamma \Rightarrow \phi, \chi, \psi, \Delta}{\phi \leftrightarrow \chi, \Gamma \Rightarrow \Delta, \psi}\, L_{\leftrightarrow} \qquad \psi, \Pi \Rightarrow \Delta}{\phi \leftrightarrow \chi, \Gamma, \Pi \Rightarrow \Theta, \Delta}\, cut$$

After the transformation in which the cut is moved upwards, we again obtain two applications of cut. The value of the cut-height on the left branch is equal to $h_l^1 + h_r$, whereas the cut-height of the right branch equals $h_l^2 + h_r$, therefore the objective has been met.

$$\dfrac{\dfrac{\phi, \chi, \Gamma \Rightarrow \Delta, \psi \quad \psi, \Pi \Rightarrow \Theta}{\phi, \chi, \Gamma, \Pi \Rightarrow \Theta, \Delta}\, cut \qquad \dfrac{\Gamma \Rightarrow \Delta, \phi, \chi, \psi \quad \psi, \Pi \Rightarrow \Theta}{\Gamma, \Pi \Rightarrow \Theta, \Delta, \phi, \chi}\, cut}{\phi \leftrightarrow \chi, \Gamma, \Pi \Rightarrow \Theta, \Delta}\, L_{\leftrightarrow}$$

($L_{\equiv}^1$) In this case the height of the (cut) application is equal to $h_l + h_r + 1$. The original derivation is the following:

$$\dfrac{\dfrac{\phi \equiv \phi, \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \psi}\, L_{\equiv}^1 \qquad \psi, \Pi \Rightarrow \Theta}{\Gamma, \Pi \Rightarrow \Theta, \Delta}\, cut$$

After the transformation in which the cut is moved upwards, the value of the cut-height is equal to $h_l + h_r$, therefore the objective has been met.

$$\dfrac{\dfrac{\phi \equiv \phi, \Gamma \Rightarrow \Delta, \psi \quad \psi, \Pi \Rightarrow \Theta}{\phi \equiv \phi, \Gamma, \Pi \Rightarrow \Theta, \Delta}\, cut}{\Gamma, \Pi \Rightarrow \Theta, \Delta}\, L_{\equiv}^1$$

($L_{\equiv}^2$) In this case the height of the (cut) application is equal to $h_l + h_r + 1$. The original derivation is the following:

$$\dfrac{\dfrac{\phi \equiv \chi, \neg\phi \equiv \neg\chi, \Gamma \Rightarrow \Delta, \psi}{\phi \equiv \chi, \Gamma \Rightarrow \Delta, \psi}\, L_{\equiv}^2 \qquad \psi, \Pi \Rightarrow \Theta}{\phi \equiv \chi, \Gamma, \Pi \Rightarrow \Theta, \Delta}\, cut$$

After the transformation in which the cut is moved upwards, the value of the cut-height is equal to $h_l + h_r$, therefore the objective has been met.

$$\dfrac{\dfrac{\neg\phi \equiv \neg\chi, \phi \equiv \chi, \Gamma \Rightarrow \Delta, \psi \quad \psi, \Pi \Rightarrow \Theta}{\neg\phi \equiv \neg\chi, \phi \equiv \chi, \Gamma, \Pi \Rightarrow \Theta, \Delta}\ cut}{\phi \equiv \chi, \Gamma, \Pi \Rightarrow \Theta, \Delta}\ L^2_{\equiv}$$

($L^3_{\equiv}$) In this case the height of the (*cut*) application is equal to $h_l + h_r + 1$. The original derivation is the following:

$$\dfrac{\dfrac{\phi \leftrightarrow \chi, \phi \equiv \chi, \Gamma \Rightarrow \Delta, \psi}{\phi \equiv \chi, \Gamma \Rightarrow \Delta, \psi}\ L^3_{\equiv} \quad \psi, \Pi \Rightarrow \Theta}{\phi \equiv \chi, \Gamma, \Pi \Rightarrow \Theta, \Delta}\ cut$$

After the transformation in which the cut is moved upwards, the value of the cut-height is equal to $h_l + h_r$, therefore the objective has been met.

$$\dfrac{\dfrac{\phi \leftrightarrow \chi, \phi \equiv \chi, \Gamma \Rightarrow \Delta, \psi \quad \psi, \Pi \Rightarrow \Theta}{\phi \leftrightarrow \chi, \phi \equiv \chi, \Gamma, \Pi \Rightarrow \Theta, \Delta}\ cut}{\phi \equiv \chi, \Gamma, \Pi \Rightarrow \Theta, \Delta}\ L^3_{\equiv}$$

($L^4_{\equiv}$) In this case the height of the (*cut*) application is equal to $h_l + h_r + 1$. The original derivation is the following:

$$\dfrac{\dfrac{(\phi \otimes \delta) \equiv (\chi \otimes \gamma), \phi \equiv \chi, \delta \equiv \gamma, \Gamma \Rightarrow \Delta, \psi}{\phi \equiv \chi, \delta \equiv \gamma, \Gamma \Rightarrow \Delta, \psi}\ L^4_{\equiv} \quad \psi, \Pi \Rightarrow \Theta}{\phi \equiv \chi, \delta \equiv \gamma, \Gamma, \Pi \Rightarrow \Theta, \Delta}\ cut$$

After the transformation in which the cut is moved upwards, the value of the cut-height is equal to $h_l + h_r$, therefore the objective has been met.

$$\dfrac{\dfrac{(\phi \otimes \delta) \equiv (\chi \otimes \gamma), \phi \equiv \chi, \delta \equiv \gamma, \Gamma \Rightarrow \Delta, \psi \quad \psi, \Pi \Rightarrow \Theta}{(\phi \otimes \delta) \equiv (\chi \otimes \gamma), \phi \equiv \chi, \delta \equiv \gamma, \Gamma, \Pi \Rightarrow \Theta, \Delta}\ cut}{\phi \equiv \chi, \delta \equiv \gamma, \Gamma, \Pi \Rightarrow \Theta, \Delta}\ L^4_{\equiv}$$

The transformations in Point (3.2) are analogous – the right premise with non-principal cut formula is considered.

The final step is to consider the last subcase, *i.e.*, (3.3), in which the cut-formula is a principal formula in both left and right premiss of the (*cut*).

We consider five logical connectives:

(1) $\psi = \neg\phi$

(2) $\psi = \phi \wedge \chi$

(3) $\psi = \phi \vee \chi$

(4) $\psi = \phi \to \chi$

(5) $\psi = \phi \leftrightarrow \chi$

We do not consider ψ of the form $\phi \equiv \chi$ as we do not have any right-sided identity rules in the rule set for $\mathsf{G3_{SCI}}$.

In these cases we refer to the induction hypothesis concerning the weight of the cut-formula.

(1) The following derivation

$$\dfrac{\dfrac{\phi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg\phi}\, R_{\neg} \quad \dfrac{\Pi \Rightarrow \Theta, \phi}{\neg\phi, \Pi \Rightarrow \Theta}\, L_{\neg}}{\Gamma, \Pi \Rightarrow \Theta, \Delta}\, cut$$

is replaced with the following one, where the cut-height is lesser than in the original derivation.

$$\dfrac{\Pi \Rightarrow \Theta, \phi \quad \phi, \Gamma \Rightarrow \Delta}{\Gamma, \Pi \Rightarrow \Theta, \Delta}\, cut$$

(2) The following derivation

$$\dfrac{\dfrac{\Gamma \Rightarrow \Delta, \phi \quad \Gamma \Rightarrow \Delta, \chi}{\Gamma \Rightarrow \Delta, \phi \wedge \chi}\, R_{\wedge} \quad \dfrac{\phi, \chi, \Pi \Rightarrow \Theta}{\phi \wedge \chi, \Pi \Rightarrow \Theta}\, L_{\wedge}}{\Gamma, \Pi \Rightarrow \Theta, \Delta}\, cut$$

is replaced with the following one, where the upper cut-height is lesser than in the above derivation. For the lower cut application we refer to the induction hypothesis on the weight of the formula. Moreover, the contraction is used (since we know that contraction is admissible in $\mathsf{G3_{SCI}}$).

$$\dfrac{\dfrac{\Gamma \Rightarrow \Delta, \phi \quad \dfrac{\Gamma \Rightarrow \Delta, \chi \quad \chi, \phi, \Pi \Rightarrow \Theta}{\phi, \Gamma, \Pi \Rightarrow \Theta, \Delta}\, cut}{\Gamma, \Gamma, \Pi \Rightarrow \Theta, \Delta, \Delta}\, cut}{\Gamma, \Pi \Rightarrow \Theta, \Delta}\, ctr$$

(3) The following derivation

$$\dfrac{\dfrac{\Gamma \Rightarrow \phi, \chi, \Theta}{\Gamma \Rightarrow \phi \vee \chi, \Theta}\, R_{\vee} \quad \dfrac{\phi, \Pi \Rightarrow \Delta \quad \chi, \Pi \Rightarrow \Delta}{\phi \vee \chi, \Pi \Rightarrow \Delta}\, L_{\vee}}{\Gamma, \Pi \Rightarrow \Theta, \Delta}\, cut$$

is replaced with one where the upper cut application is of a lesser height. For the lower cut application we refer to the induction hypothesis on the weight of the formula.

$$\dfrac{\dfrac{\dfrac{\Gamma \Rightarrow, \phi, \chi, \Theta \quad \chi, \Pi \Rightarrow \Delta}{\Gamma, \Pi \Rightarrow \phi, \Theta, \Delta}\, cut \quad \phi, \Pi \Rightarrow \Delta}{\Gamma, \Pi, \Pi \Rightarrow \Theta, \Delta, \Delta}\, cut}{\Gamma, \Pi \Rightarrow \Theta, \Delta}\, L_{ctr}, R_{ctr}$$

(4) The following derivation

$$\dfrac{\dfrac{\phi, \Gamma \Rightarrow \chi, \Delta}{\Gamma \Rightarrow \phi \to \chi, \Delta}\, R_{\to} \quad \dfrac{\Pi \Rightarrow \phi, \Theta \quad \chi, \Pi \Rightarrow \Theta}{\phi \to \chi, \Pi \Rightarrow \Theta}\, L_{\to}}{\Gamma, \Pi \Rightarrow \Theta, \Delta}\, cut$$

we replace with the following one, where the height of the upper cut is lesser than in the original derivation. For the lower derivation we refer to the induction hypothesis on the weight of the formula.

$$\dfrac{\dfrac{\Pi \Rightarrow \Theta, \phi \quad \dfrac{\phi, \Gamma \Rightarrow \Delta, \chi \quad \chi, \Pi \Rightarrow \Theta}{\phi, \Gamma, \Pi \Rightarrow \Theta, \Delta}\, cut}{\Gamma, \Pi, \Pi \Rightarrow \Theta, \Theta, \Delta}\, cut}{\Gamma, \Pi \Rightarrow \Theta, \Delta}\, ctr$$

(5) We introduce the original derivation

$$\dfrac{\dfrac{\phi, \Gamma \Rightarrow \Delta, \chi \quad \chi, \Gamma \Rightarrow \Delta, \phi}{\Gamma \Rightarrow \Delta, \phi \leftrightarrow \chi}\, R_{\leftrightarrow} \quad \dfrac{\phi, \chi, \Pi, \Rightarrow \Theta \quad \Pi \Rightarrow \Theta, \phi, \chi}{\phi \leftrightarrow \chi, \Pi \Rightarrow \Theta}\, L_{\leftrightarrow}}{\Gamma, \Pi \Rightarrow \Theta, \Delta}\, cut$$

and we transform it in the following way, thus obtaining derivation, where the two upper cut-heights are lesser than in the above derivation. For the lower cut application we refer to the induction hypothesis on the formula weight.

$$\dfrac{\dfrac{\dfrac{\dfrac{\Pi \Rightarrow \Theta, \phi, \chi \quad \chi, \Gamma \Rightarrow \Delta, \phi}{\Gamma, \Pi \Rightarrow \Theta, \Delta, \phi, \phi}\, cut}{\Gamma, \Pi \Rightarrow \Theta, \Delta, \phi}\, R_{ctr} \quad \dfrac{\dfrac{\phi, \Gamma \Rightarrow \Delta, \chi \quad \chi, \phi, \Pi \Rightarrow \Theta}{\phi, \phi, \Gamma, \Pi \Rightarrow \Theta, \Delta}\, cut}{\phi, \Gamma, \Pi \Rightarrow \Theta, \Delta}\, L_{ctr}}{\Gamma, \Gamma, \Pi, \Pi \Rightarrow \Theta, \Theta, \Delta, \Delta}\, cut}{\Gamma, \Pi \Rightarrow \Theta, \Delta}\, L_{ctr}, R_{ctr}$$

Thanks to the admissibility of contraction (Theorem 42) the objective of the proof has been met. □

7.2 Cut issues in $\mathsf{G3_{WB}}$

Even though for the classical and SCI-dedicated part of the $\mathsf{G3_{WB}}$ we can prove the cut elimination theorem, the addition of $R^B_{\equiv}$ either forces us to keep the cut rule in the system or to eliminate it with the cost of keeping the weakening rule. The reason for this lies in its particular shape. We can show the failure of cut elimination in two ways. For the cut elimination procedure we use the notions of height of the derivation and cut-height, as were presented in Section 7.1

Let us discuss the issues concerning the cut elimination first by referring to the standard cut-elimination procedure originally proposed by Dragalin and later reconstructed by Negri [36], as we described in the previous section.

Rule $R^B_{\equiv}$ is excluded from cases (3.1) and (3.2) (below we can see the original derivations, before transformation; R_x denotes an exemplary right-sided rule).

$$
\dfrac{\begin{array}{c} D_1 \\ \vdots\, R_x \\ \overset{I}{\Rightarrow} \phi \equiv \chi, \psi \end{array} \quad \begin{array}{c} D_2 \\ \vdots \\ \psi \overset{I}{\Rightarrow} \end{array}}{\overset{I}{\Rightarrow} \phi \equiv \chi}\,cut
\qquad
\dfrac{\begin{array}{c} D \\ \vdots \\ \overset{I}{\Rightarrow} \psi \end{array} \quad \begin{array}{c} D \\ \vdots\, R_x \\ \psi \overset{I}{\Rightarrow} \phi \equiv \chi \end{array}}{\overset{I}{\Rightarrow} \phi \equiv \chi}\,cut
$$

Of course, we cannot consider $R^B_{\equiv}$ in these cases, because by adding the cut-formula, the contexts are not empty, which in turn prohibits us from applying $R^B_{\equiv}$.

Let us consider the last case, *i.e.*, (3.3), in which the cut-formula is a principal formula in both the left and right premiss of the (*cut*). Suppose the cut-formula is $\phi \equiv \chi$.

In these cases we refer to the induction hypothesis on the weight of the formula—we want to transform the original derivation through the use of cut-formulae of a lesser weight.

We introduce the original derivation, in which $\phi \equiv \chi$ was obtained through $L^3_{\equiv}$ (even though in the full proof we would have to consider other cases, in which $\phi \equiv \chi$ was obtained through $L^2_{\equiv}$ and $L^4_{\equiv}$), in which we notice that there is no reduction of the weight of formulae. We can additionally consider reduction of the cut-height on the left premiss of the cut and the right leaf in the derivation, but we would be left with additional equivalence on the left side.

$$
\dfrac{\dfrac{\overset{C}{\Rightarrow} \phi \leftrightarrow \chi}{\overset{I}{\Rightarrow} \phi \equiv \chi}\,R_{\equiv} \quad \dfrac{\phi \leftrightarrow \chi, \phi \equiv \chi, \Gamma \overset{I}{\Rightarrow} \Delta}{\phi \equiv \chi, \Gamma \overset{I}{\Rightarrow} \Delta}\,L^3_{\equiv}}{\Gamma \overset{I}{\Rightarrow} \Delta}\,cut
$$

Moreover, if we were to examine the case in which $L_{\leftrightarrow}$ and $R_{\leftrightarrow}$ are applied (which results in the reduction of the weight of formulae), we would encounter problems with labels.

$$
\dfrac{\dfrac{\dfrac{\phi \overset{C}{\Rightarrow} \chi \quad \chi \overset{C}{\Rightarrow} \phi}{\overset{C}{\Rightarrow} \phi \leftrightarrow \chi}\,R_{\leftrightarrow}}{\overset{I}{\Rightarrow} \phi \equiv \chi}\,R_{\equiv} \quad \dfrac{\dfrac{\Gamma, \phi, \chi, \phi \equiv \chi \overset{I}{\Rightarrow} \Delta \quad \Gamma, \phi \equiv \chi \overset{I}{\Rightarrow} \phi, \chi, \Delta}{\phi \leftrightarrow \chi, \phi \equiv \chi, \Gamma \overset{I}{\Rightarrow} \Delta}\,L_{\leftrightarrow}}{\phi \equiv \chi, \Gamma \overset{I}{\Rightarrow} \Delta}\,L^3_{\equiv}}{\Gamma \overset{I}{\Rightarrow} \Delta}\,cut
$$

Application of cut to $\phi \overset{C}{\Rightarrow} \chi$ and $\Gamma, \phi, \chi, \phi \equiv \chi \overset{I}{\Rightarrow} \Delta$ (in order to cut formula χ) is problematic since sequents have different labels.

We will now discuss which WB theorems require *cut*.

$R^B_{\equiv}$ works in a restricted way and does not allow us to prove theorems from other extensions of SCI. We are able to prove theorem

$$
(\phi \equiv (\phi \wedge \chi)) \leftrightarrow ((\phi \vee \chi) \equiv \chi) \tag{7.1}
$$

of WB [51; 68], but at the same time we cannot prove the following theorem of WT: $(\phi \equiv (\phi \wedge \chi)) \equiv ((\phi \vee \chi) \equiv \chi)$ as shown below:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ \phi \equiv (\phi \wedge \chi) \overset{C}{\Rightarrow} (\phi \vee \chi) \equiv \chi\end{array} \quad \begin{array}{c}\vdots\\ (\phi \vee \chi) \equiv \chi \overset{C}{\Rightarrow} \phi \equiv (\phi \wedge \chi)\end{array}}{\overset{C}{\Rightarrow} (\phi \equiv (\phi \wedge \chi)) \leftrightarrow (\phi \vee \chi) \equiv \chi}\, R_{\leftrightarrow}}{\overset{I}{\Rightarrow} (\phi \equiv (\phi \wedge \chi)) \equiv ((\phi \vee \chi) \equiv \chi)}\, R^B_{\equiv}$$

We will now examine cut issues in certain formulae. We look at the proof of formula $(\phi \equiv (\phi \wedge \chi)) \leftrightarrow ((\phi \vee \chi) \equiv \chi)$ of WB (here, shown in two separate derivations due to the size of the overall proof; later on, formula $(\phi \equiv (\phi \wedge \chi)) \rightarrow ((\phi \vee \chi) \equiv \chi)$ will be referred to as $(\clubsuit)$). To minimize the size of the proof, we will use the following short-cut transitivity rule (which is a rule derivable in $\mathsf{G3_{WB}}$ with sequents labelled by I, as it was shown it is derivable in sequent calculus for SCI, the proof can be found in [4]).

$$\dfrac{\phi \equiv \psi, \phi \equiv \chi, \chi \equiv \psi, \Gamma \overset{I}{\Rightarrow} \Delta}{\phi \equiv \chi, \chi \equiv \psi, \Gamma \overset{I}{\Rightarrow} \Delta}\, trans$$

We will present the proof of the formula in two derivations of its two implications:

- $(\phi \equiv (\phi \wedge \chi)) \rightarrow ((\phi \vee \chi) \equiv \chi)$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}D_1\\ \vdots\end{array}}{\overset{I}{\Rightarrow} ((\phi \wedge \chi) \vee \chi) \equiv \chi}\, R^B_{\equiv} \quad \dfrac{\Gamma, (\phi \equiv (\phi \wedge \chi)), (\phi \vee \chi) \equiv \chi \overset{I}{\Rightarrow} (\phi \vee \chi) \equiv \chi}{\Gamma, (\phi \equiv (\phi \wedge \chi)) \overset{I}{\Rightarrow} (\phi \vee \chi) \equiv \chi}\, trans}{(\phi \vee \chi) \equiv ((\phi \wedge \chi) \vee \chi), (\phi \equiv (\phi \wedge \chi)), \chi \equiv \chi \overset{I}{\Rightarrow} (\phi \vee \chi) \equiv \chi}\, cut}{\phi \equiv (\phi \wedge \chi), \chi \equiv \chi \overset{I}{\Rightarrow} (\phi \vee \chi) \equiv \chi}\, L^4_{\equiv}}{\phi \equiv (\phi \wedge \chi) \overset{I}{\Rightarrow} (\phi \vee \chi) \equiv \chi}\, L^1_{\equiv}}{\overset{I}{\Rightarrow} (\phi \equiv (\phi \wedge \chi)) \rightarrow ((\phi \vee \chi) \equiv \chi)}\, R_{\rightarrow}$$

where Γ stands for $\{((\phi \wedge \chi) \vee \chi) \equiv \chi, (\phi \vee \chi) \equiv ((\phi \wedge \chi) \vee \chi), \chi \equiv \chi\}$ and D_1:

$$\dfrac{\dfrac{\dfrac{\phi, \chi \overset{C}{\Rightarrow} \chi}{\phi \wedge \chi \overset{C}{\Rightarrow} \chi}\, L_{\wedge} \quad \chi \overset{C}{\Rightarrow} \chi}{((\phi \wedge \chi) \vee \chi) \overset{C}{\Rightarrow} \chi}\, L_{\vee} \quad \dfrac{\chi \overset{C}{\Rightarrow} \phi \wedge \chi, \chi}{\chi \overset{C}{\Rightarrow} (\phi \wedge \chi) \vee \chi}\, R_{\vee}}{\overset{C}{\Rightarrow} ((\phi \wedge \chi) \vee \chi) \leftrightarrow \chi}\, R_{\leftrightarrow}$$

- $((\phi \vee \chi) \equiv \chi) \rightarrow (\phi \equiv (\phi \wedge \chi))$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\begin{matrix} D_2 \\ \vdots \end{matrix}}{\overset{I}{\Rightarrow} \delta}\, R^{B}_{\equiv} \qquad \dfrac{\phi \equiv (\phi \wedge \chi), \Gamma \overset{I}{\Rightarrow} \phi \equiv (\phi \wedge \chi)}{\delta, \omega, (\phi \vee \chi) \equiv \chi, \phi \equiv \phi \overset{I}{\Rightarrow} \phi \equiv (\phi \wedge \chi)}\, \mathit{trans}}{(\phi \wedge (\phi \vee \chi)) \equiv (\phi \wedge \chi), (\phi \vee \chi) \equiv \chi, \phi \equiv \phi \overset{I}{\Rightarrow} \phi \equiv (\phi \wedge \chi)}\, \mathit{cut}}{(\phi \vee \chi) \equiv \chi, \phi \equiv \phi \overset{I}{\Rightarrow} \phi \equiv (\phi \wedge \chi)}\, L^{4}_{\equiv}}{(\phi \vee \chi) \equiv \chi \overset{I}{\Rightarrow} \phi \equiv (\phi \wedge \chi)}\, L^{1}_{\equiv}}{\overset{I}{\Rightarrow} ((\phi \vee \chi) \equiv \chi) \rightarrow (\phi \equiv (\phi \wedge \chi))}\, R_{\rightarrow}
$$

where δ stands for $\phi \equiv (\phi \wedge (\phi \vee \chi))$, ω stands for $(\phi \wedge (\phi \vee \chi)) \equiv (\phi \wedge \chi)$, Γ stands for $\{\phi \equiv (\phi \wedge (\phi \vee \chi)), \omega, ((\phi \vee \chi) \equiv \chi), \phi \equiv \phi\}$ and D_2:

$$
\dfrac{\dfrac{\phi \overset{C}{\Rightarrow} \phi \qquad \dfrac{\phi \overset{C}{\Rightarrow} \phi, \chi}{\phi \overset{C}{\Rightarrow} \phi \vee \chi}\, R_{\vee}}{\phi \overset{C}{\Rightarrow} \phi \wedge (\phi \vee \chi)}\, R_{\wedge} \qquad \dfrac{\phi, \phi \vee \chi \overset{C}{\Rightarrow} \phi}{\phi \wedge (\phi \vee \chi) \overset{C}{\Rightarrow} \phi}\, L_{\wedge}}{\overset{C}{\Rightarrow} \phi \leftrightarrow (\phi \wedge (\phi \vee \chi))}\, R_{\leftrightarrow}
$$

After the application of $R^{B}_{\equiv}$ the labels changed from $\overset{I}{\Rightarrow}$ to $\overset{C}{\Rightarrow}$ (looking bottom-up), which means we are no longer able to apply any identity-dedicated rules.

What we see in the case of $((\phi \wedge \chi) \equiv \chi) \leftrightarrow ((\phi \vee \chi) \equiv \chi)$ is the fact that certain WB theorems require the application of *cut*. Other formulae of this kind are the following:

1. $((\phi \rightarrow \chi) \equiv \top) \leftrightarrow ((\phi \wedge \neg\chi) \equiv \bot)$
2. $((\phi \rightarrow \chi) \equiv \top) \rightarrow ((\phi \equiv \top) \rightarrow (\chi \equiv \top))$
3. $((\phi \wedge \chi) \equiv \top) \leftrightarrow ((\phi \equiv \top) \wedge (\chi \equiv \top))$
4. $(\phi \equiv \chi) \leftrightarrow ((\phi \leftrightarrow \chi) \equiv \top)$

These formulae can give us insight into issues regarding cut elimination. Let us additionally consider a proof of the formula $((\phi \rightarrow \chi) \equiv \top) \leftrightarrow ((\phi \wedge \neg\chi) \equiv \bot)$ (again, shown in two separate derivations due to the size of the proof):

- $((\phi \rightarrow \chi) \equiv \top) \rightarrow ((\phi \wedge \neg\chi) \equiv \bot)$

$$
\dfrac{\dfrac{\dfrac{\begin{array}{c}D_1\\ \vdots\\ \xRightarrow{I} \neg\top\equiv\bot\end{array} \quad \dfrac{\dfrac{\begin{array}{c}D_2\\ \vdots\\ \xRightarrow{I}\delta\end{array}\quad \dfrac{(\phi\wedge\neg\chi)\equiv\bot,\delta,\neg(\phi\to\chi)\equiv\bot,\Gamma\xRightarrow{I}(\phi\wedge\neg\chi)\equiv\bot}{\delta,\neg(\phi\to\chi)\equiv\bot,\Gamma\xRightarrow{I}(\phi\wedge\neg\chi)\equiv\bot}\,trans}{\neg(\phi\to\chi)\equiv\bot,\Gamma\xRightarrow{I}(\phi\wedge\neg\chi)\equiv\bot}\,cut}{\neg\top\equiv\bot,\neg(\phi\to\chi)\equiv\neg\top,(\phi\to\chi)\equiv\top\xRightarrow{I}(\phi\wedge\neg\chi)\equiv\bot}\,trans}{\neg(\phi\to\chi)\equiv\neg\top,(\phi\to\chi)\equiv\top\xRightarrow{I}(\phi\wedge\neg\chi)\equiv\bot}\,cut}{\dfrac{(\phi\to\chi)\equiv\top\xRightarrow{I}(\phi\wedge\neg\chi)\equiv\bot}{\xRightarrow{I}((\phi\to\chi)\equiv\top)\to((\phi\wedge\neg\chi)\equiv\bot)}\,R_{\to}}\,L^2_{\equiv}
$$

where δ stands for $(\phi\wedge\neg\chi)\equiv\neg(\phi\to\chi)$, Γ stands for $\{\neg\top\equiv\bot,\neg(\phi\to\chi)\equiv\neg\top,(\phi\to\chi)\equiv\top\}$.

- $((\phi\wedge\neg\chi)\equiv\bot)\to((\phi\to\chi)\equiv\top)$

$$
\dfrac{\dfrac{\dfrac{\begin{array}{c}D_3\\ \vdots\\ \xRightarrow{I} \neg\bot\equiv\top\end{array} \quad \dfrac{\dfrac{\begin{array}{c}D_4\\ \vdots\\ \xRightarrow{I}\omega\end{array}\quad \dfrac{(\phi\to\chi)\equiv\top,\omega,\neg(\phi\wedge\neg\chi)\equiv\top,\Gamma'\xRightarrow{I}(\phi\to\chi)\equiv\top}{\omega,\neg(\phi\wedge\neg\chi)\equiv\top,\Gamma'\xRightarrow{I}(\phi\to\chi)\equiv\top}\,trans}{\neg(\phi\wedge\neg\chi)\equiv\top,\Gamma'\xRightarrow{I}(\phi\to\chi)\equiv\top}\,cut}{\neg\bot\equiv\top,\neg(\phi\wedge\neg\chi)\equiv\neg\bot,(\phi\wedge\neg\chi)\equiv\bot\xRightarrow{I}(\phi\to\chi)\equiv\top}\,trans}{\neg(\phi\wedge\neg\chi)\equiv\neg\bot,(\phi\wedge\neg\chi)\equiv\bot\xRightarrow{I}(\phi\to\chi)\equiv\top}\,cut}{\dfrac{(\phi\wedge\neg\chi)\equiv\bot\xRightarrow{I}(\phi\to\chi)\equiv\top}{\xRightarrow{I}((\phi\wedge\neg\chi)\equiv\bot)\to((\phi\to\chi)\equiv\top)}\,R_{\to}}\,L^2_{\equiv}
$$

where ω stands for $(\phi\to\chi)\equiv\neg(\phi\wedge\neg\chi)$, Γ' stands for $\{\neg\bot\equiv\top,\neg(\phi\wedge\neg\chi)\equiv\neg\bot,(\phi\wedge\neg\chi)\equiv\bot\}$ and D_1–D_4 are derivations of sequents $\xRightarrow{I}\delta_i$, where δ_i stands for different theorems of WB. In order to construct proofs for these sequents application of $R^B_{\equiv}$ is necessary, followed (looking bottom-up) by applications of different classical rules (these are obvious).

Sequent labels can prevent us from applying certain rules. Moreover, $R^B_{\equiv}$ requires us to keep the antecedent empty while the succedent consisting of a singular equation (in the conclusion of the rule). Suppose we have WB theorem of the following form $\phi\leftrightarrow\chi$ (similarly for $\phi\to\chi$), where at least one of the main components is an equation. We start building a proof with the sequent $\xRightarrow{I}\phi\leftrightarrow\chi$ to which we can apply $R_{\leftrightarrow}$. As a result we have two branches, one of which is labelled with $\phi\xRightarrow{I}\chi$, and the other with $\chi\xRightarrow{I}\phi$. In both cases, in order to apply $R^B_{\equiv}$, we can either apply weakening rules (which are not invertible) or apply the *cut* rule, which enables the division of contexts.

Equivalences $\phi\leftrightarrow\chi$, where both ϕ and χ are theorems of WB, do not require *cut* application in the derivation, *e.g.*, derivation of the following structure:

$$\dfrac{\dfrac{\dfrac{\dfrac{\phi,\phi \stackrel{C}{\Rightarrow} \phi}{\phi\wedge\phi \stackrel{C}{\Rightarrow} \phi}\, L_\wedge \quad \dfrac{\phi \stackrel{C}{\Rightarrow} \phi \quad \phi \stackrel{C}{\Rightarrow} \phi}{\phi \stackrel{C}{\Rightarrow} \phi\wedge\phi}\, R_\wedge}{\stackrel{C}{\Rightarrow} (\phi\wedge\phi)\leftrightarrow\phi}\, R_\leftrightarrow}{\stackrel{I}{\Rightarrow} (\phi\wedge\phi)\equiv\phi}\, R^B_\equiv \qquad (\phi\wedge\phi)\equiv\phi, (\phi\vee\phi)\equiv\phi \stackrel{I}{\Rightarrow} (\phi\wedge\phi)\equiv\phi}{(\phi\vee\phi)\equiv\phi \stackrel{I}{\Rightarrow} (\phi\wedge\phi)\equiv\phi}\, cut$$

can be modified to exclude the use of *cut*. In this case we instead apply the weakening rule and obtain the following derivation:

$$\dfrac{\dfrac{\dfrac{\dfrac{\phi,\phi \stackrel{C}{\Rightarrow} \phi}{\phi\wedge\phi \stackrel{C}{\Rightarrow} \phi}\, L_\wedge \quad \dfrac{\phi \stackrel{C}{\Rightarrow} \phi \quad \phi \stackrel{C}{\Rightarrow} \phi}{\phi \stackrel{C}{\Rightarrow} \phi\wedge\phi}\, R_\wedge}{\stackrel{C}{\Rightarrow} (\phi\wedge\phi)\leftrightarrow\phi}\, R_\leftrightarrow}{\stackrel{I}{\Rightarrow} (\phi\wedge\phi)\equiv\phi}\, R_\equiv}{(\phi\vee\phi)\equiv\phi \stackrel{I}{\Rightarrow} (\phi\wedge\phi)\equiv\phi}\, wk$$

however in this particular case we need to know in advance that the components of (for example) equivalence are indeed valid in WB.

Rule $R^B_\equiv$ is the main reason for non-eliminable cut. Even if we were to consider a different set of sequent calculus identity-dedicated rules as a base for the SCI part of the calculus, we would be left with the same results. Let us consider the identity rule proposed by Michaels in [35] with additional labels, to fit into our framework:

$$\dfrac{\Gamma^{(n)}, \alpha_1, \beta_1, \Pi \stackrel{I}{\Rightarrow} \Delta^{(r)}, \Theta \quad \Gamma^{(n)}, \Pi \stackrel{I}{\Rightarrow} \Delta^{(r)}, \alpha_1, \beta_1, \Theta}{\Gamma^{(n)}, \alpha_1\equiv\beta_1, \ldots, \alpha_m\equiv\beta_m \stackrel{I}{\Rightarrow} \Delta^{(r)}, \gamma_1\equiv\eta_1, \ldots, \gamma_s\equiv\eta_s}\, L_\equiv$$

where $\Gamma^{(n)}$ and $\Delta^{(r)}$ are sequents of propositional variables:

$$\Gamma^{(n)} = \quad p_1, \ldots, p_n, \qquad \Delta^{(r)} = \quad q_1, \ldots, q_r$$

Π and Θ are sequences of equations: $\Pi :\quad \alpha_2\equiv\beta_2, \ldots, \alpha_m\equiv\beta_m, \alpha_1\equiv\beta_1, \beta_1\equiv\alpha_1, (\alpha_2\equiv\beta_2)[\alpha_1/\beta_1], \ldots,, (\alpha_m\equiv\beta_m)[\alpha_1/\beta_1]$,
$\Theta :\quad \gamma_1\equiv\eta_1, \ldots, \gamma_s\equiv\eta_s, (\gamma_1\equiv\eta_1)[\alpha_1/\beta_1], \ldots, (\gamma_s\equiv\eta_s)[\alpha_1/\beta_1]$
consisting of all possible results of substitutions of at least one occurrence of α_1 with β_1.

Proof of formula (♣):

$$\dfrac{\dfrac{\begin{array}{cc}\vdots & \vdots \\ \Gamma^1 \overset{I}{\Rightarrow} \Delta^1 & \Gamma^2 \overset{I}{\Rightarrow} \Delta^2\end{array}}{(\phi \equiv (\phi \wedge \chi)) \overset{I}{\Rightarrow} ((\phi \vee \chi) \equiv \chi)}\, L_{\equiv}}{\overset{I}{\Rightarrow} (\phi \equiv (\phi \wedge \chi)) \to ((\phi \vee \chi) \equiv \chi)}\, R_{\to}$$

where $\Gamma^1 = \{\phi, \phi \wedge \chi, \phi \equiv (\phi \wedge \chi), (\phi \wedge \chi) \equiv \phi, (\phi \wedge \chi) \equiv (\phi \wedge \chi), \phi \equiv ((\phi \wedge \chi) \wedge \chi), (\phi \wedge \chi) \equiv ((\phi \wedge \chi) \wedge \chi)\}$, $\Delta^1 = \{(\phi \vee \chi) \equiv \chi, ((\phi \wedge \chi) \vee \chi) \equiv \chi\}$ and $\Gamma^2 = \{\phi \equiv (\phi \wedge \chi), (\phi \wedge \chi) \equiv \phi, (\phi \wedge \chi) \equiv (\phi \wedge \chi), \phi \equiv ((\phi \wedge \chi) \wedge \chi), (\phi \wedge \chi) \equiv ((\phi \wedge \chi) \wedge \chi)\}$, $\Delta^2 = \{\phi, \phi \wedge \chi, (\phi \vee \chi) \equiv \chi, ((\phi \wedge \chi) \vee \chi) \equiv \chi\}$. In both cases the result is the same as the result of an application of $L^2_{\equiv}$, as we have non-empty antecedents and succedents, which means we are still unable to use $R^B_{\equiv}$. Therefore we would still be forced to use cut, although this issue could possibly be overcome through a modified/extended version of the substitution rule. $\Gamma^1 \Rightarrow \Delta^1$ and $\Gamma^2 \Rightarrow \Delta^2$ can both be obtained through the weakening of sequent $\Rightarrow ((\phi \wedge \chi) \vee \chi) \equiv \chi$. $((\phi \wedge \chi) \vee \chi) \equiv \chi$ is a theorem of WB, therefore the mentioned sequent can be proved with the use of $R^B_{\equiv}$. WB can be therefore formalized without the cut rule.

In lights of the issues discussed above, we can pose the following question: can we in some way control the use of *cut*, or at least restrict the content of the set of possible cut-formulae? We can discuss the issue of the subformula property regarding $\mathsf{G3_{WB}}$. The standard definition for the subformula property provides a property that cannot be met for the mentioned calculus, as identity-dedicated rules consist of formulae not meeting the conditions in it. In [60] it was shown that we can use an extended subformula property to show that the sequent calculus $\mathsf{G3_{ISCI}}$ (or rather its more minimalistic variant) is a decidability procedure.

The following measure used in [60] differs from the weight of the formula in value specified in condition 1.

Definition 111 (Complexity of a formula)**.** *By complexity of a formula of* $\mathcal{L}$ *we mean the following value:*

- $c(\phi) = 0$, *if* $\phi \in \mathsf{Var}$ *or* $\phi = \bot$ *or* $\phi = \top$*;*
- *when* ϕ *is of the form* $\neg\chi$*, then* $c(\phi) = c(\chi) + 1$*;*
- *when* ϕ *is of the form* $\chi \otimes \psi$*, with* $\otimes \in \{\wedge, \vee, \to, \leftrightarrow, \equiv\}$*, then* $c(\phi) = c(\chi) + c(\psi) + 1$*.*

Definition 112 ($ex.sub^{\mathsf{WB}}()$)**.** *Let* $\phi \in L$*,*

1. $\phi \in ex.sub^{\mathsf{WB}}(\phi)$*,*
2. *if* $\psi \in ex.sub^{\mathsf{WB}}(\phi)$ *and* χ *is a subformula of* ψ*, then* $\chi \in ex.sub^{\mathsf{WB}}(\phi)$*,*
3. *if* $\psi \in ex.sub^{\mathsf{WB}}(\phi)$ *and* $c(\psi \equiv \psi) \leqslant c(\phi)$*, then* $\psi \equiv \psi \in ex.sub^{\mathsf{WB}}(\psi)$*,*

4. *if* $\psi \equiv \chi \in ex.sub^{\mathsf{WB}}(\phi)$ *and* $c(\neg\psi \equiv \neg\chi) \leqslant c(\phi)$*, then* $\neg\psi \equiv \neg\chi \in ex.sub^{\mathsf{WB}}(\phi)$*,*

5. *if* $\psi_1 \equiv \psi_2 \in ex.sub^{\mathsf{WB}}(\phi), \chi_1 \equiv \chi_2 \in ex.sub^{\mathsf{WB}}(\phi)$ *and for* $\otimes \in \{\wedge, \vee, \rightarrow, \leftrightarrow, \equiv\}$*,* $c((\psi_1 \otimes \chi_1) \equiv (\psi_2 \otimes \chi_2)) \leqslant c(\phi)$*, then* $(\psi_1 \otimes \chi_1) \equiv (\psi_2 \otimes \chi_2) \in ex.sub^{\mathsf{WB}}(\phi)$*,*

Each element of $ex.sub^{\mathsf{WB}}(\phi)$ *is called an extended subformula of* ϕ*.*

In order to bypass the use of cut rule we can consider a new left-sided identity rule, which allows us to introduce any formula that falls under any of the Boole algebra axiom schemata:

$$\frac{\phi, \Gamma \stackrel{I}{\Rightarrow} \Delta}{\Gamma \stackrel{I}{\Rightarrow} \Delta}\ L^B_{\equiv}$$

where ϕ is one of the Boole axioms. Completeness of label-free sequent calculus with rule $L^B_{\equiv}$ has been shown in [59].

$L^B_{\equiv}$ can be obtained in $\mathsf{G3_{SCI}}$ through the application of $L^1_{\equiv}$, $L^3_{\equiv}$ and $L^4_{\equiv}$ or through the use of the cut rule, but in both cases we are forced to use weakening on both branches. For example, suppose ϕ is one of the Boole algebra axioms:

$$\cfrac{\cfrac{\cfrac{\phi, \Gamma \stackrel{I}{\Rightarrow} \Delta}{\phi, \phi, \phi \equiv \phi, \Gamma \stackrel{I}{\Rightarrow} \Delta}\ L_{wk} \qquad \cfrac{\begin{matrix} D_1 \\ \vdots \\ \stackrel{I}{\Rightarrow} \phi \end{matrix}}{\phi \equiv \phi, \Gamma \stackrel{I}{\Rightarrow} \Delta, \phi, \phi}\ L_{wk}, R_{wk}}{\cfrac{\phi \leftrightarrow \phi, \phi \equiv \phi, \Gamma \stackrel{I}{\Rightarrow} \Delta}{\phi \equiv \phi, \Gamma \stackrel{I}{\Rightarrow} \Delta}\ L^3_{\equiv}}\ L_{\leftrightarrow}}{\Gamma \stackrel{I}{\Rightarrow} \Delta}\ L^1_{\equiv}$$

D_1 is of course, a proof.

Now suppose we consider a quasi-analytic version of cut, where cut formula ϕ can only be one of the Boolean algebra axioms:

$$\frac{\stackrel{I}{\Rightarrow} \phi \quad \phi, \Gamma \stackrel{I}{\Rightarrow} \Delta}{\Gamma \stackrel{I}{\Rightarrow} \Delta}\ cut$$

The derivation seen above shows us that we can also simulate quasi-analytical cut by using left-sided identity rules, weakening and $R^B_{\equiv}$. Moreover, it is evident that in both rules, that is quasi-analytical cut and $L^B_{\equiv}$, we can consider not only Boolean algebra axioms, but also other Boolean algebra theorems.

We can now modify the above derivations of $(\phi \equiv (\phi \wedge \chi)) \leftrightarrow ((\phi \vee \chi) \equiv \chi)$

$$
\begin{prooftree}
\AxiomC{D_1}
\noLine
\UnaryInfC{$\vdots$}
\AxiomC{$\phi, \chi \overset{C}{\Rightarrow} \chi$}
\RightLabel{$L_\wedge$}
\UnaryInfC{$\phi \wedge \chi \overset{C}{\Rightarrow} \chi$}
\AxiomC{$\chi \overset{C}{\Rightarrow} \chi$}
\RightLabel{$L_\vee$}
\BinaryInfC{$(\phi \wedge \chi) \vee \chi \overset{C}{\Rightarrow} \chi$}
\AxiomC{$\chi \overset{C}{\Rightarrow} \phi \wedge \chi, \chi$}
\RightLabel{$R_\vee$}
\UnaryInfC{$\chi \overset{C}{\Rightarrow} (\phi \wedge \chi) \vee \chi$}
\RightLabel{$R_\leftrightarrow$}
\BinaryInfC{$\overset{C}{\Rightarrow} ((\phi \wedge \chi) \vee \chi) \leftrightarrow \chi$}
\RightLabel{$R^B_\equiv$}
\UnaryInfC{$\overset{I}{\Rightarrow} ((\phi \wedge \chi) \vee \chi) \equiv \chi$}
\RightLabel{L_{wk}, R_{wk}}
\UnaryInfC{$\delta, \phi \equiv (\phi \wedge \chi) \overset{I}{\Rightarrow} (\phi \vee \chi) \equiv \chi, \Delta$}
\RightLabel{$L_\leftrightarrow$}
\BinaryInfC{$\omega, \delta, \phi \equiv (\phi \wedge \chi) \overset{I}{\Rightarrow} (\phi \vee \chi) \equiv \chi$}
\RightLabel{$L^3_\equiv$}
\UnaryInfC{$\delta, \phi \equiv (\phi \wedge \chi) \overset{I}{\Rightarrow} (\phi \vee \chi) \equiv \chi$}
\RightLabel{$L^1_\equiv$}
\UnaryInfC{$\phi \equiv (\phi \wedge \chi) \overset{I}{\Rightarrow} (\phi \vee \chi) \equiv \chi$}
\RightLabel{$R_\to$}
\UnaryInfC{$\overset{I}{\Rightarrow} (\phi \equiv (\phi \wedge \chi)) \to ((\phi \vee \chi) \equiv \chi)$}
\end{prooftree}
$$

where δ stands for $(((\phi \wedge \chi) \vee \chi) \equiv \chi) \equiv (((\phi \wedge \chi) \vee \chi) \equiv \chi)$, ω stands for $(((\phi \wedge \chi) \vee \chi) \equiv \chi) \leftrightarrow (((\phi \wedge \chi) \vee \chi) \equiv \chi)$, $\Delta = \{((\phi \wedge \chi) \vee \chi) \equiv \chi, ((\phi \wedge \chi) \vee \chi) \equiv \chi\}$. Moreover, derivation D^1 is the following one:

$$
\begin{prooftree}
\AxiomC{$(\phi \vee \chi) \equiv \chi, (\phi \vee \chi) \equiv ((\phi \wedge \chi) \vee \chi), \chi \equiv \chi, \Delta, \delta, \phi \equiv (\phi \wedge \chi) \overset{I}{\Rightarrow} (\phi \vee \chi) \equiv \chi$}
\RightLabel{$trans$}
\UnaryInfC{$(\phi \vee \chi) \equiv ((\phi \wedge \chi) \vee \chi), \chi \equiv \chi, \Delta, \delta, \phi \equiv (\phi \wedge \chi) \overset{I}{\Rightarrow} (\phi \vee \chi) \equiv \chi$}
\RightLabel{$L^4_\equiv$}
\UnaryInfC{$\chi \equiv \chi, \Delta, \delta, \phi \equiv (\phi \wedge \chi) \overset{I}{\Rightarrow} (\phi \vee \chi) \equiv \chi$}
\RightLabel{$L^1_\equiv$}
\UnaryInfC{$\Delta, \delta, \phi \equiv (\phi \wedge \chi) \overset{I}{\Rightarrow} (\phi \vee \chi) \equiv \chi$}
\end{prooftree}
$$

Finally, at the end suppose we modify the right identity rule in the following way, which encompasses both weakening rules and $R^B_\equiv$:

$$
\begin{prooftree}
\AxiomC{$\overset{C}{\Rightarrow} \phi \leftrightarrow \chi$}
\RightLabel{$R^*_\equiv$}
\UnaryInfC{$\Gamma \overset{I}{\Rightarrow} \Delta, \phi \equiv \chi$}
\end{prooftree}
$$

The obtained rule is correct, it preserves the validity of a premiss and the proof can be constructed analogously as in the case of its previous version. However, unsurprisingly, the obtained rule is not invertible. Sequent calculus for WB can be therefore formalized without the cut rule and the weakening rule, using $R^*_\equiv$ instead of $R^B_\equiv$.

7.3 Cut issues in $\mathsf{G3_{WT}}$

For cut elimination procedure we use the base of the proof presented in Section 7.1. We will extend the proof by adding cases including identity-dedicated rules. However, ultimately we will show that for several cases cut cannot be eliminated.

If the cut-formula δ is an equation $\alpha \equiv \beta$, we could consider the following scenario for $R^T_{\equiv}$:

$$\dfrac{\dfrac{D_1}{\Gamma^{\equiv} \Rightarrow \delta} \quad \dfrac{\dfrac{D_2}{\delta, \Gamma^{\equiv'} \Rightarrow \phi \leftrightarrow \chi}}{\delta, \Gamma^{\equiv'} \Rightarrow \phi \equiv \chi}\, R^T_{\equiv}}{\Gamma^{\equiv}, \Gamma^{\equiv'} \Rightarrow \phi \equiv \chi}\, cut$$

where cut height is $h_1 + h_2 + 1$. We then transform the derivation in the following manner:

$$\dfrac{\dfrac{\dfrac{D_1}{\Gamma^{\equiv} \Rightarrow \delta} \quad \dfrac{D_2}{\delta, \Gamma^{\equiv'} \Rightarrow \phi \leftrightarrow \chi}}{\Gamma^{\equiv}, \Gamma^{\equiv'} \Rightarrow \phi \leftrightarrow \chi}\, cut}{\Gamma^{\equiv}, \Gamma^{\equiv'} \Rightarrow \phi \equiv \chi}\, R^T_{\equiv}$$

and obtain cut height $h_1 + h_2$.

We move to case (3.3) and consider the cut-formula principal in both premisses of cut; the cut-formula is of the form $\alpha \equiv \beta$.

$$\dfrac{\dfrac{\vdots}{\Gamma^{\equiv} \Rightarrow \alpha \equiv \beta} \quad \dfrac{\vdots}{\alpha \equiv \beta, \Gamma \Rightarrow \Delta}}{\Gamma^{\equiv}, \Gamma \Rightarrow \Delta}\, cut$$

For the left premiss of the cut we have only one possible rule to apply: $R^T_{\equiv}$. However, for the right premiss of the cut we can consider three subcases:

(3.3a) $\alpha \equiv \beta, \Gamma \Rightarrow \Delta$ has been obtained from $\alpha \equiv \beta, \neg\alpha \equiv \neg\beta, \Gamma \Rightarrow \Delta$ by means of rule $L^2_{\equiv}$,

(3.3b) $\alpha \equiv \beta, \Gamma \Rightarrow \Delta$ has been obtained from $\alpha \equiv \beta, \alpha \leftrightarrow \beta, \Gamma \Rightarrow \Delta$ by means of rule $L^3_{\equiv}$,

(3.3c) $\alpha \equiv \beta, \psi \equiv \omega, \Gamma' \Rightarrow \Delta$ (where $\psi \equiv \omega \cup \Gamma' = \Gamma$) has been obtained from $\alpha \equiv \beta, \psi \equiv \omega, (\alpha \otimes \psi) \equiv (\beta \otimes \omega), \Gamma \Rightarrow \Delta$ by means of rule $L^4_{\equiv}$.

For all of the listed cases we cannot assume how the previous steps of the derivation have been achieved. If it is the case that equations introduced to the derivation through cut application are not modified in the previous steps, we are unable to eliminate cut through the process of replacing $\alpha \equiv \beta$ with less complex formulae, as we do not have a rule in the system that allows us to synthesize equations (when applying rules bottom up). There is one rule which does that, $L^1_{\equiv}$, but it only allows reflexive equations to be introduced. We are unable to synthesize an arbitrary equation (in this case cut-formula $\alpha \equiv \beta$) through subsequent use of $L^1_{\equiv}$ and other left-sided identity rules.

Let us consider the following formula $(\phi \equiv (\phi \wedge \chi)) \to \delta$ and its proof:

$$\dfrac{\dfrac{\begin{array}{c}D_1\\ \vdots\\ \Rightarrow (\phi \equiv \chi) \equiv (\chi \equiv \phi)\end{array} \quad \dfrac{\delta, \Gamma \Rightarrow \delta}{(\phi \equiv \chi) \equiv (\chi \equiv \phi), \phi \equiv (\phi \wedge \chi) \Rightarrow \delta}\, L^4_{\equiv}}{(\phi \equiv (\phi \wedge \chi)) \Rightarrow \delta}\, \mathit{cut}}{\Rightarrow (\phi \equiv (\phi \wedge \chi)) \to \delta}\, R_{\to}$$

where $\Gamma = \{(\phi \equiv \chi) \equiv (\chi \equiv \phi), \phi \equiv (\phi \wedge \chi)\}$, $\delta = ((\phi \equiv (\phi \equiv \chi)) \equiv ((\phi \wedge \chi) \equiv (\chi \equiv \phi)))$ and D_1 is a proof, since $\vdash_{\mathsf{WT}} (\phi \equiv \chi) \equiv (\chi \equiv \phi)$. If we had not applied the cut rule and if our proof building procedure had been focused on the systematic use of rules aimed at obtaining less and less complex formulae, we would not have been able to construct a proof. The above example shows us that for certain formulae our goal should not be to obtain propositional variables, but to construct equations in the antecedent of a sequent of the same structure as the identity in the succedent of a sequent.

Here we arrive at similar point as in the case of G3$_{\mathsf{WB}}$: we can overcome this issue, but not without a cost. In the derivation above through *cut* we introduced formula $(\phi \equiv \chi) \equiv (\chi \equiv \phi)$. We can also introduce it through the joint application of $L^1_{\equiv}$, $L^3_{\equiv}$ and $R_{\leftrightarrow}$. However, as a result we obtain more formulae in the succedent of two obtained premisses.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\delta, \alpha, \alpha, \alpha \equiv \alpha, \phi \equiv (\phi \wedge \chi) \Rightarrow \delta}{\alpha, \alpha, \alpha \equiv \alpha, \phi \equiv (\phi \wedge \chi) \Rightarrow \delta}\, L^4_{\equiv} \quad \dfrac{\begin{array}{c}D_1\\ \vdots\\ \Rightarrow \alpha\end{array}}{\alpha \equiv \alpha, \phi \equiv (\phi \wedge \chi) \Rightarrow \delta, \alpha, \alpha}\, L_{wk}, R_{wk}}{\alpha \leftrightarrow \alpha, \alpha \equiv \alpha, \phi \equiv (\phi \wedge \chi) \Rightarrow \delta}\, L_{\leftrightarrow}}{\alpha \equiv \alpha, \phi \equiv (\phi \wedge \chi) \Rightarrow \delta}\, L^3_{\equiv}}{\phi \equiv (\phi \wedge \chi) \Rightarrow \delta}\, L^1_{\equiv}}{\Rightarrow (\phi \equiv (\phi \wedge \chi)) \to \delta}\, R_{\to}$$

where $\alpha = (\phi \equiv \chi) \equiv (\chi \equiv \phi)$. Therefore, we are yet again required to use the weakening rule, so we apply $R^T_{\equiv}$ to one of the premisses. We propose a solution similar to that we offered for G3$_{\mathsf{WB}}$—modified right-identity rule which will be correct, but not invertible (due to the built-in weakening on both sides):

$$\dfrac{\Gamma^{\equiv} \Rightarrow \phi \equiv \chi}{\Gamma, \Gamma^{\equiv} \Rightarrow \phi \equiv \chi, \Delta}$$

The standard approach to the *cut* admissibility procedure, in which we move the application of *cut* upwards until we reach leaves, is not applicable in G3$_{\mathsf{WT}}$. However, through analysis of the set of extended subformulae of our initial problem, we can use identity-dedicated rules to introduce these extended subformulae into our derivation, thus omitting the necessity of using *cut*. If we were to construct such a set for formula δ, we would consider the set of extended formulae of δ, it would contain $\alpha, \alpha \equiv \alpha$ and $\alpha \leftrightarrow \alpha$.

7.4 Cut issues in $\mathsf{G3_{WH}}$

We have built $\mathsf{G3_{WH}}$ on the basis of $\mathsf{G3_{WT}}$, therefore we know that certain issues regarding the admissibility of structural rules will be inherited; formulae which were problematic for $\mathsf{G3_{WT}}$ will remain problematic in $\mathsf{G3_{WH}}$. It is worth noting, though, that the added rule $L^5_\equiv$ does not create new problems in the proof for cut elimination. In a standard cut elimination procedure we described in previous chapters, we consider the following situation, where the cut-formula is not principal in the left premiss of cut:

$$\dfrac{\dfrac{\Gamma, (\phi \equiv \chi) \equiv \top \Rightarrow \Delta, \delta \quad \Gamma, (\phi \equiv \chi) \equiv \bot \Rightarrow \Delta, \delta}{\Gamma \Rightarrow \Delta, \delta}\, L^5_\equiv \qquad \delta, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\, cut$$

and we can change the order of the applied rules, thus lowering the overall height of the cut application(s).

$$\dfrac{\dfrac{\Gamma, (\phi \equiv \chi) \equiv \top \Rightarrow \Delta, \delta \quad \delta, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma', (\phi \equiv \chi) \equiv \top \Rightarrow \Delta, \Delta'}\, cut \quad \dfrac{\Gamma, (\phi \equiv \chi) \equiv \bot \Rightarrow \Delta, \delta \quad \delta, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma', (\phi \equiv \chi) \equiv \bot \Rightarrow \Delta, \Delta'}\, cut}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\, L^5_\equiv$$

The case in which the cut-formula is principal in the left premiss only is analogous, and we start with the following derivation:

$$\dfrac{\Gamma' \Rightarrow \Delta', \delta \qquad \dfrac{\delta, \Gamma, (\phi \equiv \chi) \equiv \top \Rightarrow \Delta \quad \delta, \Gamma, (\phi \equiv \chi) \equiv \bot \Rightarrow \Delta}{\delta, \Gamma \Rightarrow \Delta}\, L^5_\equiv}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\, cut$$

and modify it into:

$$\dfrac{\dfrac{\Gamma' \Rightarrow \Delta', \delta \quad \delta, \Gamma, (\phi \equiv \chi) \equiv \bot \Rightarrow \Delta}{\Gamma, \Gamma', (\phi \equiv \chi) \equiv \bot \Rightarrow \Delta, \Delta'}\, cut \quad \dfrac{\Gamma' \Rightarrow \Delta', \delta \quad \delta, \Gamma, (\phi \equiv \chi) \equiv \top \Rightarrow \Delta}{\Gamma, \Gamma', (\phi \equiv \chi) \equiv \top \Rightarrow \Delta, \Delta'}\, cut}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\, L^5_\equiv$$

where the cut application height(s) is (are) lesser than in the original derivation. We also know that these are the only two cases we would have to consider regarding $L^5_\equiv$'s role in the cut elimination procedure. We therefore know that if in our derivations we are using $L^5_\equiv$ but not $R^T_\equiv$, then we can avoid the unnecessary use of *cut*, as it is rule $R^T_\equiv$ which has an adverse effect on the cut admissibility procedure. Again, we can consider a similar option with modification of the aforementioned troublesome rule, but the alternative approach pressures us to use the (built-in or separate) weakening rule. Worth noting is that in standard sequent formalization of modal logic $\mathsf{S5}$ [37], to which WH corresponds, cut elimination theorem cannot be proved. In $\mathsf{G3_{WH}}$ the rule that determines its similarity to $\mathsf{S5}$ is, in this particular context, harmless.

Chapter 8

Final remarks

The aim of the work was to develop and examine system sequents for three non-Fregean theories proposed by Roman Suszko: WB, WT, and WH. Non-Fregean systems reject the so-called Fregean Axiom

$$(\phi \leftrightarrow \chi) \rightarrow (\phi \equiv \chi),$$

which equates equivalence with the identity connective. Suszko introduced the identity connective to mark the difference between semantic correlates; in Fregean systems, sentences are names of two truth values, whereas in non-Fregean systems we rely on the notion of situation (stemming from the ontology of Wittgenstein's *Tractatus*). The weakest non-Fregean system proposed by Suszko is SCI, and the three theories examined in this thesis are its axiomatic extensions.

Sequent calculi for these three non-Fregean theories have not been proposed before; in literature we can find proof systems mostly for SCI [4; 14; 16; 19; 28; 35; 41; 45; 46; 59; 66; 67], as well as its intuitionistic counterpart, ISCI [5; 6; 11; 60]. The systems $\mathsf{G3}_{\mathsf{WB}}$, $\mathsf{G3}_{\mathsf{WT}}$, and $\mathsf{G3}_{\mathsf{WH}}$ were built on the basis of sequent calculus $\ell\mathsf{G3}_{\mathsf{SCI}}$ [4], which has been adapted to slightly different language, where we use negation and equivalence. In both $\mathsf{G3}_{\mathsf{WB}}$ and $\mathsf{G3}_{\mathsf{WT}}$, one right-sided rule has been added, as a formalization of the consequence operation-based definition of the two systems. In $\mathsf{G3}_{\mathsf{WB}}$ we additionally use two labels, $\overset{C}{\Rightarrow}$ and $\overset{I}{\Rightarrow}$, to have better control over the application of the rules. These labels play a crucial role in the formalization of right-sided identity rule $R^B_{\equiv}$:

$$\frac{\overset{C}{\Rightarrow} \phi \leftrightarrow \chi}{\overset{I}{\Rightarrow} \phi \equiv \chi}\ R^B_{\equiv}$$

which is a formalization of $\mathsf{WB} = C(\{\phi \equiv \chi : (\phi \leftrightarrow \chi) \in \mathsf{TFT}\})$. Application (bottom-up) of $R^B_{\equiv}$ to a sequent changes its label from $\overset{I}{\Rightarrow}$ to $\overset{C}{\Rightarrow}$, which disables the possibility of further use of other identity rules. This mechanism makes it impossible to create proofs for formulae from other extensions. However, this particular shape of the rule also makes it difficult to omit the cut application (in some particular cases we discussed in the previous chapter). We therefore obtain a system that is sound and complete with regard to Boole algebra semantics, but lacks the cut elimination property.

Similar issues are encountered in the case of $\mathsf{G3_{WT}}$. In this system we omit the labels, but we add the similar right-sided identity rule $R^T_\equiv$:

$$\frac{\Gamma^\equiv \Rightarrow \phi \leftrightarrow \chi}{\Gamma^\equiv \Rightarrow \phi \equiv \chi}\ R^T_\equiv$$

which is a formalization of $\mathsf{WT} = C(\{\phi \equiv \chi : (\phi \leftrightarrow \chi) \in C(\emptyset)\})$. We now allow the antecedent of a sequent to be non-empty, but it can only consist of other equations. The aim of this restriction is to disable the possibility of proving the Fregean Axiom. The fact that the right-sided rule is obtained by means of some other strategy (in the case of $\mathsf{G3_{SCI}}$ upon which it is based, the rules characterizing identity are directly linked to axioms characterizing identity, and are not linked to the overall definition of the logic), seems to complicate the cut elimination procedure. We do not claim that the cut elimination theorem does not hold for $\mathsf{G3_{WT}}$, but we have identified formulae whose proofs contain the application of the cut rule. As a result we again obtain a system which is sound and complete with regard to topological Boole algebra semantics, but most likely lacks the cut elimination property.

In $\mathsf{G3_{WH}}$ we use $\mathsf{G3_{WT}}$ as the base and come back to the strategy of obtaining sequent rules from axioms. As a result we add the new left-sided identity rule $L^5_\equiv$

$$\frac{(\phi \equiv \chi) \equiv \top, \Gamma \Rightarrow \Delta \quad (\phi \equiv \chi) \equiv \bot, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ L^5_\equiv$$

where, looking bottom-up, at any point in the derivation we can add two pieces of information to the antecedent of each premiss: that either $\phi \equiv \chi$ is necessary or impossible. This rule shares similarities with the cut rule, although its presence in the system does not cause any issues with the cut elimination procedure. However, as the system is based on $\mathsf{G3_{WH}}$, it inherits the same cut elimination issues linked to the rule $R^T_\equiv$. Even though the cut elimination is still problematic, the overall system is sound and complete with regard to Henle algebra.

Non-Fregean theories constitute an interesting field for further examination. WT's and WH's correspondence to modal logics, respectively, $\mathsf{S4}$ and $\mathsf{S5}$ points to the fact that cut elimination could be proved for both systems. We could therefore examine other variants of sequent systems, such as labelled systems or hypersequent calculi, to examine whether these modifications would make it possible to build cut-free proofs. Suszko did not point out whether logic WB corresponds to some modal system. It would be worth pursuing this analysis, as it could shed some light on the encountered issues. It would also be interesting to study intermediate systems between the three main extensions.

There are of course numerous other directions for non-Fregean research. In the case of theories analyzed in this thesis we focus on the extensional language originating from CPC. Identity can also be studied in other non-extensional languages and in the context of non-classical systems, e.g., in intuitionistic setting. There are proof systems for ISCI, but the analysis concerned with intuitionistic counterparts of the main axiomatic extensions of SCI have been initiated as well. In these intuitionistic counterparts we have to study the behavior of the identity connective, particularly in the axioms; certain axioms would have to be omitted or reformulated to make sure the law of excluded middle is not smuggled under the cover of equation content. Non-Fregean identity in an intuitionistic context, gradually adopting different properties, would certainly provide an intriguing research direction within the realm of structural proof theory.

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