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Report on the scientific achievement
"Dynamical properties of operators acting on spaces of smooth and holomorphic functions"
by Adam Przestacki

The work presented by Adam Przestacki for the conferment of the degree of doctor habilitated makes important contributions to the dynamics of linear operators, an area of functional analysis and operator theory that has originated from the invariant subspace problem at the end of the last century and which has since then been studied intensively. The main notions in linear dynamics are that of a hypercyclic operator, that is, an operator that admits a dense orbit, and that of a chaotic operators, that is, a hypercyclic operator that admits a dense set of periodic points. Further notable concepts are that of a mixing operator and that of a frequently hypercyclic operator, both of which are considerably stronger than hypercyclicity. While there exists a myriad of hypercyclic operators, many of them fall into one of three categories: weighted shifts on sequence spaces, weighted composition operators and differential operators on function spaces.

Adam Przestacki presents six papers, [A] - [F], which show both a remarkable coherence of his work and a great breadth of his expertise. Five of these papers concern weighted composition operators, while one is dedicated to the invariant subspace problem. We briefly discuss these papers.

The papers [A], [C]-[F], study dynamical properties of weighted composition operators on spaces of smooth and holomorphic functions. A weighted composition operator, acting on a function f, is given by $C_{w,\psi}f = w(f \circ \psi)$, that is, a composition of f with a function ψ, followed by a multiplication by a weight function w.

In [A], Adam characterizes when a weighted composition operator is hypercyclic on the space $C^\infty(\Omega)$ of smooth functions on an open set $\Omega \subset \mathbb{R}^d$; he also characterizes when it is mixing. Surprisingly, not even the unweighted case, that is, when $w \equiv 1$, has been studied in the literature before. Adam's study was subsequently generalized by Kalmes (2019), and it motivated Albanese et al. (2022) to study further dynamical properties of these operators.

In [D], Adam continues the work started in [A]. He shows that every hypercyclic weighted composition operator on $C^\infty(\mathbb{R})$ automatically has the much stronger property of being frequently hypercyclic. The proof is not at all straightforward. One important tool is the study of solutions of the Abel functional equation, which permits to pass from the simpler case of the symbol $\psi(x) = x + 1$ to general symbols ψ.

In [E], Adam studies weighted composition operators on the space $H(\Omega)$ of holomorphic functions on a domain $\Omega \subset \mathbb{C}$. While the hypercyclicity of unweighted composition operators had previously been characterized, the weighted case had only been studied for simply connected domains, which was obtained by Bès (2014). Adam shows that, for the latter domains, every hypercyclic weighted composition operator is even frequently hypercyclic. For domains of the form $\Omega^* = \Omega \setminus \{0\}$, where Ω is simply connected and contains $\{0\}$, and for infinitely connected domains, Adam characterizes the hypercyclic weighted composition operators. In particular, he answers a question of Bès (2014) by showing that $\mathbb{C}^*$ supports such an operator. And he shows that no other domain, that is, no finitely connected domain that is not conformally isomorphic to some Ω^*, can support a hypercyclic weighted composition operator.

We note that papers [A], [D] and [E] were the starting point of the thesis by Foster (2024) under the guidance of Bès. In a subsequent paper, Bès and Foster (2025) solve a problem posed by Adam in [E].

In [C], Adam studies weighted translation operators on the Schwartz space $\mathcal{S}(\mathbb{R})$, the space of rapidly decreasing functions; in that case, the symbol is $\psi(x) = x + 1$. He characterizes when such an operator is hypercyclic, when it is mixing, and when it is chaotic. It is interesting to note that, as Adam shows, the space $\mathcal{S}(\mathbb{R})$ does not admit any hypercyclic (unweighted) composition operator.

In [F], Adam turns to the space $\mathcal{O}_M(\mathbb{R})$ of smooth functions of moderate growth on $\mathbb{R}$, which appears as the multiplier space of $\mathcal{S}(\mathbb{R})$. This space is of particular interest because, unlike the other spaces studied by Adam so far, it is not a Fréchet space: it is a non-metrizable locally convex space on which the Baire category theorem is no longer available. As a result, sequential hypercyclicity, hypercyclicity and topological transitivity no longer necessarily coincide, which calls for new proof techniques. As his main result, Adam obtains a characterization when a composition operator is mixing on $\mathcal{O}_M(\mathbb{R})$. He shows that the mixing property is closely related to the solvability of the Abel functional equation. In addition, the translation operator is shown to be sequentially hypercyclic on $\mathcal{O}_M(\mathbb{R})$.

Finally, paper [B] is dedicated to the invariant subspace problem. This problem asks whether, on a given separable Banach or Fréchet space, every operator admits a non-trivial closed invariant subspace. While the answer was shown to be negative on some Banach spaces by Enflo and Read in the 1980s, the problem seems to remain open in Hilbert spaces, in spite of a recent effort by Enflo (2023). The case of non-normable Fréchet spaces was studied thoroughly by Menet (2018). At the end of his paper, Menet asked whether the invariant subspace problem has a positive answer for Fréchet spaces with a Schauder basis whose topology is defined by an increasing sequence $(p_j)_j$ of seminorms so that, for every $j \geq 1$, ker p_{j+1} is of infinite codimension in ker p_j. In his paper, Adam gives a negative answer to this question by constructing an operator without non-trivial closed invariant subspaces on the space $C^\infty(\mathbb{R})$ of smooth functions. Adam's result was subsequently generalized by Menet (2021). For paper [B], Adam obtained the 2021 Ames Award, a best-paper award by the *Journal of Mathematical Analysis and Applications.* This is quite remarkable for a young author, even more so in view of the paucity of awards in Mathematics.

Adam's six papers contain a wealth of deep results that are of immense interest to the international community in linear dynamics. Many of his results are definitive answers to natural problems, some of which had previously been posed in the literature. A nice additional feature is that throughout his papers, Adam suggests or explicitly poses open problems. As we have seen, some of them have already been answered. Future work can certainly be expected.

It is fair to say that, with his papers, Adam Przestacki has become the international authority on weighted composition operators on spaces of smooth and holomorphic functions, and a recognized expert on the invariant subspace problem on Fréchet spaces.

In conclusion, I strongly recommend to award the degree of doktor habilitowany (habilitated doctor) to dr Adam Przestacki.

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