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Topics on Topological robotics

On topological complexity of Eilenberg-MacLane spaces and effective topological complexity.

A doctoral dissertation in natural sciences
in the area of mathematics
Advisor: prof. dr. hab. Wacław Bolesław Marzantowicz
Associate advisor: dr. Zbigniew Błaszczyk

Tematy dotyczące robotyki topologicznej

O złożoności topologicznej przestrzeni Eilenberga-MacLane'a i efektywnej złożoności topologicznej.

Rozprawa doktorska z nauk ścisłych i przyrodniczych w zakresie matematyki
Promotor: prof. dr hab. Wacław Bolesław Marzantowicz
Promotor pomocniczy: dr Zbigniew Błaszczyk

Wydział Matematyki i Informatyki
Uniwersytet im. Adama Mickiewicza w Poznaniu
2024

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On topological complexity of $K(G, 1)$-spaces and effective topological complexity.

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Abstract

In this dissertation we work on several problems concerning the relationship between topological complexity and sectional category, and groups. The work presented here can be divided in two main branches.

In the first part of the thesis, we investigate topics related with the description of the topological complexity of Eilenberg-MacLane spaces. First we develop the notion of sectional category of group monomorphisms, as a more general framework of study and which contains the original problem, and we provide a generalization of a characterization from Farber, Grant, Lupton and Oprea of TC of a group in terms equivariant maps to the classifying space of full families of subgroups.

We also develop a relative canonical class in this setting, and study its properties. Additionally, we introduce the notion of Adamson cohomology theory into the study of $\mathrm{secat}(H \hookrightarrow G)$. We will proceed as well to generalize the notion of essential cohomology classes to arbitrary group monomorphisms, and to build a more general version of the Farber-Mescher spectral sequence in order to get a new bound for $\mathrm{secat}(H \hookrightarrow G)$, which we will specialize to obtain new lower bounds of sequential and fiberwise TC. To finish this first part, we provide a characterization of TC of a group G in terms of the $\mathcal{A}$-genus in the sense of Clapp and Puppe.

In the second part, we switch our point of view, and consider, instead of $K(G, 1)$-spaces, actions of groups over spaces, and so we investigate some properties of the effective topological complexity of Błaszczyk and Kaluba. First we develop a notion of effective LS-category, and then we observe the relationship between the effective TC and cat and the orbit map with respect to the action in some situations, giving several computations and examples. We will finish by providing cohomological arguments to determine cases in which such effective TC is non-zero in dimension two.

Tematy dotyczące robotyki topologicznej

Na złożoność topologiczna $K(G,1)$-przestrzeni i efektywnej złożoności topologicznej.

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Abstrakt

W przedstawionej rozprawie doktorskiej omawiany kilka dotyczących relacji pomiędzy złożonością topologiczną , kategorią sekcyjną i grupami. Merytorycznie treść pracy podzielona jest na dwa główne nurty tematyczne.

W pierwszej części dysertacji, badamy zagadnienia związane z opisem złożoności topologicznej , oznaczanej przez TC, przestrzeni Eilenberga-MacLaina. Po pierwsze wprowadzamy pojęcie kategorii sekcyjnej monomorfizmów grup jako ogólne narzędzie do badań, które pozwala też opisać postawiony pierwotnie problem. Następnie uzyskujemy uogólnienie charakteryzacji Farbera, Grant, Luptona i Oprei TC grupy w terminach odwzorowań współzmienniczych do przestrzeni klasyfikujących pełne rodziny podgrup.

Używając wprowadzonych pojęć określamy kanoniczną relatywną klasę i badamy jej własności. Dodatkowo pokazujemy, że do badania $\mathrm{secat}(H \hookrightarrow G)$ można wykorzystać pojęcie kohomologii Adamsona. Kolejno, uogólniamy określenie pojęcia istotnych klas kohomologii do przypadku dowolnych monomorfizmów grup, i konstruujemy bardziej ogólną wersję ciągu spektralnego Farbera-Meschera. To ostatnie pozwala uzyskać nowe ograniczenie na $\mathrm{secat}(H \hookrightarrow G)$, które wykorzystujemy aby otrzymać nowe ograniczenie dolne na ciągową i włóknistą złożoność topologiczną. Na koniec tej części podajemy charakteryzację TC grupy G w terminach $\mathcal{A}$-genusu w sensie Clapp i Puppe.

W drugiej części przedstawiamy pewne własności efektywnej topologicznej złożoności w sensie Błaszczyka i Kaluby dla przestrzeni z działaniem grupy. Po pierwsze wprowadzamy pojęcie efektywnej LS-kategorii, a następnie opisujemy związki pomiędzy efektywną złożonością topologiczną, kategorią i odwzorowaniem rzutowania na przestrzeń orbit w wybranych przypadkach podając obliczenia i przykłady. Na zakończenie podajemy warunki kohomologiczne pozwalające określić w jakich przypadkach efektywna złożoność topologiczna jest niezerowa w wymiarze dwa.

Cuando eres niño te advierten:
«Limítate a contemplarlas.
Si las tocas, las espectrales
te dejarán su quemadura,
la marca a fuego, el estigma
de quien codicia lo prohibido.»
Quizá dijiste en silencio:
«Pretendo asir la marea,
acariciar lo imposible.»
— José Emilio Pacheco, *Las flores del mar*

A mis padres y a mi amada Magdalena
que son, respectivamente
cimientos y luz de mi existencia

Acknowledgements

I would like to start, as it could not be in any other way, by expressing my deepest gratitude to my advisor, Prof. Wacław Marzantowicz, for all his steady support through the elaboration of this work, his invaluable help, mentorship and advice in many different aspects (both mathematical and practical), his insight and expertise, and his enduring patience (specially during the last months of the preparation of this manuscript). The present work would not have been possible in any way without his crucial support, and it has been a real privilege to develop my research under his supervision.

I want to extend my gratitude as well to my coadvisor, Dr. Zbigniew Błaszczyk, who introduced me to many of the problems and topics that have inspired the present work. His ideas and influence have been fundamental throughout my whole research so far, and his personal and practical support were crucial, particularly during the tough beginnings of my PhD studies in Poznan (specially during that infamous first year marked by the pandemic!). The very same gratitude is due, as well, to Dr. José Gabriel Carrasquel Vera, who first introduced me to the realm of Topological Complexity, who first gave me the opportunity to internationalize my career, and to grow abroad as a researcher, and who taught me in practice so many things of what, in daily praxis, means to be a mathematician. Even if both of them are no longer involved in pure mathematics research (and I do surely miss their presence there!) I know I can always count on their honest and friendly advice and their genuine interest and support.

I also want to express the deepest of gratitudes to Prof. Antonio Viruel Arbáizar, to whom I hold a mathematical debt of difficult, if even possible, retribution. Since the very first time I met him, he has always been a source of mathematical inspiration, learning and insight, and I can not thank enough all these years of support, mentorship and encouragement, of selfless advice and, of course, of most valuable mathematical discussion and collaboration, which I surely hope it will keep going for many years to come.

I am most thankful as well to Dr. Stephan Mescher for his enthusiastic collaboration and his genuine mathematical and personal interest since I first presented to him my ideas for a joint project. I am very grateful for how seriously he took our collaboration since the very beginning, for his many stimulating ideas, insights and mathematical discussions, for his honest encouragement and his continued interest, both mathematical and personal and, of course, for his unlimited archive of jokes! I am looking forward to develop the projects we already have, and to see the ideas that we may develop in the future.

I am indebted as well to Professors Michael Farber and John Oprea, for their good disposition to collaborate, their willingness to review and discuss many ideas and drafts, even before our formal collaboration was stablished, and their insights and expertise. It has been a true privilege to be able to collaborate with such leading experts of the field at this early stage of my career.

My thanks as well to both Universidad de Málaga and Martin-Luther-Universität Halle-Wittenberg for their hospitality during the respective research visits I conducted at such institutions, and where different parts of this work were carried out.

I feel the need to express my gratitude to Prof. Enrique Pardo Espino, who was my main personal reference of what a mathematician must be during my bachelor days, and who guided my first steps into the world of Homological Algebra and Group Cohomology, both fundamental fields for the methods used in this dissertation. Also, I must not forget Julio César Barrios Santamaría and María Jesús Santamaría, my primary and high school teachers, for so much care and encouragement to always pursue what I truly wanted. My thanks also to Ms. Magdalena Stachowiak, for her nearly unlimited patience and her invaluable expertise and guidance through the dense jungle that is sometimes the polish bureocracy.

I had the immense luck to be blessed in my life with some truly great friends who, despite the (geographical) distance, have been at my side all along this journey, and this dissertation is also partly yours: Jesús, with his unlimited faith in me and support that I will never be able to thank enough; Guillermo, always eager to discuss from the most simple to the deepest of topics, and to bring a smile in the bleakest moments with his jokes; Ana and Amelia, always there offering a helping hand in the toughest times, a reassuring opinion, and a fascinating conversation; Óscar and Miguel Ángel, never passing an opportunity to ask how the day went and to offer support; Eme, María and Pontones, comrades and friends through so many experiences, always giving me the best advice, and the most enjoyable of ”welcome back meetings"; Paula, with whom I can always voice my most honest thoughts and doubts, and reminisce of so many moments lived together; Juanjo, Alberto, Taray and Carlos, my childhood friends who have journeyed at my side for so many years, despite the large geographic distance that separate us now; my bachelor friends Daniel, Rubén, Luis, Israel, Manuel... even though life separated some of us already, I always remember the bright days together.

Since the very moment she came to my life, my beloved Magdalena has been filling it with light, hope and happiness, and her unlimited love and support have been my biggest source of strength during these years. She has been always there to listen to all my worries, to give me her shoulder in the most difficult times, to embrace my soul with her love and encourage me to try to give always the best version of myself. This dissertation is as much hers as it is mine.

Finally, I want to thank my brother for these years of trust, support and encouragement in his specific way and, most importantly, I want to thank my parents, who have always been the strongest pillars of my life. They brought me to this world but, more significantly, they have always given me their inconditional love, support and trust, their most honest advice and encouragement, and have done as much as was in their power to make sure I could always pursue the path I wanted in life. None of this could have been possible without them. I owe to them the foundations of all my virtues; my sins are mine alone.

To all of you, thank you!

This work has been financially supported by a scholarship of the Doctoral School of Exact Sciences of Adam Mickiewicz University. I have also received the support of the Narodowy Centrum Nauki (Polish National Science Center) research grant UMO-2022/45/N/ST1/02814 PRELUDIUM, and the Excellence Initiative - Research University project 021/13/UAM/0051.

Outcome of this thesis

From the work presented in this document, the following papers have stemmed, in different stages of the publication process:

1. *On the sectional category of subgroup inclusions and Adamson cohomology theory.* Joint with Z. Błaszczyk and J.G. Carrasquel-Vera. Published in Journal of Pure and Applied Algebra Volume 226, Issue 6, June 2022, 106959.
2. *Sectional category and sequential topological complexity of aspherical spaces as* $\mathcal{A}$-genus In latter stage of preparation.
3. *On properties of effective topological complexity and effective Lusternik-Schnirelmann category.* Joint with Z. Błaszczyk and A. Viruel. Submitted.
4. *Sequential topological complexity of aspherical spaces and sectional category of subgroup inclusions.* Joint with M. Farber, S. Mescher and J. Oprea. Submitted.

Additionally and unrelated to the lines of work explored in this dissertation, the following paper is currently in preparation:

- *Lefschetz number of equivariant mapping defined by equivariant cohomology theory.* Joint with W. Marzantowicz.

Contents

CHAPTER 1

Introduction

> *"Bo nie ma cofanki. Bo dla nas kierunek ruchu jest jeden: naprzód. Bardzo możliwe że po kole, czy po elipsie, czy śrubowej spirali - ale na przód."*
>
> — Edward Stachura, *Się*

> *"No se pasa de la ceguera a la luz, no se entra en los soberanos dominios del sol como quien entra en un teatro. Es este un nacimiento en que hay también mucho dolor"*
>
> — Benito Pérez Galdós, *Marianela*

The term *robot* comes from the slavic root of "robot-", with meanings connected to work in different slavic languages (such as the polish *robota*). The first account of the word to designate some kind of "artificial worker" was due to Karel Čapek in 1921, in his theatre play *R.U.R* (*Rossum's Universal Robots*), while the term *robotics* was coined by the famous science-fiction writer Isaac Asimov to refer to the rising scientific discipline concerned with the study of robots. Interestingly, it is not easy to define precisely what a robot is, as the word means different things to different people, with even roboticists themselves having discrepancies about what could or could not be considered as a robot. The Robot Institute of America, in 1979, suggested the following definition:

> *A robot is a reprogrammable, multifunctional manipulator designed to move material, parts, tools, or specialized devices through various programmed motions for the performance of a variety of task.*

Discussion over the specifics of the definition aside, it is generally accepted that the ultimate objective in the field of robotics is, in the words of J.C. Latombe ([84]), the creation of fully autonomous robots, which accept high-level descriptions of tasks and execute them without further human intervention. Such an endeavor raises numerous and complex problems with plenty of ramifications and applications on the field. Certainly, one of the most fundamental of such problems is the so called *motion planning problem*. Picture a robot displacing over a space, starting from a point A and striving to reach a determined endpoint B, naturally avoiding all the possible obstacles that could be present in such space. The motion planning problem consists in providing a path, connecting the two known points, that our robot can follow to fulfill its mission. Here we have to take into account that "robot" is understood, in consonance with the aforementioned objective, in a broader sense, as any kind of mechanical or digital device capable of automatic process, with a determined space of possible “states". Such space is known as the *configuration space* of the system, and it is naturally equipped with a topology, which opens up the way to consider a mathematical (specifically topological) approach to study the different robotical problems and, as such, conduces naturally to the growing field of Topological Robotics.

As such, given any topological space X, the motion planning problem over X consists on providing an algorithm which, given any two points $x, y \in X$ as input, returns as an output a path γ with $\gamma(0) = x$ and $\gamma(1) = y$. Naturally, one would hope for such a motion planning to be “stable¨, i.e. continuous on the pair of points. Such continuity means that small changes in either of the extreme state points translates into a predictable change of the path followed by the robot. Unfortunately, this is hardly possible: the only configuration spaces for which such a continuous motion planning algorithm is possible are those who are contractible, and this is usually not the case, save for very basic situations. Of course, the fact that this stability of the system is only possible for such simple (topologically speaking) spaces indicates a relationship between the appearence of discontinuities in the system and topological features of the base configuration space. Therefore, the question arise on how to make this relationship precise.

In order to adress such question, Michael Farber developed in 2003 the notion of *topological complexity*, denoted by $\mathrm{TC}(X)$, as a topological tool to measure this degree of instability of mechanical systems. He was indeed inspired by previous work on the topological study of the complexity of algorithms carried out by Smale [112] and Vassiliev [115]. As all the information about the discointinuity of the motion planning algorithm is encoded in the topological features of the space, this topological complexity is an homotopy invariant of it, and, as such, constitues not only a tool of interest from the point of view of robotics, but also a valuable invariant from a purely mathematical point of view. Indeed, the topological complexity has a close connection with classic homotopy invariants previouly known, such as the *Lusternik-Schnirelmann category*, or the *sectional category* (originally named genus) of a fibration. Since its inception, the notion of topological complexity has attracted a lot of interest, and a fruithful line of work has sprouted from considering different problems on the field, including the development of several variations of the original notion, aimed at measuring different aspects or properties of the motion planning problem or of the base configuration spaces.

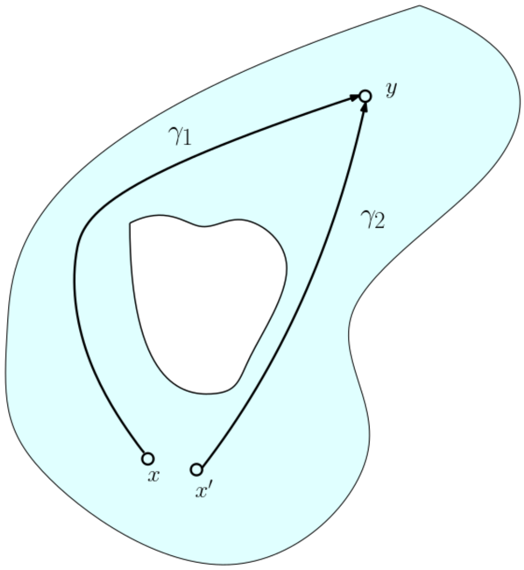


Figure 1.1: A topological feature of the configuration space inducing instability on the motion planning

1.1 Structure and contents of this dissertation

In this thesis, we intend to contribute to the study of the relationship between topological complexity and groups. In this sense, the work is divided into two clearly distinct parts, regarding the role played by the groups. On one hand, we have the group as the subject of interest per se, i.e. the study of the topological complexity of spaces that are determined, up to homotopy, by their fundamental groups, named as *Eilenberg-MacLane spaces*, *aspherical spaces* or *classifying spaces* (and denoted by $K(G, 1)$). In their celebrated article [47], S. Eilenberg and T. Ganea provided, save fringe cases that were later worked out ([113], [114]) a purely algebraic characterization of the Lusternik-Schnirelmann category of aspherical spaces. Following the close connection between the two invariants, M. Farber posed the question on whether such characterization was possible in the realm of TC. The answer seems elusive so far, but the problem has sparkled a fruithful line of research, to which this work humbly intends to contribute.

On the other hand we are interested in how symmetries that may be found in the configuration space affect the motion planning problem. Those symmetries are codified as actions on groups on the space and, as such, the main focus switch from groups themselves to their possible actions. That brings us to the line of research of equivariant notions of topological complexity. There have been many non-equivalent approaches to a notion of topological complexity in the equivariant world, but we are interested, specifically, in one of them, the known as *effective topological complexity*, first introduced by Z. Błaszczyk and M. Kaluba. The key point of such topological complexity is that, while it does not arise as the most natural notion of equivariant TC, is the one that actually tries to reduce the complexity of the motion planning by taking advantage of the symmetries of the system. Unfortunately, this leads to a variety of TC that it is particularly difficult to work with, and so far, original paper aside, it has been poorly investigated and understood. As such, we want to contribute to deepen the understanding of this invariant, by investigating some of its properties.

As mentioned just before this introduction, the work presented in this thesis has been materialized in four different papers, and the purpose of this memory is to present them in the frame of a cohesive narrative. Each of those papers corresponds, essentially, with a chapter of this thesis, with some variations in exposition.

Chapter 2 constitutes a small survey of basic tools that will be used throughout the rest of the dissertation, mostly notions of homological algebra, group cohomology and equivariant cohomology.

Chapter 3 is of purely expositive nature. We recall the main formal definitions associated with the motion planning problem, as well as the definition of topological complexity and sectional category. We outline some of its main properties, providing some useful basic examples that will come at hand later on.

The original contributions of the present work are distributed amongst chapters 4 to 7. We will proceed to give a brief outline of each chapter, highlighting the main results.

Chapter 4

In Chapter 4 we present, with some significant additions, the results of a joint project with Z. Błaszczyk and J.G. Carrasquel Vera, published in [14]. Given a non-torsion group G, and H one of its subgroups, we introduce the notion of sectional category of the subgroup inclusion $H \hookrightarrow G$, $\mathrm{secat}(H \hookrightarrow G)$. Studying this invariant constitutes a natural generalization of the problem of determining the topological complexity of a group G, given that $\mathrm{TC}(G)$ can be understood as $\mathrm{secat}(\Delta G \hookrightarrow G \times G)$. We obtain a generalization of a characterization from Farber, Grant, Lupton and Oprea (see [56]) of topological complexity in terms of maps between the universal space of G and the classifying space with respect to the semi-full family generated by the subgroup:

Theorem (Theorem 4.1.4)**.** *The sectional category of $H \hookrightarrow G$ coincides with the minimal integer $n \geq 0$ such that the G-equivariant map $\rho\colon EG \to E_{\langle H \rangle}G$ can be G-equivariantly factored up to G-homotopy as*

$$\begin{array}{ccc} EG & \xrightarrow{\quad\rho\quad} & E_{\langle H\rangle}G \\ & \searrow \quad \nearrow & \\ & (E_{\langle H\rangle}G)_n, & \end{array}$$

where $(E_{\langle H\rangle}G)_n$ denotes the n-skeleton of $E_{\langle H\rangle}G$.

We also describe and develop a "relative canonical class" analogous to the one developed by Berstein and Schwarz for the study of Lusternik–Schnirelmann category theory. In particular, we can prove a generalization of the Costa and Farber theorem for the powers of the canonical class, presented in [35]:

Theorem (Theorem 4.1.11). *If $n = \operatorname{cd} G \geq 3$, then $\operatorname{secat}(H \hookrightarrow G) \leq n-1$ if and only if $\omega^n = 0$.*

Then, we introduce a new actor into the fray, the Adamson cohomology theory of the pair (G, H). First defined by Adamson in [1], and later systematized by G. Hochschild in the more general setting of relative homological algebra (see [75]), we find a canonical class in Adamson cohomology, which turns out to be universal, and relate it with the previously defined relative canonical class. Of particular interest is the possibility of characterizing Adamson cohomology groups in terms of zero divisors in usual cohomology for a suitable choice of coefficient systems:

Theorem (Proposition 4.2.8). *For any G-module M and $n \geq 1$, we have*

$$H^n([G:H], M) = \ker\left[H^1(G, \operatorname{Hom}_{\mathbb{Z}}(I^{\otimes n-1}, M)) \to H^1(H, \operatorname{Hom}_{\mathbb{Z}}(I^{\otimes n-1}, M))\right].$$

In particular,

$$H^1([G:H], M) = \ker\left[H^1(G, M) \to H^1(H, M)\right].$$

We also find a spectral sequence associated to the pair (G, H) which contains the information of Adamson cohomology in its second page, and recast Adamson cohomology in terms of Bredon cohomology, of which we give a new and independent proof.

Theorem (Theorem 4.2.14). *Given a G-module M, let $\underline{M}$ be the $\operatorname{Or}_{\langle H \rangle} G$-module defined by setting $\underline{M}(G/K) = M^K$. Then*

$$H^*\big([G:H], M\big) \cong H^*_{\langle H \rangle}(E_{\langle H \rangle} G, \underline{M}).$$

In particular, $\operatorname{cd}[G:H] \leq \operatorname{cd}_{\langle H \rangle} G$.

Chapter 5

The contents of Chapter 5 constitute part of a joint work with M. Farber, S. Mescher and J. Oprea, see [49]. Here we present mainly the algebraic part of such work, in line with the approach taken in Chapter 4.

Continuing with the ideas developed in Chapter 4, we start by providing alternative characterizations of relative Berstein-Schwarz classes with respect to subgroup inclusions. We also generalize the notion of essential cohomology classes, first introduced by Farber and Mescher in [57], to arbitrary group monomorphisms, and we use them to give new bounds for the sectional category of inclusions of normal subgroups, summarized in the following theorem:

Theorem (Theorem 5.1.6 and Theorem 5.1.7). *Let $N \triangleleft G$ be a normal subgroup, put $Q := G/N$ for the quotient group and let $\pi : G \to Q$ denote the projection.*

a) Let $\omega \in H^1(G; I)$ be the Berstein-Schwarz class of G relative to N and let $\beta \in H^1(Q; I_Q)$ be the Berstein-Schwarz class of Q, where $I_Q \subset \mathbb{Z}[Q]$ denotes the augmentation ideal of Q. Then

$$\pi^*\beta = \omega.$$

b) Let A be a left $\mathbb{Z}[Q]$-module and let $n \in \mathbb{N}$. A cohomology class $u \in H^n(G; \pi^* A)$ with $u \neq 0$ is essential relative to N if and only if there exists $v \in H^n(Q; A)$ with $\pi^* v = u$.

c) If there exists a left $\mathbb{Z}[Q]$-module A for which $\pi^* : H^{\mathrm{cd}(Q)}(Q; A) \to H^{\mathrm{cd}(Q)}(G; A)$ is non-zero, then

$$\mathrm{secat}(N \hookrightarrow G) = \mathrm{cd}(Q).$$

Once this is accomplished, we will proceed to develop a thorough generalization of the construction of a spectral sequence to sectional categories of subgroup inclusions, that was carried out for the topological complexity of aspherical spaces in [57]. The properties of such spectral sequence are summarized as follows:

Theorem (Theorem 5.2.4)**.** *Let $n \in \mathbb{N}$ and let $u \in H^n(G; A)$ with $u \neq 0$.*

a) The class u is essential relative to H if and only if $u \in D_n^{n,0}$.

b) $D_1^{n,0} = \ker[\iota^* : H^n(G; I) \to H^n(H; \widetilde{I})]$, where ι^* is induced by the inclusion $\iota : H \hookrightarrow G$.

c) Let $s \in \{0, 1, \ldots, n-1\}$. Then $u \in D_{s+1}^{n,0}$ if and only if

$$u \in D_s^{n,0} \qquad \text{and} \qquad u \in \ker\left[j_s : D_s^{n,0} \to E_s^{n-s,s}\right].$$

Of particular interest is the expression of the zeroth-page of such spectral sequence in terms of products of cohomology groups of isotropy subgroups with respect to certain action of H on G/H. Namely, if $\mathcal{C}_s'(G/H)$ is the set of orbits of $(G/H) \setminus \{H\}$ with respect to said action, we find the following decomposition of $E_0^{r,s}$.

Theorem (Theorem 5.2.5)**.** *Let $s \in \mathbb{N}$. For each $C \in \mathcal{C}_s'(G/H)$ fix a representative $x_C \in C$ and let $N_C := H_{x_C}$ be the isotropy group of x_C. Then, for any $\mathbb{Z}[G]$-module A we have*

$$E_0^{r,s} \cong \prod_{C \in \mathcal{C}_s'(G/H)} H^r(N_C, \mathrm{Res}_{N_C}^G(A)) \quad \forall r \in \mathbb{N}.$$

We will use the power of this spectral sequence (in particular of the decomposition indicated just above) to derive a new lower bound for sectional category of arbitrary subgroup inclusions in terms of the cohomological dimension of the isotropy groups with respect to the prescribed left H-action on the coset space G/H.

Theorem (Theorem 5.3.3)**.** *Let G be a geometrically finite group and let $H \leq G$ be a subgroup. For each $x \in G$, we let H_x denote the isotropy group of the left H-action on G/H in xH and put*

$$\kappa_{G,H} := \sup\{\mathrm{cd}(H_x) \mid x \in G \setminus H\}.$$

Then we get the lower bound

$$\mathrm{secat}(H \hookrightarrow G) \geq \mathrm{cd}(G) - \kappa_{G,H}.$$

We carry out the consequences of this very general result for sequential topological complexity and for the parametrized topological complexity of group epimorphisms.

Theorem (Theorem 5.4.2 and Theorem 5.4.8). *a) Let π be a geometrically finite group and let $r \in \mathbb{N}$ with $r \geq 2$. Then*

$$\mathrm{TC}_r(K(\pi,1)) \geq r \cdot \mathrm{cd}(\pi) - k(\pi),$$

where $k(\pi) = \max\{\mathrm{cd}(C(g)) \mid g \in \pi \setminus \{1\}\}$.

b) Let G and Q be geometrically finite groups and let $\rho : G \twoheadrightarrow Q$ be an epimorphism. Then

$$\mathrm{TC}[\rho : G \to Q] \geq \mathrm{cd}(G \times_Q G) - k(\rho),$$

where

$$k(\rho) = \max\{\mathrm{cd}(C(g)) \mid g \in \ker \rho,\ g \neq 1\}.$$

As an application, we will also show that for a free amalgamated product of the form $\pi_1 *_H \pi_2$, where H is malnormal both in π_1 and in π_2, we obtain that

$$\mathrm{TC}_r(K(\pi_1 *_H \pi_2, 1)) \geq r \cdot \mathrm{cd}(\pi_1 *_H \pi_2) - \max\{k(\pi_1), k(\pi_2)\} \qquad \forall r \geq 2.$$

Chapter 6

To close our study of the topological complexity of aspherical spaces, in Chapter 6 we return to some of the ideas we explored at the beginning of Chapter 4. The objective of this chapter is to recast the sectional category of subgroup inclusions and, in particular, the sequential topological complexities of aspherical spaces in the lenguage of another category-like homotopy invariant, the $\mathcal{A}$-genus. This will be accomplished through the more general identification of the sectional category of connected covers of a CW-complex X to $\mathcal{A}$-genus(X) for a suitable choice of family $\mathcal{A}$.

Theorem (Theorem 6.2.1, Corollary 6.2.2 and Proposition 6.2.4). *Let X be a path connected CW-complex with $\pi_1(X) = \pi$. If $q : \widehat{X} \to X$ is a connected covering, then*

$$\mathrm{secat}(q) = \mathcal{A}\text{-genus}(\widetilde{X})$$

where $\mathcal{A} = \{\pi/\pi_1(\widehat{X})\}$.

In particular we have the following:

(1) Let G a torsion-free group, and $H \leqslant G$. Then we have $\mathrm{secat}(H \hookrightarrow G) = \mathcal{A}\text{-genus}(EG)$ *where $\mathcal{A} = \{G/H\}$.*

(2) $\mathrm{TC}_r(X) \geq \mathcal{A}\text{-genus}(\widetilde{X}^r)$ *for $\mathcal{A} := \{\pi^k/\Delta_{\pi,r}\}$.*

(3) Furthermore, if X is aspherical, then $\mathrm{TC}_r(X) = \mathcal{A}\text{-genus}(\widetilde{X}^r)$.

Such characterization, coupled with the already known properties of $\mathcal{A}$-genus, allow us to derive new bounds for sectional category and sequential topological complexities.

Theorem (Proposition 6.2.6). *Let G be a torsion-free group, $H \leqslant G$ and $\mathcal{A} = \{G/H\}$.*

(a) Given $\mathcal{F}$ a full family of subgroups of G we have that $\mathrm{secat}(H \hookrightarrow G) \leq \mathcal{A}\text{-genus}(E_{\mathcal{F}}(G))$.

(b) *For any subgroup* $K \leqslant G$ *subconjugate to* H *such that* $\mathrm{cd}_{\langle K\rangle}G \geq 3$ *we have* $\mathrm{secat}(H \hookrightarrow G) \leq \mathrm{cd}_{\langle K\rangle}G$.

(c) *Under the hypothesis of (b), if* $K \trianglelefteq G$ *then* $\mathrm{secat}(H \hookrightarrow G) \leq \mathrm{cd}(G/K)$.

In particular, we obtain some bounds in terms of subgroups of the given group, which in the case of sequential topological complexities provide a new explicit measure of the lack of monotonicity of topological complexity with respect to inclusion of subgroups, and allow us to give a new bound for the case of the semidirect product of groups.

Theorem (Corollary 6.2.8 and Corollary 6.2.9)**.** *Let* π *be a torsion free group with subgroups* $H, K \leqslant \pi$*, and* $J \leqslant H$*. The following inequality holds*

$$\mathrm{secat}(J \hookrightarrow H) \leq (\mathrm{secat}(K \hookrightarrow \pi)+1)(\mathcal{B}\text{-genus}((\pi/K))+1)-1$$

for $\mathcal{B} = \{H/J\}$*.*

In particular, for the specific case of the iterated diagonal inclusions $\Delta_{H,r} \hookrightarrow H^r$ *and* $\Delta_{\pi,r} \hookrightarrow \pi^r$ *the above inequality yields*

$$\mathrm{TC}_r(H) \leq (\mathrm{TC}_r(\pi)+1)(\mathcal{B}\text{-genus}(\pi^r/\Delta_{\pi,r})+1)-1$$

where $\mathcal{B} = \{H^r/\Delta_{H,r}\}$ *and* $r \in \mathbb{N}$ *with* $r \geq 2$*.*

Moreover, we have $\mathrm{TC}_r(H \rtimes K) \geq \mathrm{TC}_r(K)$*.*

We close the chapter with some quick thoughts about a new notion of topological complexity for proper actions and $\mathcal{A}$-genus with respect to the family of finite subgroups.

Chapter 7

In Chapter 7, the last part of this dissertation, we change our point of view about the role that groups play: instead of the topological complexity of spaces whose homotopy type is wholly determined by the isomorphism type of their fundamental groups (the Eilenberg-MacLane spaces), we care about more general kind of spaces, but such that they present symmetries, i.e group actions, that, while undetected by the classic version of topological complexity, might be put to use to reduce the complexity of the motion planning algorithms. That was the foundational idea behind the notion of effective topological complexity, first introduced by Z. Błaszczyk and M. Kaluba in [16]. In this chapter, we further investigate this variant of topological complexity, investigating some of its properties.

We start our analysis by studying the relationship between the different broken path spaces, and introducing the notion of the global effective path space, as a limit of a chain of inclusions. Then we proceed define an effective version of the Lusternik-Schnirelmann category, which will play an analogous role to the classic LS-category in this setting. In particular, we obtain the effective version of the classic bound of TC in terms of LS-cat, and other crucial properties summarized in the following theorem.

Theorem (Theorem 7.4.2, Proposition 7.4.4 and Corollary 7.5.4). *Let X be a G-space. The following statements hold:*

(1) $\mathrm{cat}^{G,\infty}(X) \leq \mathrm{TC}^{G,\infty}(X) \leq \mathrm{cat}^{G\times G,\infty}(X \times X) \leq 2\mathrm{cat}^{G,\infty}(X)$.

(2) Let $\rho_X \colon X \to X/G$ be the orbit map with respect to the action of G. Then $\mathrm{cat}(\rho_X) \leq \mathrm{cat}^{G,\infty}(X)$.

(3) $\mathrm{cat}^{G,\infty}(X) = 0$ *if and only if* $\mathrm{TC}^{G,\infty}(X) = 0$.

Afterwards, we will turn our attention to discussing the problem of determining the kind of G-spaces with $\mathrm{TC}^{G,\infty}(X) = 0$. The relationship hinted there between the orbit projection map $\rho_X \colon X \to X/G$ and the effective TC will be adressed further later on, through the study of the relationship between ρ_X and $\mathrm{TC}^{G,\infty}(X)$ in two distinct cases: when the orbit projection map has a strict section, and when it is a fibration. Our findings are summarized in the following theorem.

Theorem (Theorem 7.6.1 and Theorem 7.6.4). *Let X be a G-space. If $\rho_X : X \to X/G$ has a strict section $s : X/G \to X$, the following holds:*

(1) $\mathrm{cat}^{G,\infty}(X) = \mathrm{cat}(X/G)$.

(2) $\mathrm{TC}^{G,\infty}(X) = \mathrm{TC}(X/G)$.

If the orbit map ρ_X is a fibration instead, then we have:

(1) $\mathrm{cat}^{G,\infty}(X) = \mathrm{cat}^{G,2}(X) = \mathrm{cat}(\rho_X) \leq \mathrm{cat}(X/G)$.

(2) $\mathrm{TC}^{G,\infty}(X) = \mathrm{TC}^{G,2}(X) \leq \mathrm{TC}(X/G)$.

We will discuss plenty of examples in both situations, mostly concerning actions of compact Lie groups, and some consequences of the above result.

In the final section of this chapter, we will show how the broken path space at stage two $\mathcal{P}_2(X)$ is homotopically equivalent to the saturated diagonal $\daleth(X)$, and we will make use of this information to derive some dimensional conditions for the non-vanishing of the stage 2 effective topological complexity for compact G-ANR with G finite.

Theorem (Theorem 7.7.3, Corollary 7.7.4). *Let G be a finite group, and X a compact G-ANR such that* $\mathrm{cd}(X^H) \leq \mathrm{cd}(X)$ *for all non-trivial subgroup $H \leqslant G$. Then, for any L list of elements of G,* $\mathrm{cd}(\daleth_L(X)) \leq \mathrm{cd}(X) + |L| - 1$. *In particular, we have*

$$\mathrm{cd}(\daleth(X)) \leq \mathrm{cd}(X) + |G| - 1.$$

Under these assumptions, if $|G| \leq \mathrm{cd}(X)$, then it holds that $\mathrm{TC}^{G,2}(X) > 0$.

Part I

Preliminaries

CHAPTER 2

A small preliminary toolbox

In this chapter we will introduce the basic necessary framework that will be instrumental during this dissertation, and thoroughly used therein. Naturally, we hold no claim of originality over any of the contents presented in this chapter, and the experienced reader might very well skip it altogether, should they be already familiar with the material. At the beginning of each section we will suggest some standard bibliographic references for the interested reader.

2.1 Basic definitions of homological algebra

We will provide a very short summary of the very basic notions of homological algebra, that are instrumental in the study of group cohomology, and that we will use extensively through a significant part of this dissertation. By no means we intend to present a complete survey on the matter, and for the reader interested in increasing their knowledge of the matter we suggest classic manuals on the topic, such as [116], [107] or the absolute all-time classic seminal book of H. Cartan and S. Eilenberg, [26].

2.1.1 Chain complexes and homology

We will start by reviewing the most basic definitions, that constitutes the bedrock of the whole theory.

Definition 2.1.1. Let R be a ring. A *chain complex* (C_*, d_*) is a sequence of R-modules C_i with R-module homomorphisms $d_i \colon C_i \to C_{i-1}$ such that $d_{i-1} \circ d_i = 0$. Likewise, by a *cochain complex* (C_*, d_*) we understand a sequence of R-modules C_i and of R-module homomorphisms $d_i \colon C_i \to C_{i+1}$ such that $d_{i+1} \circ d_i = 0$. In both cases, the d_i maps are called the *differentials* of the (co)chain complex.

Whenever the situation is general enough, we will write the arrows orientation in terms of chain complexes. Bear in mind though that everything is immediately translatable into cochain complexes just by properly reversing the direction of the differentials.

Obviously, the idea that composition of the successive diferentials is zero implies that the image of any differential map lies in the kernel of the next. In particular, we can define the key notion of exactness of a sequence of R-modules.

Definition 2.1.2. We say that a sequence of R-modules

$$\cdots \to M_n \xrightarrow{f_n} M_{n+1} \xrightarrow{f_{n+1}} M_{n+2} \xrightarrow{f_{n+2}} \cdots$$

is *exact* if $\operatorname{Im} f_n = \ker f_{n+1}$ for all n. In particular, exact sequences of the form

$$0 \to L \to M \to N \to 0$$

are called *short exact* sequences.

Definition 2.1.3. Given two (co)chains (C_*, d_*) and (C'_*, d'_*) we define a *chain map* $f_* \colon C_* \to C'_*$ to a collection of R-module homomorphisms $f_i \colon C_i \to C'_*$ that commutes with the respective differentials, that is, such that for every i the following diagram is commutative

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{d_{i+2}} & C_{i+1} & \xrightarrow{d_{i+1}} & C_i & \xrightarrow{d_i} & C_{i-1} & \xrightarrow{d_{i-1}} & \cdots \\
 & & \downarrow{\scriptstyle f_{i+1}} & & \downarrow{\scriptstyle f_i} & & \downarrow{\scriptstyle f_{i-1}} & & \\
\cdots & \xrightarrow[d'_{i+2}]{} & C'_{i+1} & \xrightarrow[d'_{i+1}]{} & C_i & \xrightarrow[d'_i]{} & C_{i-1} & \xrightarrow[d'_{i-1}]{} & \cdots
\end{array}$$

We can recall now the definition of (co)homology of (co)chain complexes.

Definition 2.1.4. The n-dimensional (co)homology group of the (co)chain complex (C_i, d_i) is defined as

$$H_n(C_*, d_*) = {\ker d_n}\big/{\operatorname{Im} d_{n+1}} \qquad H^n(C_*, d_*) = {\ker d_n}\big/{\operatorname{Im} d_{n-1}}$$

respectively. As in the usual (co)homology of topological spaces, the elements of $\ker d_i$ are called (co)cycles, while those in $\operatorname{Im} d_i$ are known as (co)boundaries.

We are interested in defining a notion of equivalence between (co)chain complexes, that will make them impossible to distinguish from the point of view of homology. This role will be played by the chain homotopies.

Definition 2.1.5. Given two (co)chain complexes (C_*, d_*) and (C'_*, d'_*), we say that two chain maps $f_*, g_* \colon C_* \to C'_*$ are *chain homotopic*, denoted (in analogy to the topological case) $f \simeq g$, if there exists a diagonal chain map $h_* \colon C_* \to C'_*$ i.e. for each degree i we have $h_i \colon C_i \to C'_{i+1}$ for chain complexes (and $h_i \colon C_i \to C_{i-1}$ for cochain ones) such that

$$h_i \circ d'_{i+1} + h_{i-1} \circ d_i = f_i - g_i \qquad (d'_{i-1} \circ h_i + h_{i+1} \circ d_i = f_i - g_i \quad \text{for cochain complexes}).$$

By the commutativity of the (co)chain map and the differentials at each degree, it is straightforward to see that a (co)chain map always induces an associated homomorphism in (co)homology. In particular, it is just an exercise of diagram chasing to verify that the following statement holds.

Proposition 2.1.6. *Two chain homotopic maps $f_*, g_* \colon C_* \to C'_*$ induce the same homomorphism in (co)homology*

$$H_*(f) = H_*(g) \qquad H^*(f) = H^*(g).$$

Given three (co)chain complexes A, B and C we say that they form a *short exact sequence* of (co)chain complexes

$$0 \to A \to B \to C \to 0$$

if, for every index i, the following commutative diagram has exact rows

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A_{i+1} & \longrightarrow & B_{i+1} & \longrightarrow & C_{i+1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A_i & \longrightarrow & B_i & \longrightarrow & C_i & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A_{i-1} & \longrightarrow & B_{i-1} & \longrightarrow & C_{i-1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

(with the corresponding change of direction for cochains). It is an exercise of diagram chasing derived from the snake lemma to show that this short exact sequence of (co)chains determine long exact sequences in (co)homology

$$\cdots \to H_{n+1}(C) \xrightarrow{\delta} H_n(A) \to H_n(B) \to H_n(C) \to \cdots$$

$$\cdots \to H^n(C) \xrightarrow{\delta} H^{n+1}(A) \to H^{n+1}(B) \to H^{n+1}(C) \to \cdots$$

where the δ maps are called (in analogy to the topological case) *connecting homomorphisms*.

Recall that the bifunctor $\mathrm{Hom}_R(\cdot, \cdot)$ is covariant in the first component and contravariant in the second. As such, if we fix an R-module A, and we apply the functor $\mathrm{Hom}_R(\cdot, A)$ to a chain complex (C_*, d_*) we obtain a cochain complex where, for each i, the functor inverts the direction of the differential, i.e. we get

$$\mathrm{Hom}_R(C_i, A) \xrightarrow{\mathrm{Hom}_R(\cdot, A)(d_i)} \mathrm{Hom}_R(C_{i+1}, A).$$

Recall also that $\mathrm{Hom}_R(\cdot, A)$ is not an exact functor (i.e. it doesn't preserve exact sequences of R-modules) but it is instead left exact. That means that, for any right exact sequence

$$M' \to M \to M'' \to 0$$

the induced sequence

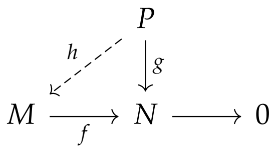
$$0 \to \mathrm{Hom}_R(M'', A) \to \mathrm{Hom}_R(M, A) \to \mathrm{Hom}_R(M', A)$$

is exact. We will use both such fact in a short while to define the Ext groups and, consequently, the cohomology of groups.

2.1.2 Projective modules and Ext groups

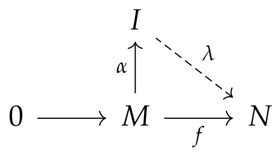
We need a specific kind of R-modules and chain complexes in order to give a precise definition of the cohomology of the group. In this subsection we will introduce the notion of projective modules and projective resolutions, and we will use them to build the Ext groups.

Definition 2.1.7. Let R be a ring, and P an R-module. We say that P is a *projective* module if, for every R-linear map $P \xrightarrow{g} N$ and any short exact sequence of R-modules $M \xrightarrow{f} N \to 0$ there exists an unique R-linear map $h\colon P \to N$ making the following diagram commutative:



In particular, every free R-module F is projective.

Let now I be an R-module. We say that I is an *injective* module if, for any short exact sequence of R-modules $0 \to M \xrightarrow{f} N$ and any R-linear map $\alpha M \to I$ there exists an unique R-linear map $\lambda\colon M \to I$ making the following diagram commutative:



Let P be a R-module. It is not difficult to show that the following are equivalent conditions of projectiveness of P to that of the definition above.

- There exists a R-module Q such that $P \oplus Q$ is a free R-module.
- Every exact sequence of R-modules $0 \to A \to B \xrightarrow{f} P \to 0$ is split exact, i.e. there is a R-module homomorphism $g\colon P \to B$ such that $f \circ g = \mathrm{id}_P$.
- The functor $\mathrm{Hom}_R(P, \cdot)$ is exact.

Definition 2.1.8. Let R be a ring, and M a (left) R-module.

We say that a *projective resolution* of M is an exact sequence

$$\mathcal{P}\colon \ \cdots \to P_2 \to P_1 \to P_0 \to M \to 0$$

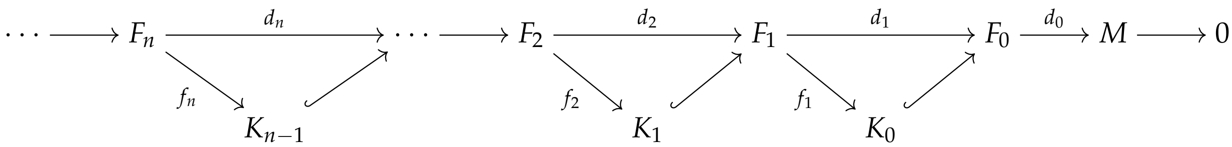
such that P_i is a projective R-module for every $i \geq 0$.

Conversely, for an *injective resolution* of M we understand an exact sequence

$$\mathcal{J}: 0 \to M \to J_0 \to J_1 \to J_2 \to \cdots$$

where J_i is an injective R-module for every $i \geq 0$.

Given an arbitrary R-module M we may always construct an associated projective resolution in a very straightforward manner. Start choosing an epimorphism $F_0 \xrightarrow{d_0} M$, where F_0 is a free module that surjects onto M, and consider its kernel $K_0 = \ker d_0$. Now choose another free module F_1 with an epimorphism $F_1 \xrightarrow{f_1} K_0$, and consider in turn its kernel $K_1 = \ker f_1$. Proceeding iteratively, we obtain a projective (in this case actually free) resolution of the form



The kernels K_i are called the *syzygies* of the resolution (colourful name borrowed from Greek by Cayley but mostly popularized later on by Hilbert), and are key pieces for many techniques of studying projective resolutions in classic homological algebra. As for the injective case, it is also possible to prove that for every R-module it is possible to construct an injective resolution, though in this case the proof is less straightforward.

Of course, as it becomes apparent from the procedure just described, projective resolutions are by no means unique. In fact, each step of the construction involved a particular choice, which means that the range of possible resolutions of that sort that can be constructed for a giving module can be significantly large. As it turns out, it matters not, as all of them are chain homotopy equivalent.

Theorem 2.1.9 (The comparison theorem). *Let $\mathcal{P}_\bullet \to M$ and $\mathcal{Q}_\bullet \to M$ two different projective resolutions of an R-module M. Then $\mathcal{P}_\bullet$ and $\mathcal{Q}_\bullet$ are chain homotopy equivalent, i.e. there exist chain maps*

$$\phi\colon \mathcal{P}_\bullet \to \mathcal{Q}_\bullet \qquad \wedge \qquad \psi\colon \mathcal{Q}_\bullet \to \mathcal{P}_\bullet$$

lifting the identity on M and such that

$$\psi \circ \phi \simeq \mathrm{id}_{\mathcal{P}_\bullet} \qquad \wedge \qquad \phi \circ \psi \simeq \mathrm{id}_{\mathcal{Q}_\bullet}.$$

We can now proceed to define the Ext groups.

Definition 2.1.10. For a ring R the n-th *extension bifunctor*

$$\mathrm{Ext}_R(\cdot,\cdot)\colon R\text{-Mod} \to \mathrm{Ab}$$

from the category of R-modules to the category of abelian groups is defined as the right-derived functor of $\mathrm{Hom}_R(\cdot,\cdot)$. Specifically, let M and N be two left R-modules, and $\mathcal{P}_\bullet \to M$ a projective resolution of M. We define the n^{th}*-extension group* of M and N by

$$\mathrm{Ext}^n_R(M,N) = H^n(\mathrm{Hom}_R(\mathcal{P}_\bullet, N))$$

i.e. the n-th cohomology group of the cochain complex $\mathrm{Hom}_R(\mathcal{P}_\bullet, N)$.

Proposition 2.1.11 (Basic properties of Ext)**.** *Let M, M_1, M_2 and N, N_1, N_2 be left R-modules, and let $n \in \mathbb{N}$. Then the following properties hold:*

(a) $\operatorname{Ext}^0_R(M, N) \cong \operatorname{Hom}_R(M, N)$.

(b) Any homomorphism of R-modules $f\colon M_1 \to M_2$ induces a group homomorphism

$$f^*\colon \operatorname{Ext}^n_R(M_2, N) \longrightarrow \operatorname{Ext}^n_R(M_1, N).$$

Conversely, any homomorphism of R-modules $g\colon N_1 \to N_2$ induces a group homomorphism

$$g^*\colon \operatorname{Ext}^n_R(M, N_1) \longrightarrow \operatorname{Ext}^n_R(M, N_2).$$

(c) Consider a short exact sequence of R-modules

$$0 \to A \to B \to C \to 0.$$

As a consequence of the long exact sequence in cohomology, there are exact sequences

$$\begin{aligned} 0 &\to \operatorname{Hom}_R(M, A) \to \operatorname{Hom}_R(M, B) \to \operatorname{Hom}_R(M, C) \\ &\to \operatorname{Ext}^1_R(M, A) \to \operatorname{Ext}^1_R(M, B) \to \ldots \end{aligned}$$

$$\begin{aligned} 0 &\to \operatorname{Hom}_R(C, N) \to \operatorname{Hom}_R(B, N) \to \operatorname{Hom}_R(A, N) \\ &\to \operatorname{Ext}^1_R(C, N) \to \operatorname{Ext}^1_R(B, N) \to \ldots \end{aligned}$$

(d) If P is a projective R-module, then $\operatorname{Ext}^n_R(P, N) = 0 \;\; \forall n \geq 1$.

(e) If J is an injective R-module, then $\operatorname{Ext}^n_R(M, J) = 0 \;\; \forall n \geq 1$.

(f) For any collection $\{Q_1, \cdots, Q_n\}_{i \in I}$ of R-modules we have

$$\operatorname{Ext}^n_R(\bigoplus_{i \in I} Q_i, M) \cong \prod_{i \in I} \operatorname{Ext}^n_R(Q_i, M) \qquad \operatorname{Ext}^n_R(M, \bigoplus_{i \in I} Q_i) \cong \bigoplus_{i \in I} \operatorname{Ext}^n_R(M, Q_i)$$

Consider two rings R, S with an inclusion $R \subset S$. It is obvious that any S module can be regarded as an R module just by restriction of scalars. However, it is possible to go also in the opposite direction. There are two ways of performing this, depending of the interpretation of what "enlarging" should mean, but here we will just make use of one of them, the so called *extension of scalars*. Given an R-module M, form the tensor product $S \otimes_R M$, where S is seen as a right R-module, and then use the left operation of S on itself to turn the tensor product into an S-module, by putting

$$s \cdot (s' \otimes m) := ss' \otimes m$$

for $s, s' \in S$ and every $m \in M$. We say that $S \otimes_R M$ is the S-module obtained from M by extension of scalars. It is an obvious enlargement of M in the sense that there exists a canonical R-module homomorphism

$$i\colon M \to S \otimes_R M \qquad m \mapsto 1 \otimes m.$$

It is characterized by means of an universal property: given N an S-module, and an R-module homomorphism $f\colon M \to N$ there is an unique S-module homomorphism $g\colon S \otimes_R M \to N$ making the following diagram commutative

$$\begin{array}{ccc} M & \xrightarrow{i} & S \otimes_R M \\ {\scriptstyle f}\downarrow & \swarrow{\scriptstyle g} & \\ N & & \end{array} \tag{2.1.1}$$

2.2 Group cohomology

In this section we will recall the very basics of the vast field of group cohomology. The standard bibliographic reference for the interested non-expert reader is the famous and notoriously beautiful textbook by K. Brown [22]. We begin, as it is mandatory, by recalling the definition of ring group and group modules.

Definition 2.2.1. Let G be a group (written here multiplicatively). The *(integral) group ring* of G is the ring $\mathbb{Z}[G]$ whose underlying abelian group is the free $\mathbb{Z}$-module $\bigoplus_{g\in G}\mathbb{Z}$ freely generated by G, and whose multiplication is the $\mathbb{Z}$-linear extension of the operation in G, i.e.

$$\begin{aligned} \cdot\colon \mathbb{Z}[G]\times\mathbb{Z}[G] &\longrightarrow \mathbb{Z}[G] \\ \left(\sum_{h\in G} m_h h, \sum_{k\in G} n_k k\right) &\mapsto \sum_{g\in G}\left(\sum_{hk=g} m_h n_k\right) g. \end{aligned}$$

Definition 2.2.2. A (left) $\mathbb{Z}[G]$-module (also sometimes called G-module) consists of an abelian group A which carries a G-representation, i.e. such that there exists an action of the group G on A through a map $\cdot\colon G \times A \to A$ satisfying

$$1\cdot a = a, \qquad g\cdot(a+b) = g\cdot a + g\cdot b, \qquad (g\cdot h)\cdot a = g\cdot(h\cdot a) \qquad \forall g,h\in G \wedge a,b\in A.$$

It is straightforward to check that this action extends to an operation of the group ring $\mathbb{Z}[G]$ on A endowing A with a module structure with respect to the ring $\mathbb{Z}[G]$ in the usual sense, hence justifying the definition just provided. Naturally, a homomorphism of $\mathbb{Z}[G]$-modules $\alpha \cdot A \to M$ is just a homomorphism between abelian groups that satisfies

$$\alpha(ga) = g\alpha(a) \qquad \forall g\in G, a\in A.$$

We say that a G-module A has a trivial module structure if $ga = a$ for all $g \in G$ and all $a \in A$. Of particular relevance is the so called *augmentation map*

$$\begin{aligned} \varepsilon\colon \mathbb{Z}[G] &\longrightarrow \mathbb{Z} \\ \sum_{g\in G} m_g g &\mapsto \sum_{g\in G} m_g \end{aligned}$$

where $\mathbb{Z}$ is regarded as a trivial $\mathbb{Z}[G]$-module. The kernel of this module homomorphism is called the augmentation ideal of $\mathbb{Z}[G]$

$$K = \ker \varepsilon = \mathrm{Span}(\{g - 1 | \, g \in G\}).$$

One way to construct G-modules that will be of particular interest for us is by means of linearization of permutation representations. For any G-set X, one can form the free abelian group generated by X, denoted by $\mathbb{Z}[X]$, and extend the action of G on X to a linear action of G on $\mathbb{Z}[X]$, in a manner completely analogous to the one in the definition of the group ring. In fact, it is obvious that the group ring is just a particular case when the G-set consider is G itself with the natural action on itself. The $\mathbb{Z}[G]$-module that results from it is called a *permutation module* of X.

There is one example of such modules that will be crucial for us in the future.

Example 2.2.3. Let G be a group, and $H \leq G$. The set of lateral cosets G/H is a natural G set with G acting by left translation. As such, one can construct the associated permutation module $\mathbb{Z}[G/H]$.

Furthermore, notice that the operation of disjoint union of G-sets corresponds with the direct sum operation in the category of $\mathbb{Z}[G]$-modules. Consequently $\mathbb{Z}[\coprod_{i \in J} X_i] = \bigoplus_{i \in J} \mathbb{Z}[X_i]$. As the action of G on a G-set X induce a natural decomposition as a disjoint union of orbits, we see that every permuation module decomposes as $\mathbb{Z}[X] \cong \bigoplus_{x \in X} \mathbb{Z}[G/G_x]$ where x ranges over the representatives of orbits of the G-action. and G_x stands for the isotropy subgroup of x (see (2.3.1) for a precise definition). As a consequence, for every free G-set X, the permutation module $\mathbb{Z}[X]$ is a free $\mathbb{Z}[G]$-module. In particular, for every subgroup $H \leqslant G$ it follows that $\mathbb{Z}[G]$ is a free $\mathbb{Z}[H]$-module with basis the set of representatives of the H-cosets of G. Indeed, every free $\mathbb{Z}[G]$-module F can be seen as a free $\mathbb{Z}[H]$-module, just consider $F \cong \bigoplus_J \mathbb{Z}[G] \cong \bigoplus_J (\bigoplus_E \mathbb{Z}[H])$, where E here stands for the set of representatives of H-cosets. As every projective $\mathbb{Z}[G]$-module P is a direct summand of a free $\mathbb{Z}[G]$-module, $F = P \oplus P'$, we observe that P is also $\mathbb{Z}[H]$-projective, hence it follows that any projective $\mathbb{Z}[G]$-projective resolution is also a $\mathbb{Z}[H]$-projective resolution.

We can now recall the standard definition of cohomology of a group G, in terms of the $\mathrm{Ext}_{\mathbb{Z}[G]}$ groups of $\mathbb{Z}[G]$-projective resolutions of $\mathbb{Z}$.

Definition 2.2.4. Let G be a group, A a $\mathbb{Z}[G]$-module and $n \in \mathbb{Z}$. Take a $\mathbb{Z}[G]$-projective resolution of $\mathbb{Z}$ as a trivial $\mathbb{Z}[G]$-module $\mathcal{P}_\bullet \to \mathbb{Z}$. We define the n-th cohomology group of G with coefficients in A, written $H^n(G, A)$, by

$$H^n(G, A) := H^n(\mathrm{Hom}_{\mathbb{Z}[G]}(\mathcal{P}_\bullet, A)) = \mathrm{Ext}^n_{\mathbb{Z}[G]}(\mathcal{P}_\bullet, A).$$

By the comparison theorem, 2.1.9, it is straightforward that such notion is well defined, as the chain homotopy between two projective resolutions of $\mathbb{Z}$ prescribed by the comparison theorem induces a (canonical) isomorphism at the level of Ext groups. Notice that, given the contravariance of the $\mathrm{Hom}_{\mathbb{Z}[G]}$-functor on the second coordinate, if we look at the first terms of a $\mathbb{Z}[G]$-projective resolution of $\mathbb{Z}$ under its transformation by $\mathrm{Hom}_{\mathbb{Z}[G]}(\cdot, A)$, we have a left exact sequence

$$0 \to \mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, A) \to \mathrm{Hom}_{\mathbb{Z}[G]}(P_0, A) \to \mathrm{Hom}_{\mathbb{Z}[G]}(P_1, A)$$

from which we can easily infer the identification

$$H^0(G, A) = \mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, A) = A^G \tag{2.2.1}$$

where A^G just stands for the invariants of A with respect to the action of G (see next section for the precise definition).

Remark 2.2.5. For any $\mathbb{Z}[G]$-module A (with group structure denoted here additively) define the notion of *crossed homomorphism* as a map $\phi\colon G \to A$ satisfying the condition

$$\phi(gh) = \phi(g) + (g \cdot \phi(h)) \qquad \forall g, h \in G.$$

The crossed homomorphisms are sometimes called *derivations*, hence the classic notation of the set of all crossed homomorphisms with respect to A as $\mathrm{Der}(G, A)$. A *principal homomorphism*, in turn, it is defined as a map $\phi\colon G \to A$ satisfying $\phi(g) = g \cdot a - a$ for some $a \in A$. It is immediate to see that every principal homomorphism is also a crossed homomorphism. The set of all principal homomorphism to A is denoted by $P(G, A)$.

It is known that the first cohomology group of G with coefficients in A corresponds with the equivalence classes of crossed homomorphisms with respect to principal homomorphisms, i.e

$$H^1(G, A) \cong \mathrm{Der}(G, A) \big/ P(G, A)$$

(see for example [22, Chapter IV, Section 2]). Later in the dissertation (see Proposition 4.2.12) we will give an analogue to this characterization for relative cohomology.

For any pair of $\mathbb{Z}[G]$-modules M, N there exists a natural diagonal action of G on the groups $\mathrm{Hom}_{\mathbb{Z}}(M, N)$, exploiting the bifunctoriality of $\mathrm{Hom}_{\mathbb{Z}}(\cdot, \cdot)$ and the fact that G acts on both coordinates. Such action is explicitely determined by the expression

$$(gf)(m) := gf(g^{-1}n) \qquad g \in G, f \in \mathrm{Hom}_{\mathbb{Z}}(M, N), m \in M.$$

From this action, the following identification is straightforward

$$\mathrm{Hom}_{\mathbb{Z}[G]}(M, N) = \mathrm{Hom}_{\mathbb{Z}}(M, N)^G. \tag{2.2.2}$$

In particular, by virtue of the well-known Hom-Tensor adjunction, for any other $\mathbb{Z}[G]$-module Q we know that

$$\mathrm{Hom}_{\mathbb{Z}}(M \otimes Q, N)^G \cong \mathrm{Hom}_{\mathbb{Z}}(M, \mathrm{Hom}_{\mathbb{Z}}(Q, N))^G$$

so observe that, as a consequence of (2.2.2) we have the isomorphism

$$\mathrm{Hom}_{\mathbb{Z}[G]}(M \otimes Q, N) \cong \mathrm{Hom}_{\mathbb{Z}[G]}(M, \mathrm{Hom}_{\mathbb{Z}}(Q, N)). \tag{2.2.3}$$

The diagonal G-action over the Hom groups defined above also translates into a helpful property for working with the Ext-groups.

Proposition 2.2.6 ([22] Proposition III.2.2). *Let M and N be G-modules. If M is $\mathbb{Z}$-torsion free then*

$$\operatorname{Ext}^*_{\mathbb{Z}[G]}(M,N) \cong H^*(G, \operatorname{Hom}_{\mathbb{Z}}(M,N))$$

where G acts diagonally on $\operatorname{Hom}_{\mathbb{Z}}(M,N)$.

Consider a subgroup inclusion $H \hookrightarrow G$. As we mentioned in the first section of this chapter, any $\mathbb{Z}[G]$-module A inherits an obvious structure as $\mathbb{Z}[H]$-module just by restriction of scalars. We will call this the *restriction module* with respect to H, denoted by $\operatorname{Res}^G_H(A)$. Moreover, given a $\mathbb{Z}[H]$-module M, from the obvious ring inclusion $\mathbb{Z}[H] \hookrightarrow \mathbb{Z}[G]$ coming from the subgroup inclusion we can, as well, consider the process of extension of scalars. The output is what is known as the *induction module* with respect to H

$$\operatorname{Ind}^G_H(M) := \mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} M.$$

2.2.1 $K(G,1)$-spaces

Definition 2.2.7. Let G be a group. In the following we will say that a space X is of type $K(G,1)$ if its homotopy groups satisfy

$$\pi_i(X) \cong \begin{cases} G & \text{if } i = 1, \\ \{0\} & \text{if } i \neq 1. \end{cases}$$

If the space X is taken as a CW complex (as it will always be our case), an immediate consequence of Whitehead's theorem is that the homotopy type of X is determined by the isomorphism class of G, hence any two $K(G,1)$ CW complexes are homotopically equivalent. As a result, by abuse of notation, we will also write $K(G,1)$ as a space instead of X if our considerations are independent of the chosen model space of type $K(G,1)$. We say that such space $K(G,1)$ is an *aspherical space*, or an *Eilenberg-MacLane space* associated to G or, equivalently, the *classifying space* of G, and it is sometimes also denoted by BG. The universal cover of BG will be denoted by EG, and it is sometimes called the *total space* of G. The classifying property that gives BG its name states that, for any principal G-bundle $f\colon X \to Y$ (see precise definition later this chapter) there exists a map $\varphi\colon Y \to BG$ (called *classifying map*) such that the following commutative square is a pullback diagram

$$\begin{array}{ccc} X & \longrightarrow & EG \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle p} \\ Y & \xrightarrow[\varphi]{} & BG. \end{array}$$

One way to visualize a CW structure on EG is by regarding it as the Δ-complex (or weak simplicial complex) having for n-simplices the ordered $(n+1)$-tuples of elements of G, $[g_0, g_1, \cdots, g_n]$. The attaching of such a n-simplex to $(n-1)$-simplices of the form $[g_0, \cdots, \hat{g}_i, \cdots, g_n]$ (where $\hat{g}_i$ denotes that the vertex g_i is removed) is performed in the same way a standard n-simplex attach to its faces. Notice that the following assignment

$$[g_0, g_1, \cdots, g_n] \to [1, g_0, g_1, \cdots, g_n]$$

defines an extra degeneracy in EG

$$s_{-1}\colon EG_n \to EG_{n+1}$$

and so EG is a contractible space (see, for example, [65, Lemma III.5.1]). There is an obvious left action of G on EG, defined, for any $g \in G$, by

$$g([g_0, g_1, \cdots . g_n]) = [gg_0, gg_1, \cdots, gg_n].$$

It is immediately seen that only the identity element sends any tuple to itself, and so it is easy to check that this constitutes a covering space action. Therefore, the space EG is the universal cover of its quotient under the orbit map by this action. Thus we can define, without loss of generality, $BG = EG/G$. Given that the left G-action on EG consist basically on freely permuting the simplices, the Δ-complex structure of EG is naturally inherited by BG. Since all the vertices of EG are identified to each other through the action of G, we can consider BG as having just one vertex.

Remark 2.2.8. Recall that every Δ-complex X can be understood as a CW-complex in which each cell e^n_α is equipped with a distinguished characteristic map of the form

$$\sigma_\alpha\colon \Delta^n \to X$$

satisfying that its restriction to each face Δ^{n-1} of Δ is the corresponding distinguished map σ_β for some $(n-1)$-dimensional cell e^{n-1}_β. As such, we can just look at Δ-complexes as CW-complexes with an extra layer of combinatorial information.

This construction is usually very big and, as such, it can be useful to have other possible models for EG in mind. One of the most prevalent in the literature is the so called *Milnor construction*, first presented in the celebrated article [100]. This model is defined as the infinite join of the group G with itself, i.e.

$$EG := \operatorname{colim}_n(\underbrace{G * G * \cdots * G}_{n}) = *^\infty G \tag{2.2.4}$$

Regardless of the model considered, by the contractibility of EG, the associated cellular chain complex of EG augmented over $\mathbb{Z}$ is, obviously, a $\mathbb{Z}[G]$ free (hence projective) resolution of the trivial $\mathbb{Z}[G]$-module $\mathbb{Z}$. Therefore, by the comparison theorem, we immediately observe that the purely algebraically defined notion of cohomology of a group G with coefficients in a $\mathbb{Z}[G]$-module A admits an obvious topological interpretation as the singular (cellular) cohomology of the associated $K(G,1)$-space with local coefficient system A

$$H^*(G, A) = H^*(K(G,1); A)$$

In particular, there is an obvious free resolution $F_* \to \mathbb{Z}$ of $\mathbb{Z}$ coming from the Δ-complex structure of EG, known as the *standard resolution*, $\mathcal{F}_* = \{F_*, \partial_*\}$. For each $n > 0$ the module

F_n is the free $\mathbb{Z}$-module generated by the $(n+1)$-tuples $[g_0, g_1, \cdots, g_n]$ of elements of G, with the previously defined G-action, and the boundary operator is defined as

$$\partial_n \colon F_n \to F_{n-1} \qquad \partial_n = \Sigma_{i=0}^{n}(-1)^i d_i$$

where the d_i are induced by the face maps of the original Δ-complex, i.e.

$$d_i([g_0, g_1, \cdots, g_n]) = [g_0, g_1, \cdots, \hat{g}_i, \cdots, g_n].$$

Definition 2.2.9. Let G be a group, we define the *cohomological dimension* of G, denoted by $\mathrm{cd}(G)$, as one of the following equivalent conditions:

$$\mathrm{cd}(G) = \begin{cases} \inf\{n \in \mathbb{Z} | \ \mathbb{Z} \text{ admits a } \mathbb{Z}[G]-\text{projective resolution of length } n\} \\ \sup\{n \in \mathbb{Z} | \ H^n(G, A) \neq 0 \text{ for some } \mathbb{Z}[G]-\text{module } M\} \end{cases}$$

The *geometric dimension* of G, denoted by $\mathrm{gd}(G)$ is defined as the minimal dimension of a model of BG.

Remark 2.2.10. Notice that if G is a discrete group with torsion, it contains an isomorphic copy of a non trivial cyclic group. But any such cyclic group has infinite periodic projective resolution (see for example [22, Example III.1.2]), hence infinite cohomological dimension. Therefore, only groups without torsion can be cohomologically finite.

Since the cellular chain complex of EG yields a free (hence projective) resolution of $\mathbb{Z}$ as trivial $\mathbb{Z}[G]$-module, we clearly have the inequality

$$\mathrm{cd}(G) \leq \mathrm{gd}(G).$$

What about the equality? The exact relationship between both dimensions is described by a beautiful result due to S. Eilenberg and T. Ganea, appearing as Theorem 1 in their celebrated article [47]. As Eilenberg and Ganea limited themselves to state the theorem, without demonstration, we refer the readers to [22, Theorem VIII.7.1] for a comprehensive and thoroughly detailed proof.

Theorem 2.2.11 (Eilenberg-Ganea theorem for cohomological dimension)**.** *Let G be a discrete group and put $n := \max\{\mathrm{cd}(G), 3\}$. Then there exists an n-dimensional $K(G,1)$-space. That is, for $\mathrm{cd}(G) \geq 3$ we always have the equality*

$$\mathrm{gd}(G) = \mathrm{cd}(G).$$

It is important to remark that, by the Stallings-Swan theorem (see [113] and [114]) we know that a group G has $\mathrm{cd}(G) = 1$ if and only if G is free, and any free group has a one dimensional model for BG, given by a wedge of circles, so the equality between both dimensions is preserved. Consequently, only the case $\mathrm{cd}(G) = 2$ is still unknown. This leads to the formulation of the famous Eilenberg-Ganea conjecture.

Conjecture 2.2.12 (The Eilenberg-Ganea conjecture)**.** *If* $\mathrm{cd}(G) = 2$ *then there is a* 2*-dimensional* $K(G,1)$.

The investigation of the Eilenberg-Ganea conjecture goes way beyond the scope of this dissertation. For our purposes, it just suffices to keep in mind that, in some situations, we have to adress the possible occurence of the problematic case of $\mathrm{cd}(G) = 2$ in our formulation of dimension-related hypothesis on groups.

In the setting of topological complexity, the main groups that will be involved are those whose cohomological dimension is finite. By our discussion above, this implies that the geometric dimension of their classifying spaces is finite as well.

Definition 2.2.13. A group G is called *geometrically finite* if there exists a finite CW complex of type $K(G,1)$.

2.2.2 The Cohomology ring

To close this section, let us very briefly remind the notion of cup product in group cohomology (we assume previous knowledge of the topological definition for cellular cohomology of topological spaces, see for example [74, Section 3.2, Chapter 3]), which turns the cohomology of a group into a (graded) commutative ring, and that will be of frequent use later on. We will just sketch the construction and mention the main properties, and we will not delve deeply into the details.

Notice that, if we have two different $\mathbb{Z}[G]$-projective resolutions $\mathcal{P}_* \to \mathbb{Z}$ and $\mathcal{Q}_* \to \mathbb{Z}$, its tensor product $\mathcal{P}_* \otimes \mathcal{Q}_* \to \mathbb{Z}$, where G is acting diagonally by components (indeed acting by restriction of scalars with respect to the diagonal inclusion $G \to G \times G$) is also a $\mathbb{Z}[G]$-projective resolution.

Let M and N be a pair of $\mathbb{Z}[G]$-modules, and $\mathcal{P}_*$ and $\mathcal{Q}_*$ two $\mathbb{Z}[G]$-projective resolutions of $\mathbb{Z}$. We can define a product at cochain level, called the *cross-product* by

$$\begin{aligned} \times\colon \mathrm{Hom}_{\mathbb{Z}[G]}(\mathcal{P}_*, M) \otimes \mathrm{Hom}_{\mathbb{Z}[G]}(\mathcal{Q}_*, N) &\longrightarrow \mathrm{Hom}_{\mathbb{Z}[G\times G]}(\mathcal{P}_* \otimes \mathcal{Q}_*, M \otimes N) \\ f \otimes g &\longmapsto (f \times g) := \Big[(x,y) \mapsto (-1)^{|g|\cdot|x|}(f(x) \otimes g(y))\Big] \end{aligned}$$

where here $|g|$ and $|x|$ denote the degree of g and x, respectively. It is easy to show that the product of two cocycles is again a cocycle whose cohomology class depends just on the classes of the given ones, so this product induces a subsequent cross product in cohomology

$$\times\colon H^m(G,M) \otimes H^n(G,N) \to H^{m+n}(G \times G, M \otimes N)$$

Consider now that $\mathcal{P}_* = \mathcal{Q}_*$. There is, a priori, not obvious map from $\mathcal{P}_*$ to $\mathcal{P}_* \otimes \mathcal{P}_*$, but by the comparison theorem we know that there exists an augmentation-preserving map $d^*\colon \mathcal{P}_* \to \mathcal{P}_* \otimes \mathcal{P}_*$, and that any two chain maps of this kind are chain homotopy equivalent. Any such map of this type is called a *diagonal approximation*. In particular, if $\mathcal{P}_*$ is taken as the standard resolution F_*, there is a well known diagonal approximation $\Delta\colon F_* \to F_* \otimes F_*$ called the *Alexander-Whitney diagonal map*, whose explicit expression is determined by the formula:

$$\Delta([g_0, \cdots, g_n]) := \sum_{k=0}^{n} [g_0, \cdots, g_k] \otimes [g_k, \cdots, g_n] \tag{2.2.5}$$

(see more details in [22, p. 108]).

The cup product at cohomology level will be induced in turn by a cochain cup product, defined by composing the cochain cross product previously introduced with a diagonal approximation. In particular, if we are using the standard resolution, for two cochains $u \in \mathrm{Hom}_{\mathbb{Z}[G]}(\mathcal{F}_*, M)$ and $v \in \mathrm{Hom}_{\mathbb{Z}[G]}(\mathcal{F}_*, M)$ we can define its product by

$$u \cup v := (u \times v) \circ \Delta$$

where here Δ denotes specifically the Alexander-Whitney diagonal map 2.2.5. We are now prepared to recall the definition of the cup product in group cohomology.

Definition 2.2.14. Let G be a group, $\Delta\colon G \to G \times G$ the diagonal inclusion, M and N be two $\mathbb{Z}[G]$-modules, and consider two cohomology classes $u \in H^p(G, M)$ and $v \in H^q(G, N)$. We define the *cup product* of u and v (denoted by $u \cup v$) to be the element

$$\Delta^*(u \times v) \in H^{p+q}(G, M \otimes N)$$

and $M \otimes N$ is regarded as a $\mathbb{Z}[G]$-module through the diagonal action of G.

As later in the text we intend to introduce a relative analogue of the cup product for a different cohomology theory, we need to summarize the properties that characterize such product. In the next theorem we list them.

Theorem 2.2.15 (Chapter V.3 of [22])**.** *The cup product satisfies the following properties:*

(1) Dimension 0: *The cup product $H^0(G, M) \otimes H^0(G, N) \to H^0(G, M \otimes N)$ is the map*

$$M^G \otimes N^G \to (M \otimes N)^G$$

induced by the inclusions $M^G \hookrightarrow M$ and $N^G \hookrightarrow N$.

(2) Naturality with respect to coefficient homomorphisms: *Given $\mathbb{Z}[G]$-module homomorphisms $f\colon M \to M'$ and $g\colon N \to N'$ and cohomology classes $u \in H^*(G, M)$ and $v \in H^*(G, N)$ we have*

$$(f \otimes g)_*(u \cup v) = f_*(u) \cup g_*(v).$$

(3) Compatibility with the connecting homomorphism: *Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of $\mathbb{Z}[G]$-modules and let N be a $\mathbb{Z}[G]$-module such that the sequence*

$$0 \to M' \otimes N \to M \otimes N \to M'' \otimes N \to 0$$

is exact. Then we have $\delta(u \cup v) = \delta(u) \cup v$ for any $u \in H^p(G, M'')$ and $v \in H^q(G, N)$, and where δ here stands for the connecting homomorphism.

Similarly, if $0 \to N' \to N \to N'' \to 0$ is a short exact sequence such that

$$0 \to M \otimes N' \to M \otimes N \to M \otimes N'' \to 0$$

is exact then $\delta(u \cup v) = (-1)^p u \cup \delta(v)$ for any $u \in H^p(G, M)$ and $H^q(G, N'')$.

(4) Existence of identity: *The element* $1 \in H^0(G,\mathbb{Z}) = \mathbb{Z}$ *satisfies* $1 \cup u = u \cup 1 = u$ *for all class* $u \in H^*(G,M)$.

(5) Associativity: *Given* $u_i \in H^*(G,M_i)$ *for* $1 \le i \le 3$ *we have* $(u_1 \cup u_2) \cup u_3 = u_1 \cup (u_2 \cup u_3)$.

(6) Graded commutativity: *For any classes* $u \in H^p(G,M)$ *and* $v \in H^q(G,N)$ *we have*

$$u \cup v = (-1)^{pq}(v \cup u).$$

2.3 G-spaces and classifying spaces of families of subgroups

We will review here the basic notions of G-spaces and equivariant maps, with a special focus on the notion of classifying space with respect to certain families of subgroups of G, and the equivariant cohomology theory due to Bredon associated to them. The standard references for the reader who wish to deepen their knowledge on the subject are the textbooks of W. Lück ([93], see also his great and compact survey on the topic [92]) and T. tom Dieck [37]. For the basics of Bredon cohomology theory, of course we refer to the seminal book of G. Bredon himself, [20].

Definition 2.3.1. A *G-space* X is a topological space equipped with a group action by G such that, for each $g \in G$ we have that $g\colon X \to X$ is a continuous map. For each subgroup $H \le G$ we denote by X^H the set of all invariant points under H, that is

$$X^H = \{x \in X \mid hx = x, \forall h \in H\}.$$

We say that a G-space X is G-connected or G-simply connected if, for each subgroup $H \le G$, the invariant space X^H is, respectively, connected or simply connected.

For each point $x \in X$ we can define the *isotropy subgroup* at x, denoted by G_x, as the subgroup of G of the form

$$G_x = \{g \in G \mid gx = x\}. \tag{2.3.1}$$

If instead of fixing a point we require fixing a whole subspace $Y \subset X$, we can consider the isotropy subgroup of Y by

$$G_Y = \{g \in G \mid gy = y, \forall y \in Y\}.$$

A G-equivariant map (also called simply G-map) $f\colon X \to Y$ between G-spaces X and Y is a continuous map satisfying $f(gx) = g \cdot f(x)$ for any $g \in G$ and $x \in X$. For any two G-equivariant maps $f_0, f_1\colon X \to Y$ we say that they are G-homotopic if there exists a G-equivariant homotopy between them, i.e. a continuous G-map

$$F\colon X \times I \to Y, \qquad F(x,0) = f_0(x) \qquad \wedge \qquad F(x,1) = f_1(x) \qquad \forall x \in X$$

where we consider G acting trivially on the unit interval I. Evidently, for two G-spaces X and Y a *G-homotopy equivalence* is a G-equivariant map $f\colon X \to Y$ with a G-homotopy inverse, i.e. a G-map in reverse direction $g\colon Y \to X$ such that $g \circ f$ is G-homotopic to id_X and $f \circ g$ is G-homotopic to id_Y.

Definition 2.3.2. A $G-CW$-complex X is a G-space together with a G-invariant filtration

$$\emptyset = X_{-1} \subseteq X_0 \subseteq X_1 \subseteq \cdots \subseteq X_n \subseteq \cdots \subseteq \bigcup_{k \geq 0} X_k = X$$

such that X is equipped with the colimit (or final) topology with respect to such filtration, and with X_n obtained from X_{n-1}, for each $n \geq 0$, by a process of attaching equivariant n-dimensional cells, that is, such that there exists a pushout diagram of G-maps of the form

$$\begin{array}{ccc} \coprod_{i \in I_n} G/H_i \times S^{n-1} & \xrightarrow{\amalg_{i \in I_n} q_i^n} & X_{n-1} \\ \downarrow & & \downarrow \\ \coprod_{i \in I_n} G/H_i \times D^n & \xrightarrow[\amalg_{i \in I_n} Q_i^n]{} & X_n \end{array} \tag{2.3.2}$$

For each n, the space X_n is called the *n-skeleton*, in complete analogy to the non-equivariant case.

There is an analogue of the classic Whitehead theorem for equivariant cellular complexes, see for example [97, Chapter 1, Theorem 3.2].

Theorem 2.3.3 (Equivariant Whitehead Theorem)**.** *Let $f\colon Y \to Z$ be a G-map between G-CW-complexes such that for each $H \leq G$ and for any basepoint $x_0 \in Y^H$ the induced map*

$$\pi_i(Y^H, x_0) \to \pi_i(Z^H, f(x_0))$$

is an isomorphism for all $i < k$ and an epimorphism for $i = k$. Then for any G-CW-complex X the induced map on the set of G-homotopy classes

$$f_* : [X, Y]_G \to [X, Z]_G$$

is an isomorphism if $\dim X < k$*, and an epimorphism if* $\dim X = k$*.*

Definition 2.3.4. Let B be a topological space, and suppose P is a G-space equipped with a G-equivariant map $p\colon P \to B$ where G acts trivially on B, (i.e. p factors uniquely through the orbit space P/G). We say that (P, p, B) is a *principal G-bundle* if B has a covering by open sets $\{U_i\}_i$ such that there exist G-equivariant homeomorphisms

$$\phi_{U_i}\colon p^{-1}(U_i) \to U_i \times G$$

making the following diagram commutative

$$\begin{array}{ccc} p^{-1}(U_i) & \xrightarrow{\phi_{U_i}} & U \times G \\ {\scriptstyle p}\downarrow & \swarrow{\scriptstyle \rho} & \\ U_i & & \end{array}$$

The requisite required in the definition above is called the local triviality condition. Here $U \times G$ has a (right) G-action determined by the natural action of G on itself, that is, defined by $(u, g)h = (u, gh)$. Notice that this is implying that G acts freely on P and that p factors through a homeomorphism $\overline{p} \colon P/G \to B$. Therefore, a principal G-bundle is just a locally trivial free G-space P with orbit space B.

Every G-space X determines an obvious principal G-bundle $(X, \rho, X/G)$ via the orbit projection map. If we consider another G-space Y, and a G-equivariant map $f \colon X \to Y$, we define the quotient map associated to f as the map induced between the orbit spaces by f, i.e.

$$\rho_f \colon X/G \to Y/G \qquad \rho_f(xG) = f(x)G.$$

Definition 2.3.5. Let (P, p, B) be a principal G-bundle, and F a (left) G-space. We can consider a (right) action of G on $P \times F$ by $(x, y)g = (xg, g^{-1}y)$ for $(x, y) \in X \times F$ and $g \in G$. Put $P_F := (P \times F)/G$ under the action just defined, and let $p_F \colon P_F \to B$ be the factorization of the composition

$$P \times F \xrightarrow{\pi_P} P \xrightarrow{p} B$$

by the projection $P \times F \to P_F$. Then we call (P_F, p_F, B) the *fibre bundle over B with fibre F*, with associated principal G-bundle (P, p, B). The group G is called the structure group of the fibre bundle.

Thye following theorem yields a criteria to measure the sections of fibre bundles, and will be instrumental later on in this dissertation.

Theorem 2.3.6 (Chapter 4, Theorem 8.1 of [78]). *Let G be a group, (P, p, B) a principal G-bundle and (P_F, p_F, B) a fibre bundle with structure group G constructed as above. The sections of the fibre bundle (P_F, p_F, B) are in bijective correspondence with maps*

$$\phi \colon P \to F \qquad \phi(xg) = g^{-1}\phi(x) \;\; x \in P, g \in G.$$

The cross section corresponding to each ϕ is defined by

$$s_\phi(xG) := (x, \phi(x))G \in P_F.$$

Given a group G and any subgroup $H \leqslant G$, consider EG, the total space of G, seen as an H-space with the action induced by the subgroup inclusion $\iota \colon H \to G$. The orbit projection map under this action $EG \to EG/H$ is a principal H-bundle, with contractible total space. Thus, one observes that EG/H is a model for the classifying space BH. Furthermore, the projection $EG/H \to BG$ with respect to the residual G-action is a fibration with fiber the coset space G/H. Consequently, we have that, up to homotopy, there exists a fibration

$$G/H \to K(H,1) \xrightarrow{K(\iota,1)} K(G,1)$$

2.3.1 Classifying spaces with respect to subgroup families and Bredon cohomology

Recall that a family $\mathcal{F}$ of subgroups of G is said to be *full* provided that it is non-empty and is closed under conjugation and the taking of subgroups. This also entails the fact that the family is closed under taking intersections as well. We will write $\langle H \rangle$ for the smallest full family of subgroups of G containing H.

Definition 2.3.7. The *classifying space of G with respect to $\mathcal{F}$* is a G-CW complex $E_{\mathcal{F}}G$ satisfying the following conditions:

- every isotropy group of $E_{\mathcal{F}}G$ belongs to $\mathcal{F}$,
- *Universal property:* for any G-CW complex X with all isotropy groups in $\mathcal{F}$ there exists a unique (up to G-equivariant homotopy) G-equivariant map $X \to E_{\mathcal{F}}G$.

In particular, there is a unique G-equivariant map $EG \to E_{\mathcal{F}}G$, where EG is the classifying space of G with respect to the family consisting of the trivial subgroup, or, in other words, the universal cover of a $K(G,1)$ space. Observe as well that for any subgroup $H \leqslant G$ the family $\mathcal{F} \cap H = \{K \in \mathcal{F} \mid K \leqslant H\}$ is, in turn, a full family of subgroups of H. Consequently, $E_{\mathcal{F}}G$ is a H-space when considering the action of H induced by the subgroup inclusion, and since $\mathcal{F}$ is closed under taking subgroups and intersections, $E_{\mathcal{F}}G$ is also a model for $E_{\mathcal{F}\cap H}H$. Further properties of $E_{\mathcal{F}}G$ are discussed at length in [92].

There are many possible ways to construct models for the classifying space $E_{\mathcal{F}}(G)$, and we will recall two of them that will be useful in Chapter 4. Let $\{H_i\}_i$ be a collection of subgroups of G, and consider the family generated by this set, $\langle \{H_i\}_i \rangle$, i.e. the family consisting of all the subgroups of the members of $\{H_i\}_i$ and all their conjugates by elements of G. Put

$$\Delta_{\langle \{H_i\} \rangle} := \coprod_i G/H_i.$$

For this family, J.A. Arciniega-Nevárez and J.L. Cisneros-Molina built the following model:

Proposition 2.3.8 (Proposition 4.16 of [4])**.** *A model for $E_{\langle \{H_i\}_i \rangle}(G)$ is the geometric realization of the simplicial set whose n-simplices are the ordered $(n+1)$-tuples $(x_0, \cdots, x_n)$ of elements of $\Delta_{\langle \{H_i\} \rangle}$. The face operators are given by*

$$d_i(x_o, \cdots, x_n) = (x_0, \cdots, \hat{x}_i, \cdots, x_n)$$

and the degeneracy operators are defined by

$$s_i(x_0, \cdots, x_n) = (x_0, \cdots, x_i, x_i, \cdots, x_n).$$

The action of any $g \in G$ on an n-simplex $(x_0, \cdots, x_n)$ gives the simplex $(gx_0, \cdots, gx_n)$.

Also, for the particular case of the family $\langle H \rangle$, as explained by J. V. Blowers in [18, Section IV], one possible model for the classifying space $E_{\langle H \rangle}(G)$ is the *Milnor-Blowers construction* which is defined, in close similarity to the original Milnor join model for EG (recall (2.2.4)) as the infinite join of the coset space by H, i.e.

$$E_{\langle H \rangle}(G) \simeq *^{\infty}(G/H) \tag{2.3.3}$$

Let us briefly review now the definition of Bredon cohomology. In order to do so, first we have to introduce the necessary formalism of modules over an orbit category associated to a group.

Definition 2.3.9. Let G be a group. Define:

- The *orbit category* of G associated to a family $\mathcal{F}$ of subgroups of G, written $\mathrm{Or}_{\mathcal{F}}G$, is a category whose objects are homogeneous G-spaces G/K for $K \in \mathcal{F}$, and morphisms are G-equivariant maps between them.
- A $\mathrm{Or}_{\mathcal{F}}G$*-module* is a contravariant functor from $\mathrm{Or}_{\mathcal{F}}G$ to the category of abelian groups.
- A $\mathrm{Or}_{\mathcal{F}}G$*-homomorphism* between $\mathrm{Or}_{\mathcal{F}}$-modules is, consequently, a natural transformation.

The category of $\mathrm{Or}_{\mathcal{F}}G$-modules inherits the structure of an abelian category from the category of abelian groups; in particular, the notion of a projective $\mathrm{Or}_{\mathcal{F}}G$-module is defined. If the family contains the trivial subgroup, the *principal component* refers to evaluating the module or morphism on the orbit object determined by the trivial group $\{1\} \subset G$, i.e. on the $G/\{1\}$ component.

For every $G/K \in \mathrm{Or}_{\mathcal{F}}$, we define the *free* $\mathrm{Or}_{\mathcal{F}}$*-module based at* G/K, denoted by

$$\mathbb{Z}[\mathrm{Map}_G(G/\cdot, G/K)]$$

as the $\mathrm{Or}_{\mathcal{F}}$-module which assigns to an orbit object G/H the free $\mathbb{Z}$-module

$$\mathbb{Z}[\mathrm{Map}_G(G/H, G/K)]$$

generated by the set of G-equivariant maps from G/H to G/K. For any other $\mathrm{Or}_{\mathcal{F}}$-module $\underline{M}$ there exists, as a consequence of Yoneda's Lemma, a natural bijection of $\mathbb{Z}$-modules

$$\mathrm{Hom}_{\mathrm{Or}_{\mathcal{F}}}(\mathbb{Z}[\mathrm{Map}_G(G/\cdot, G/K)], \underline{M}) \to \underline{M}(G/K) \qquad \text{given by } f \mapsto f(G/K)(\mathrm{id}_{G/K}).$$

We say that a $\mathrm{Or}_{\mathcal{F}}$-module is free if it is isomorphic to a direct sum of the form

$$\bigoplus_I \mathbb{Z}[\mathrm{Map}_G(G/\cdot, G/K_i)] \tag{2.3.4}$$

for an appropiate choice of objects $G/K_i \in \mathrm{Or}_{\mathcal{F}}$ and index set I.

Definition 2.3.10. Let $\mathcal{F}$ be a full family of subgroups of G. Given a G-CW complex X with isotropy groups in $\mathcal{F}$, define the $\mathrm{Or}_{\mathcal{F}}$ *n-cellular group* of X as a $\mathrm{Or}_{\mathcal{F}}G$-module $\underline{C}_n(X)$ as follows.

- $\underline{C}_n(X)(G/K) = C_n(X^K)$, where $C_n(X^K)$ denotes the group of cellular n-chains of $X^K = \{x \in X \mid kx = x \text{ for any } k \in K\}$.
- If $\varphi\colon G/K \to G/L$ is a G-equivariant map, then $\varphi(gK) = gg_0L$ for some $g_0 \in G$ such that $g_0^{-1}Kg_0 \subseteq L$. Consequently, φ induces a cellular map
$$X^L \to X^K \qquad x \mapsto g_0x,$$
which descends to the chain level to define a homomorphism
$$\underline{C}_n(\varphi)\colon C_n(X^L) \to C_n(X^K).$$

For any $n \geq 1$, there is an obvious $\mathrm{Or}_{\mathcal{F}}G$-homomorphism $\underline{d}_n\colon \underline{C}_n(X) \to \underline{C}_{n-1}(X)$, and so we have the $\mathrm{Or}_{\mathcal{F}}G$*-cellular chain complex* $(\underline{C}_*(X), \underline{d}_*)$.

According to the pushout attachments in 2.3.2 for each $n \geq 0$, and by excision, if we evaluate these cellular groups at each object $G/H \in \mathrm{Or}_{\mathcal{F}}$ we obtain the chain of isomorphisms

$$\begin{aligned} \underline{C_n}(X)(G/H) &\cong C_n(X^H) = H_n(X_n^H, X_{n-1}^H) \cong H_n(\coprod_{I_n}(G/H_i) \times (D^n, S^{n-1})) \\ &\cong \bigoplus_{I_n} H_n((G/H_i) \times (D^n, S^{n-1})) \cong \bigoplus_{I_n} H_0((G/H_i)^H) \\ &\cong \bigoplus_{I_n} \mathbb{Z}[(G/H_i)^H] \cong \bigoplus_{I_n} \mathbb{Z}[\mathrm{Map}_G(G/H, G/H_i)]. \end{aligned} \tag{2.3.5}$$

which, in view of 2.3.4, allows us to see that $\underline{C_n}(X)$ is indeed a free $\mathrm{Or}_{\mathcal{F}}$-module for every $n \geq 0$.

We can now introduce the definition of the Bredon equivariant group cohomology theory.

Definition 2.3.11. Using notation from Definition 2.3.10 above, define the *Bredon cohomology* of X with respect to the family $\mathcal{F}$ and with coefficients in a $\mathrm{Or}_{\mathcal{F}}G$-module $\underline{M}$ as

$$H^*_{\mathcal{F}}(X, \underline{M}) = H^*\big(\mathrm{Hom}_{\mathrm{Or}_{\mathcal{F}}G}(\underline{C}_*(X), \underline{M})\big).$$

The *Bredon cohomological dimension* of G with respect to $\mathcal{F}$, denoted $\mathrm{cd}_{\mathcal{F}}\, G$, is the length of the shortest possible $\mathrm{Or}_{\mathcal{F}}G$-projective resolution of $\underline{\mathbb{Z}}$, where $\underline{\mathbb{Z}}$ is a constant $\mathrm{Or}_{\mathcal{F}}G$-module which sends every morphism to $\mathrm{id}\colon \mathbb{Z} \to \mathbb{Z}$. Recall that $\langle H \rangle$ denotes the smallest full family of subgroups of G containing H.

By considering the evaluation on the principal component, we have a homomorphism of complexes

$$\mathrm{Hom}_{\mathrm{Or}_{\mathcal{F}}}(\underline{C_*}(X), \underline{A}) \to \mathrm{Hom}_{\mathbb{Z}[G]}(C_*(X), A) \tag{2.3.6}$$

Notice that if X is a free G-CW complex, it is built by attachment of G-cells without non-trivial isotropy subgroups. Therefore, the $\mathrm{Or}_{\mathcal{F}}$ cellular chain complex $\underline{C_*}(X)$ reduces to the principal component, and $\underline{C_*}(X) \cong C_*(X)$. In that situation, the homomorphism 2.3.6 induces an isomorphism

$$H^*_{\mathcal{F}}(X, \underline{A}) \to H^*(X/G, A). \tag{2.3.7}$$

The following is an analogue to Eilenberg-Ganea theorem in the setting of Bredon equivariant cohomology:

Theorem 2.3.12 ([94] Theorem 0.1, see also [92] Theorem 5.2)**.** *Let G be a discrete group, $\mathcal{F}$ a semi-full family of subgroups of G, and $d \in \mathbb{Z}$ such that $d \geq 3$. Then there exists a $G-CW$-model for $E_{\mathcal{F}}(G)$ of dimension lesser or equal to d if and only if $\operatorname{cd}_{\mathcal{F}}(G) \leq d$.*

2.4 Basic notions of spectral sequences

As some spectral sequences will be developed later on during this dissertation, we will recall the most basic definitions needed. This is just a slight glimpse into a notoriously complicated and technical topic, and we will go no further than introducing the most crucial notions to understand our arguments. For the reader interested in learning more about the subject (whose relevance in the field of algebraic topology is beyond any doubt), we refer to the classic source [98], perhaps the deepest and most thorough general textbook about the topic. For more specific approaches to the practical usage of tools coming from spectral sequences on homological algebra or group cohomology, we refer to [22], [116] or [107]. Additionally, for an elementary survey about practical usage of spectral sequences in the context of algebraic topology through examples, we recommend a particular favourite of the author of these lines, [36].

Definition 2.4.1. A differential bigraded module over a ring R is a collection of R-modules $\{E^{p,q}\}_{p,q}$ whith p and q integers, and a bi-graduated map $d\colon E^{*,*} \to E^{*,*}$ called differential, of bidegree either $(s, 1-s)$ of $(-s, s-1)$ for some integer s, satisfying $d \circ d = 0$.

The presence of a differential map allows to consider the homology of the bigraded module. In the case of cohomology (which will constitute our tool of interest) the cohomology groups associated to such differential are defined, for every value of p and q, in a natural way:

$$H^{p,q}(E^{*,*}, d) := \ker d\colon E^{p,q} \to E^{p+s,q-s+1} \Big/ \operatorname{Im} d\colon E^{p-s,q+s-1} \to E^{p,q} \,.$$

Definition 2.4.2. A *spectral sequence* of cohomological type is a collection of differential bigraded R-modules $\{E_r^{*,*}, d_r\}$ for $r \in \mathbb{N}_0$. For each r, the differential d_r is of bidegree $(r, 1-r)$, and for all values of p, q and r we have

$$E_{r+1}^{p,q} \cong H^{p,q}(E_r^{*,*}, d_r).$$

The bigraded module $E_r^{p,q}$ receives the name of *r-th page* of the spectral sequence.

If there is some integer $r > 0$ such that $E_k^{p,q} \cong E_r^{p,q}$ for all $k \geq r$, we say that the spectral sequence stabilizes at the page r. Hence, we denote $E_\infty^{p,q} := E_k^{p,q}$. This bimodule is usually called the *infinity page* of the sequence.

Definition 2.4.3. Given a graded R-module H^*, we say that a spectral sequence $\{E_r^{*,*}, d_r\}$ *converges* to H^* if there exists a filtration

$$0 = F^* H^* \subset \cdots \subset F^1 H^* \subset F^0 H^* = H^*$$

such that for all p and q there is an isomorphism

$$E_\infty^{p,q} \cong F^p H^{p+q} \Big/ F^{p+1} H^{p+q}$$

We denote it by $E_r^{p,q} \Rightarrow H^*$.

One of the richest sources for the construction of spectral sequences are double complexes, as they usually offer practical examples of differential bigraded modules. We proceed to briefly introduce them now.

Definition 2.4.4. A *double complex* over a ring R $\{M^{*,*}, d', d''\}$ is a bigraded module over R equipped with two R-linear maps, called directional differentials

$$d' \colon M^{p,q} \to M^{p+1,q} \qquad \text{and} \qquad d'' \colon M^{p,q} \to M^{p,q+1} \qquad \forall p, q$$

satisfying

$$d' \circ d' = 0 \qquad d'' \circ d'' = 0 \qquad d' \circ d'' + d'' \circ d' = 0$$

Each double complex M comes with an associated *total complex*, $\mathsf{Tot}^*(M)$ defined by putting, for each n,

$$\mathsf{Tot}^n(M) := \bigoplus_{p+q=n} M^{p,q}$$

with total differential $d = d' + d''$. The bidirectionality of the differential of the total complex allows to construct two different spectral sequences associated to its cohomology. Let $H_I^{*,*}(M) = H(M^{*,*}, d')$, i.e.

$$H_I^{p,q} = \ker d' \colon M^{p,q} \to M^{p+1,q} \Big/ \operatorname{Im} d' \colon M^{p-1,q} \to M^{p,q}$$

and, analogously $H_{II}^{*,*} = H^*(M^{*,*}, d'')$, that is

$$H_{II}^{p,q} = \ker d'' \colon M^{p,q} \to M^{p,q+1} \Big/ \operatorname{Im} d'' \colon M^{p,q-1} \to M^{p,q}$$

Both $H_I^{*,*}(M)$ and $H_{II}^{*,*}(M)$ are differential bigraded modules with differentials $\underline{d''}$ and $\underline{d'}$ induced, respectively, by d'' and d'. Therefore, we can define bigraded complexes

$$H_I^{*,*} H_{II}(M) = H(H_{II}^{*,*}(M), \underline{d'}) \qquad H_{II}^{*,*} H_I(M) = H(H_I^{*,*}(M), \underline{d''}).$$

Theorem 2.4.5 (Convergence theorem for first quadrant sequences)**.** *Given a double complex $\{M^{*,*}, d', d''\}$ there are two spectral sequences $\{F_r^{*,*}, d_r^F\}$ and $\{G_r^{*,*}, d_r^G\}$, satisfying*

$$F_2^{*,*} \cong H_I^{*,*} H_{II}(M) \qquad G_2^{*,*} \cong H_{II}^{*,*} H_I(M).$$

If $M^{p,q} = \{0\}$ for $p < 0$ or $q < 0$ then both sequences converge to $H^*(\mathsf{Tot}(M), d)$.

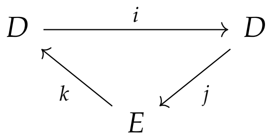
2.4.1 Exact couples

We will recall now one of the most useful and simple ways of visualizing spectral sequences, in the shape of exact couples. Every double complex can be rearranged into an exact couple, so both approaches are, actually, equivalent, and the use of each approach depends on the specific case and kind of information that we are interested in.

Definition 2.4.6. Let D and E be bigraded R-modules equipped with module homomorphisms

$$i\colon D \to D \qquad j\colon D \to E \qquad k\colon E \to D$$

such that they fit in a commutative diagram



We say that this is an *exact couple*, and we denote it by $\mathcal{C} = \{D, E, i, j, k\}$, if this diagram is exact at each group, i.e.

$$\operatorname{Im} i = \ker j \qquad \operatorname{Im} j = \ker k \qquad \operatorname{Im} k = \ker i.$$

The exactness property of the couple endowes E with a differential structure as R-module, with the differential defined as $d = j \circ k$. Indeed

$$d \circ d = (j \circ k) \circ (j \circ k) = j \circ (k \circ j) \circ k = 0.$$

As such, we can form an associated derived couple. Allow us just a moment to briefly describe the process of such formation, and the basic inner working of exact couples. Define

$$E' = H(E, d) = {}^{\ker d}\big/_{\operatorname{Im} d} \qquad D' = i(D) = \ker j.$$

Also, consider the following readjustment of the original homomorphisms present in the exact couple

$$i' = i_{|_{i(D)}}\colon D' \to D' \qquad j'\colon D' \to E' \qquad \text{with } j'(i(x)) + d(E) \in E'$$

for $x \in D$. Notice that, for $x, x' \in D$, if $i(x) = i(x')$ then $x - x' \in \ker i$, and there exists $y \in E$ satisfying $k(y) = x - x'$. Consequently

$$(j \circ k)(y) = d(y) = j(x) - j(x') \qquad \text{and} \qquad j(x) = j(x') + d(y)$$

so $j(x) + d(E) = j(x') + d(E)$ seen as cosets in E', which shows that j' is well defined. Define as well

$$k\colon E' \to D' \qquad \text{by } k'(e + d(E)) = k(e).$$

This map is also well defined: for elements $e, e' \in E$, if $e + d(E) = e' + d(E)$ then there exists some $x \in E$ such that $e' = e + d(x)$, and we see that

$$k(e') = k(e) + k(d(x)) = k(e) + (k \circ j \circ k)(x) = k(e).$$

Additionally, given that $d(e) = 0$ we know that $k(e) \in \ker j = \operatorname{Im} i = D'$. We define the *derived couple* associated to the exact couple $\mathcal{C}$ by $\mathcal{C}' = \{D', E', i', j', k'\}$. It is just an exercise of diagram chasing to see that the derived couple $\mathcal{C}'$ is itself an exact couple: notice that

$$\ker i' = \operatorname{Im} i \cap \ker i = \ker j \cap \operatorname{Im} k = k(k^{-1}(\ker j)) = k(\ker d) = k'(\ker d / \operatorname{Im} d) = \operatorname{Im} k'.$$

Given that $D' = i(D) = D / \ker i$ we also observe the chain of equalities

$$\begin{aligned} \ker j' &= j^{-1}(\operatorname{Im} d) \Big/ \ker i = j^{-1}(j(\operatorname{Im} k)) \Big/ \ker i \\ &= (\operatorname{Im} k + \ker j) \Big/ \ker i = (\ker i + \ker j) \Big/ \ker i \\ &= i(\ker j) = i(\operatorname{Im} i) = \operatorname{Im} i'. \end{aligned}$$

Finally we see that

$$\ker k' = \ker k / \operatorname{Im} d = \operatorname{Im} j / \operatorname{Im} d = \operatorname{Im} j'.$$

Once we have seen that exactness is preserved under taking derivation, given an arbitrary exact couple $\mathcal{C}$ it is just natural to iterate the process of taking derivations, which outputs the n^{th}-derived couple of $\mathcal{C}$,

$$\mathcal{C}^n = \{D^{(n)}, E^{(n)}, i^{(n)}, j^{(n)}, k^{(n)}\}.$$

Now, since by definition $E^{(n+1)} = H(E^{(n)}, d^{(n)})$ we see the appearance of an obvious spectral sequence, determined by the differential structure of the exact couple.

Theorem 2.4.7. *Let $D^{*,*}$ and $E^{*,*}$ be bigraded R-modules fitting inside an exact couple $C = \{D^{*,*}, E^{*,*}, i, j, k\}$. Then, there exists a spectral sequence $\{E_r, d_r\}$ of cohomological type with $E_r = (E^{*,*})^{(r-1)}$, the $(r-1)$-derived module of $E^{*,*}$, and $d_r = j^{(r)} \circ k^{(r)}$.*

2.5 Brief reminder of some Lie groups

To conclude this chapter, we will make a quick recollection of the definitions of some basic Lie groups that will constitute a rich source of examples in Chapter 7, for commodity of the reader, but without developing their properties. An accessible text on the matter is [73].

Definition 2.5.1. A *Lie group* (sometimes called continuous group) is a group G with the structure of a smooth manifold, such that the inversion and multiplication maps

$$G \to G,\ g \mapsto g^{-1} \qquad \text{and} \qquad G \times G \to G,\ (g, h) \mapsto gh$$

are smooth maps.

Many of the classical examples of Lie groups come from the realm of linear algebra, in the form of matrix groups. Denote the space of $n \times n$-dimensional matrices with entries on a field $\mathbb{F}$ by $M_n(\mathbb{F})$. As the reader surely knows, this is not a group under the operation of matrix multiplication, as not all of its elements admit a multiplicative inverse. Consequently, we just have to restrict such space to the largest subset admiting such a group structure.

Definition 2.5.2. The *general linear group* of order n over the field $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, denoted by $\mathrm{GL}(n, \mathbb{F})$, is the group of all $n \times n$ invertible matrices with entries in $\mathbb{F}$, i.e.

$$\mathrm{GL}(n, \mathbb{F}) := \{A \in M_n(\mathbb{F}) \mid \exists B \in M_n(\mathbb{F}) \text{ such that } AB = BA = I_n\}.$$

The *special linear group* is the (normal) subgroup $\mathrm{SL}(n, \mathbb{F}) \lhd \mathrm{GL}(n, \mathbb{F})$ of invertible matrices with determinant equal to 1. Let $\{A_k\}_{k \in J}$ be a sequence of matrices in $M_n(\mathbb{F})$. A *matrix Lie group* is a subgroup of the general linear group $G \leqslant \mathrm{GL}(n, \mathbb{C})$ with the property that for any sequence $\{A_k\}_{k \in J}$ of matrices in G convergent to a matrix A, then either $A \in G$ or A is not invertible.

Observe that the condition imposed on G is actually the same as requesting that G is a closed subset of $\mathrm{GL}(n, \mathbb{C})$. As such, we can equivalently say that a matrix Lie group is a closed subgroup of $\mathrm{GL}(n, \mathbb{C})$. Below we recall the definition of some of the most relevant of them, the (special) orthogonal and unitary Lie groups, and the symplectic group. Together with $\mathrm{GL}(n, \mathbb{F})$ and $\mathrm{SL}(n, \mathbb{F})$, they form the well-known family of classical Lie groups.

Definition 2.5.3. Define the following matrix groups:

(a) The *n-orthogonal group over* $\mathbb{F}$ as the subgroup of the n-general linear group on $\mathbb{F}$ of orthogonal matrices, i.e.

$$O(n, \mathbb{F}) = \{A \in \mathrm{GL}(n, \mathbb{F}) \mid A^T A = A A^T = I_n\}.$$

If $\mathbb{F} = \mathbb{R}$, it is denoted simply by $O(n)$. If, however, the field of choice is $\mathbb{F} = \mathbb{C}$, we denote $O(n, \mathbb{C}) = \mathrm{U}(n)$, and we call it the *unitary group* of degree n.

(b) The *special orthogonal group* of degree n by

$$\mathrm{SO}(n) := \{A \in O(n) \mid \det(A) = 1\}.$$

Respectively, the *special unitary group* of degree n, if the subgroup of $\mathrm{U}(n)$ defined by

$$\mathrm{SU}(n) := \{A \in \mathrm{U}(n) \mid \det(A) = 1\}.$$

(c) The *simplectic group of degree* $2n$ over $\mathbb{F}$ as the group of $2n \times 2n$-matrices over $\mathbb{F}$ preserving the non-degenerate skew-symmetric bilinear form defined by

$$\omega(x, y) = \sum_{i=1}^{n} (x_i y_{i+n} - y_i x_{i+n}) \ \forall x, y \in \mathbb{F}^n$$

that is $\mathrm{Sp}(2n, \mathbb{F}) = \{A \in M_{2n}(\mathbb{F}) \mid \omega(Ax, Ay) = \omega(x, y) \forall x, y \in \mathbb{F}^n\}$.

Equivalently, the form ω can be written as $\omega(x, y) = \langle \Omega x, y \rangle$, where here $\langle , \rangle$ denotes the standard symmetric bilinear form on $\mathbb{F}^n$ and

$$\Omega := \begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix}.$$

The *compact symplectic group* in degree n, in turn, can be seen as the intersection of the symplectic group $\mathrm{Sp}(2n,\mathbb{C})$ with the $2n \times 2n$-unitary group

$$\mathrm{Sp}(n) = \mathrm{Sp}(2n,\mathbb{C}) \cap \mathrm{U}(2n) = \mathrm{Sp}(2n,\mathbb{C}) \cap \mathrm{SU}(2n).$$

It is easy to see from the definitions that, for every $n \geq 2$, each of the aforementioned matrix Lie groups at degree $n-1$ are closed subgroups of their n-degree versions, i.e.

$$O(n-1) \leqslant O(n) \qquad \mathrm{U}(n-1) \leqslant \mathrm{U}(n) \qquad \mathrm{Sp}(n-1) \leqslant \mathrm{Sp}(n)$$
$$\mathrm{SO}(n-1) \leqslant \mathrm{SO}(n) \qquad \mathrm{SU}(n-1) \leqslant \mathrm{SU}(n)$$

Furthermore, as both (n) and $\mathrm{U}(n)$ are not centerless groups, we can considered their quotients by their respective centers, which gives us the *projective orthogonal* and *unitary groups*, respectively

$$\mathrm{PO}(n) := {O(n)}\big/{Z(O(n))} \qquad \mathrm{PU}(n) := {\mathrm{U}(n)}\big/{Z(\mathrm{U}(n))}\,.$$

The usual explicit construction of the *spin group*, $\mathrm{Spin}(n)$, is achieved through its identification with a subgroup of the group of invertible elements in the n-dimensional Clifford algebra. As we are just interested in a quick recollection of the definitions, we will instead describe $\mathrm{Spin}(n)$ in the geometric non-explicit way. In that sense, we can see $\mathrm{Spin}(n)$ as the Lie group that, as a manifold, is the universal cover of $\mathrm{SO}(n)$. Since $\pi_1(\mathrm{SO}(n)) = \mathbb{Z}_2$ for $n \geq 3$, this is a double cover, and $\mathrm{Spin}(n)$ fits in a short exact sequence of groups

$$\{1\} \to \mathbb{Z}_2 \to \mathrm{Spin}(n) \xrightarrow{p} \mathrm{SO}(n) \to \{1\}.$$

Indeed, the group operation can be defined in terms of path lifts: the preimage of the neutral element of $\mathrm{SO}(n)$ by the covering map, $p^{-1}(1)$ has two elements and, without loss of generality, we set one of them to be the identity, which we can denote e. For any two elements $\alpha, \beta \in \mathrm{Spin}(n)$, define their product as follows: choose paths γ_α and γ_β in $\mathrm{Spin}(n)$ such that $\gamma_\alpha(0) = \gamma_\beta(0) = e$, $\gamma_\alpha(1) = \alpha$ and $\gamma_\beta(1) = \beta$. These, in turn, define a path γ in $\mathrm{SO}(n)$ by putting $\gamma(t) := p(\gamma_\alpha(t)) \cdot p(\gamma_\beta(t))$. Denote by $\overline{\gamma}$ the unique lift of γ to $\mathrm{Spin}(n)$ through p, with $\overline{\gamma}(0) = e$. Then the product can be defined as $\alpha \cdot \beta = \overline{\gamma}(1)$.

The *exceptional group* G_2 is three simple Lie groups, in a complex form, a compact real form and a split real form. The compact real form, in particular, can be defined in two alternative but equivalent ways. Define, on the euclidean space $\mathbb{R}^7$, the associative 3-form by

$$\omega(u,v,w) := \langle u, v \times w\rangle \qquad \forall u,v,w \in \mathbb{R}^7$$

where here $\langle , \rangle$ is again the canonical bilinear form, and $v \times w$ stands for the usual cross product of vectors. Then G_2 is defined as the subgroup of $\mathrm{GL}(7,\mathbb{R})$ that preserves this 3-form. Alternatively, G_2 can be seen as the automorphism group of the octonions seen as a normed algebra.

CHAPTER 3

Topological complexity and sectional category

In this expository chapter we will recall the main definitions and properties of our main subjects of interest, those of topological complexity, Lusternik-Schnirelmann category and sectional category, that will be needed for the rest of the dissertation, and we will give some basic examples of computations. No original results are included here, and we will give plenty of bibliographic references for the interested reader.

3.1 The motion planning problem

Now we will proceed to turn the intuitive ideas that we outlined in the introduction into precise definitions. Any mechanical system S determines the array of all its possible states. Regarding each of them as a point on a space, we naturally obtain a topological space associated to the mechanical system, known as the *configuration space* of S. Without loss of generality we can assume all the spaces to be path connected (otherwise we just have to part the problem into each of the path-connected components). Obviously, each of the points of the configuration space X corresponds with a possible state of the system and, as such, continuous paths between points correspond with continuous motion between states of the system. As mentioned in the introduction, one of the crucial problems in robotics is the known as *motion planning problem*: given two different possible states x and y of the mechanical system, we strive to produce an algorithm instructing how to transition from x to y. Such an algorithm is called a *motion planner*. In the language of the configuration system, a motion planner in X is an algorithm which, given a pair of points $(x, y) \in X \times X$, outputs a path γ between x and y.

Definition 3.1.1. The *free path space* of X, denoted by PX, is defined as the set of all continuous paths in X

$$PX = \{\gamma\colon I = [0,1] \to X\}$$

with the compact open topology. The *path space fibration* of X is the fibration

$$\pi\colon PX \to X \times X$$

given by sending each path to its extreme points, i.e $\pi(\gamma) = (\gamma(0), \gamma(1))$.

In view of this definition, a motion planning algorithm can be described as a map $s\colon X \times X \to PX$ such that $\pi \circ s = \mathrm{id}_{X\times X}$. Hence, a motion planner is just a section of the path space fibration π.

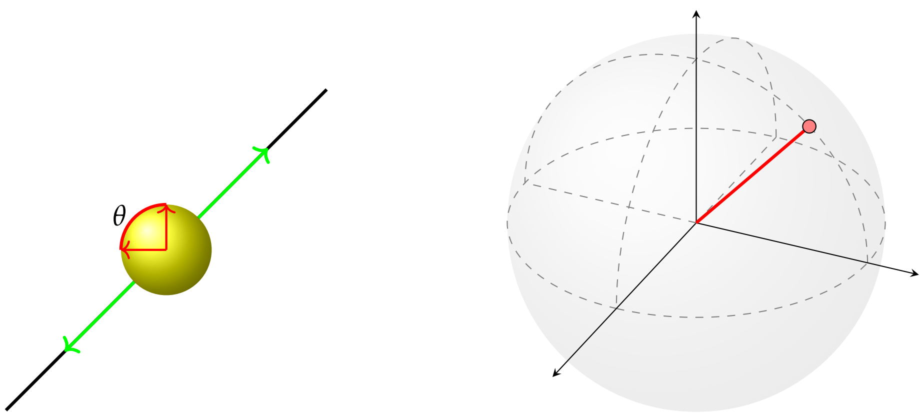
However, as we indicated in the introduction, the existence of such a section is far from the norm and, as a consequence, most spaces do not admit a continuous motion planning algorithm. If such a section exists, and we fix a $b \in X$, we can define a map $S(x,t) = s(x,b)(t)$. This gives us $S(x,0) = x$ and $S(x,1) = b$. Since s is a continuous map, S defines a deformation retract of X to a point. Therefore, only contractible spaces admit continuous motion planning algorithms. The converse of this statement, originally proved by Farber in [53, Theorem 1], is also quite straightforward: if there exists an homotopy H from X to a point $x_0 \in X$, i.e.

$$H(x,0) = x, \qquad H(x,1) = x_0, \qquad \forall x \in X$$

for every pair of points $(x,y) \in X \times X$ a continuous motion planner can be define by first moving x to x_0 through the retraction prescribed by the homotopy H and then concatenating with the inverse of the path that brings y to x_0.

Example 3.1.2. Perhaps the simplest and most obvious example of a robot that would come to mind is that of an automata moving on an open field without any kind of obstacles, such as the plane $\mathbb{R}^2$. This is indeed the case of a configuration space being contractible, and a motion planner between any two points $x, y \in \mathbb{R}^2$ is given just by taking the segment joining x and y.

As soon as we consider slightly more elaborated authomatic processes, the contractibility is usually lost. Observe the two examples in the figure below.



(a) A rotating sliding knob (b) A spherical pendulum

Figure 3.1: Some examples of simple robots

In the case of the sliding knob, any state of the robot is determined by two parameters (x, θ), where x marks the position on the axis of sliding, and θ represents the angle of rotation. As such, the configuration space corresponds with $X = \mathbb{R}^1 \times S^1$ that is, a cylinder.

The spherical pendulum is just a bar attached to a point and with a mass at the end. We observe that, during the movement, the mass can be at any possible point of a sphere of radius the length of the bar, hence the configuration space corresponds with such sphere $X = S^2$.

Example 3.1.3. Consider one of the most usual cases of mechanical systems, a robot arm. This consists of a determined number n of rigid bars attached by revolving joints. The most natural way to describe a position of the arm is to determine the angles formed by the bars at the revolving points (it is important to note that, conceptually, we are allowing self-intersections of the arm). See the example in the image.

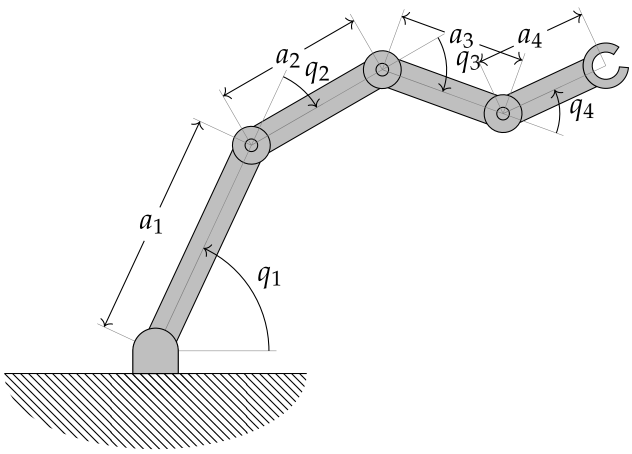


Figure 3.2: A mechanical robot arm with 4 revolving joints and a fixed grip at the extreme. Notice that, while the length of the bars may be important for engineering purposes, they are irrelevant from the point of view of topological robotics.

As such, each position of the system is determined by a n-tuple of angles $(\alpha_1, \cdots, \alpha_n)$, hence the associated configuration space is just the n-torus

$$X = \underbrace{S^1 \times \cdots \times S^1}_{n} = T^n$$

Example 3.1.4. Let X be an arbitrary topological space. We define the configuration space of a system of n particles moving through the space X and avoiding collisions between themselves by

$$F(X, n) := \{(x_1, \cdots, x_n) \in \underbrace{X \times \cdots \times X}_{n} | x_i \neq x_j \; \forall i \neq j\}.$$

A configuration space of the type $F(X, n)$ is commonly called the n^{th}*-ordered configuration space* of X. Notice that there is a naturally defined action of S^n, the symmetric group of order

n, on $F(X,n)$ by

$$\begin{aligned} S_n \times F(X,n) &\longrightarrow F(X,n) \\ (\sigma,(x_1,\cdots,x_n)) &\mapsto (x_{\sigma(1)},x_{\sigma(2)},\cdots,x_{\sigma(n)}). \end{aligned}$$

The quotient space under this action is known as the n^{th}-*unordered configuration space*

$$C(X,n) := F(X,n)/S_n$$

where we "forget" the labels of the points and we can consider them at any possible order.

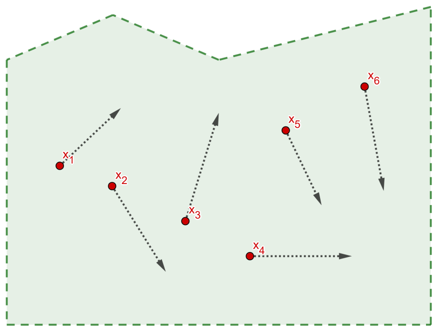


Figure 3.3: A system of particles moving on a space without colliding.

Ordered and unordered configuration spaces defined as above are very common objects in algebraic topology. In particular, the configuration spaces of the real plane $F(\mathbb{R}^2,n)$ and $C(\mathbb{R}^2,n)$ are crucial in braid theory. Indeed, they are the classifying spaces of the n-strand braid group, and the n-strand pure braid group, respectively:

$$B_n := \pi_1(C(\mathbb{R}^2,n)) \qquad P_n := \pi_1(F(\mathbb{R}^2,n))$$

Additionally, a common subject of study in the context of topological robotics are the ordered configuration spaces $F(\Gamma,n)$ where Γ is a graph, as they are used to model several objects moving along a prescribed net represented by Γ (such as possible routes on a factory floor, for example) avoiding collisions.

3.2 Topological complexity and related invariants

As the continuity of a motion planning algorithm is only possible for contractible configuration spaces, one could naturally ask how to measure the degree of instability related with the underlying topological features of the space. In order to measure such discontinuity, Michael Farber introduced in [53] the notion of topological complexity.

Definition 3.2.1. Let X be a path-connected topological space. The *topological complexity* of X, denoted by $\mathrm{TC}(X)$, is the least integer $k \geq 0$ such that there exists an open cover of $X \times X$

by a family of open subsets $\{U_i\}_{0\leq i\leq k}$ satisfying that, for each $0 \leq i \leq k$ there exists a local section of the path space fibration, i.e a map

$$s_i\colon U_i \to X \times X \qquad \text{with} \qquad \pi \circ s_i = \mathrm{id}_{U_i}.$$

If no such k exists, we say that $\mathrm{TC}(X) = \infty$.

The idea that serves as a basis for this definition is to subdivide $X \times X$ into pieces on which there are continuous motion planners, called *domains of continuity*. Provided such an open cover $\{U_i\}_{i\in I}$ by those domains, with local continuous sections $s_i\colon U_i \to PX$ of the path space fibration, and a pair of configurations $(x_1, y_1) \in X \times X$, one can organize the motion planning by finding the smallest index i_0 such that $(x_1, y_1) \in U_{i_0}$, and then give the path prescribed by $s_{i_0}(x_1, y_1)$. The discontinuity of the motion planning as a function of the state points easily becomes apparent. Indeed, suppose two open subsets U_1 and U_2 with $U_1 \cap U_2$, a pair of states $(x_1, y_1) \in U_1$ very close to the boundary, and another pair $(x_2, y_2) \in U_2 \setminus U_1$ very close to (x_1, y_1) (see the figure below). It is evident that the motion planner for (x_1, y_1), determined by $s_1(x_1, y_1)$, may be completely distinct to $s_2(x_2, y_2)$, as the local sections evaluated over the intersection, ${s_1}_{|_{U_1\cap U_2}}$ and ${s_2}_{|_{U_1\cap U_2}}$, are in general different.

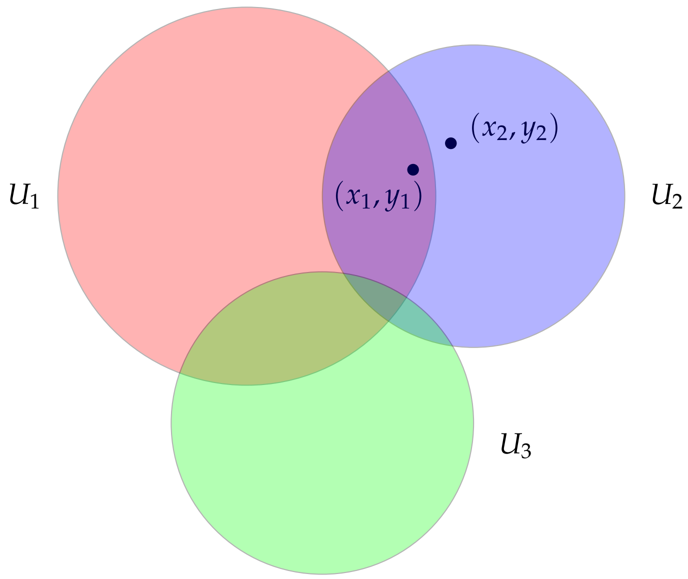


Figure 3.4: An open covering $\{U_i\}_{i\in J}$ for the topological complexity of a space.

The keen reader might wonder whether it would be possible to approach the definition by considering subdivisions of other kind, not necessarily by open covers. Indeed, Farber consider this possibility in its original article, and he gave several different characterizations of $\mathrm{TC}(X)$, which he showed to be equivalent for sufficiently nice spaces.

Certainly one of the main properties of the topological complexity of a space X is that it is an homotopy invariant of X.

Proposition 3.2.2 (Farber, [53]). *If X is a homotopy retract of Y, then $\mathrm{TC}(X) \leq \mathrm{TC}(Y)$. In particular, $\mathrm{TC}(X) = \mathrm{TC}(Y)$ if X is homotopy equivalent to Y.*

The topological complexity is closely related to another classical homotopy invariant, probaly one of the most important of the field: the Lusternik-Schnirelmann category. Originally defined by L. Lusternik and L. Schnirelmann (see [91]) as a mean to provide a lower bound on the number of critical points of any smooth function on a manifold. Even though the initial aim of this notion was analytical in nature, after it was reformulated by [64] it became a powerful and interesting tool in a broad array of topics within algebraic topology (see [34] for the universally acclaimed reference textbook on the matter, covering both properties and applications). In a moment we will make more precise the extent of the relationship between the two invariants, but the similarity of the definitions give an initial clue into it.

Definition 3.2.3. Let X be a path connected topological space. The *Lusternik-Schnirelmann category* of X, denoted by $\mathrm{cat}(X)$ is defined as the least integer $k \geq 0$ such that there exists a cover of X by open subsets, $\{U_i\}_{0\leq i\leq k}$ with the property that, for each $0 \leq i \leq k$ the inclusion map $\iota_i \colon U_i \hookrightarrow X$ is nullhomotopic.

It is also possible to consider the notion of Lusternik-Schnirelmann category of a map, instead of a space. Let $f \colon X \to Y$ a map between topological spaces. We define the LS-category of f, denoted $\mathrm{cat}(f)$, as the least integer $k \geq 0$ such that X can be covered by a family of open subsets $\{U_i\}_{0\leq i\leq k}$ satisfying that, for every $0 \leq i \leq k$, the restriction of f to U_i is nullhomotopic. Interestingly, there are several approaches in the literature to the definition of the topological complexity of a map, see for example [104], [101] or [111]. We will not delve into them in this text, as we will not be making use of such notions in the present work.

Remark 3.2.4. In the literature there exist two approaches to topological complexity, namely the non-reduced and the reduced ones, depending whether TC corresponds with the exact number of local sections or with said number minus one, respectively. In this dissertation, all the topological complexities are considered as reduced.

3.2.1 Sectional category and TC

In fact, both TC and LS-cat are particular cases of a more general invariant, called the *sectional category*. Originally conceived as *genus* of a fibration by A. Schwarz in the seminal paper [110], it was subsequently generalized to arbitrary maps by A. Fet [61] and I. Berstein and T. Ganea [12].

Definition 3.2.5. The *sectional category* of a map $f \colon X \to Y$, written $\mathrm{secat}(f)$, is defined to be the smallest integer $n \geq 0$ such that there exists an open cover $U_0, \ldots, U_n$ of Y and continuous maps $s_i \colon U_i \to X$ with the property that $f \circ s_i$ is homotopic to the inclusion $U_i \hookrightarrow Y$ for any $0 \leq i \leq n$ (i.e. s_i is a local homotopy section of f over U_i).

Remark 3.2.6. It is important to note that the condition of s_i being a local homotopy section can be strengthened if the map f is, indeed, a fibration. Under such assumption, the maps s_i are (continuous) local sections of f, i.e. $f \circ s_i = \mathrm{id}_{U_i}$.

The reader have surely noticed the similarity with Definitions 3.2.1 and 3.2.3, so we can now recast both TC and LS-cat in the lenguage of sectional category.

Definition 3.2.7. Given $\pi\colon PX \to X \times X$ the path space fibration, the topological complexity of X can be defined as
$$\mathrm{TC}(X) = \mathrm{secat}(\pi).$$

Denote by P_*X the based path space, which is the restriction of PX to paths starting at a previously fixed point $x \in X$. Define the fibration

$$\mathrm{ev}_1\colon P_*X \to X$$

which maps every path to its initial point, that is $\mathrm{ev}_1(\gamma) = \gamma(1)$. Then, we can define the Lusternik-Schnirelmann category of X as

$$\mathrm{cat}(X) = \mathrm{secat}(\mathrm{ev}_1).$$

Furthermore, it follows from [54, Corollary 18.2] that

$$\mathrm{TC}(X) = \mathrm{secat}(\Delta_X : X \to X \times X), \tag{3.2.1}$$

where $\Delta_X(x) = (x, x)$ is the diagonal embedding.

Further on we will make use of several basic properties of sectional category. Moreover, many of the classic facts about LS-category and topological complexity can be easily derived as a consequence of such properties, so we will summarize them in the next theorem. Before that, however, recall that the *fibrewise join* of a fibration $F \to E \xrightarrow{p} B$ is another fibration $p * p\colon E *_B E \to B$ whose fibre has the homotopy type of the join $F * F$ and whose total space is given by
$$E *_B E = \big\{(x, y, t) \in E \times E \times [0, 1] \,\big|\, p(x) = p(y)\big\}$$
modulo the relations $(x, y, 0) \sim (x', y, 0)$ and $(x, y, 1) \sim (x, y', 1)$. It is convenient to think of the elements of the total space of the n-fold fibrewise join of p as formal sums of the form $\sum_{i=1}^{n} t_i x_i$, where every x_i is understood to lie in the same fibre of p, and all the t_i's are non-negative real numbers such that $\sum_{i=1}^{n} t_i = 1$.

We can proceed now to state the announced theorem containing the fundamental properties of sectional category. For a detailed proof of each of the statements, we refer the interested reader to the original paper of Schwarz, [110].

Theorem 3.2.8. *Let $F \to E \xrightarrow{p} B$ a fibration, then the following properties hold:*

(a) $\mathrm{secat}(p) \leq \mathrm{cat}(B)$.

(b) If $g : X \to Y$ is homotopic to f, then $\mathrm{secat}(f) = \mathrm{secat}(g)$.

(c) Let $p\colon E \to B$ be a fibration, and $f\colon X \to B$ a map. Consider the pullback fibration f^*p over B. Then $\mathrm{secat}(f^*p) \leq \mathrm{secat}(p)$. *In particular, if $p\colon E \to B$ is a fibration and $h\colon B' \to B$ is a homotopy equivalence, then* $\mathrm{secat}(p) = \mathrm{secat}(p')$ *for p' the induced fibration of h by p.*

(d) *Given two fibrations* $p\colon E \to B$ *and* $p'\colon E' \to B$ *such that there exists a commutative diagram*

$$\begin{array}{ccc} E & \xrightarrow{f} & E' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ B & \xrightarrow[=]{} & B \end{array}$$

Then $\operatorname{secat}(p') \leq \operatorname{secat}(p)$.

(e) *Let* $k \in \mathbb{N}$*, If there are reduced cohomology classes*

$$u_i \in \ker\left[p^* : \widetilde{H}^*(B; A_i) \to \widetilde{H}^*(E; p^* A_i)\right],$$

where A_i *is a local coefficient system over* B *for each* $i \in \{1, 2, \ldots, k\}$*, such that*

$$u_1 \cup u_2 \cup \cdots \cup u_k \neq 0 \in H^*(B, A_1 \otimes A_2 \otimes \cdots \otimes A_k),$$

then we have $\operatorname{secat}(p) \geq k$.

(f) $\operatorname{secat}(p) \leq k$ *if and only if the* $(k+1)$*-fold fibrewise join of* p

$$\underbrace{p * \cdots * p}_{k+1}\colon E *_B \cdots *_B E \to B$$

has a section.

(g) *Given two fibrations* $p\colon E \to B$ *and* $p'\colon E \to B$ *consider their product*

$$p \times p'\colon E \times E' \to B \times B.'$$

We have the inequality $\operatorname{secat}(p \times p') \leq \operatorname{secat}(p) + \operatorname{secat}(p')$.

(h) *Assume* $\dim(B) = d$*, and that* F *is* $(s-1)$*-connected (i.e.* $\pi_k(F) = 0$ *for every* $k < s$*). Then we have*

$$\operatorname{secat}(p) < \frac{d+1}{s+1}.$$

Notice that both path space fibrations π and ev_1 previously defined fit to a pullback diagram of the form

$$\begin{array}{ccc} P_*X & \longrightarrow & PX \\ {\scriptstyle \mathrm{ev}_1}\downarrow & & \downarrow{\scriptstyle \pi} \\ X & \xhookrightarrow[\{x_0\}\times X]{} & X \times X. \end{array}$$

As a consequence, by (d) and (g) of Theorem 3.2.8 we obtain one of the most important bounds of topological complexity, in terms of LS-category.

Proposition 3.2.9. *If* X *is a path-connected and paracompact space we have the inequalities*

$$\operatorname{cat}(X) \leq \operatorname{TC}(X) \leq \operatorname{cat}(X \times X).$$

Let X be an $(s-1)$-connected space. It is clear from the definition that the homotopy fiber of the path space fibrations is of the homotopy type of ΩX, the loop space of X, and by the hypothesis of connectivity on X we know that ΩX is $(s-2)$-connected. The next corollary thus immediately follows from Theorem 3.2.8 (h).

Corollary 3.2.10. *Let X be an $(s-1)$-connected space, for $s > 0$. Then we have the upper bounds*

$$\mathrm{cat}(X) \leq \frac{\dim(X)}{s} \qquad \mathrm{TC}(X) \leq \frac{2\dim(X)}{s}.$$

In particular, if X is path connected

$$\mathrm{cat}(X) \leq \dim(X) \qquad \textit{and} \qquad \mathrm{TC}(X) \leq 2\dim(X).$$

Given a path-connected space X and a commutative ring A, the *cup length* of X with coefficients in A (denoted by $\mathrm{cl}_A(X)$) is defined as the maximal integer $k \geq 0$ such that there exists k reduced cohomology classes $u_1, \cdots, u_k \in \tilde{H}^*(X;A)$ such that $u_1 \cup \cdots \cup u_k \neq 0$. We say that a reduced cohomology class $u \in \tilde{H}^*(X \times X; A)$ is a *zero divisor* of X if it satisfies

$$u \in \ker\left[H^*(X \times X; A) \xrightarrow{\Delta^*} H^*(X;A)\right].$$

The zero divisors cup length of X is defined as the nilpotency of the kernel ideal of the map Δ^*, i.e. the length of the longest non-trivial product of zero divisors of X. As an immediate consequence of Theorem 3.2.8 e) and the alternative characterization 3.2.1 we have the well-known cohomological lower bounds of LS-category and topological complexity:

Corollary 3.2.11. *Let X be a path-connected space, and a commutative ring A. We have the following lower bounds.*

(a) $\mathrm{cl}_A(X) \leq \mathrm{cat}(X)$.

(b) $\mathrm{nilker}\left[H^*(X \times X; A) \xrightarrow{\Delta^*} H^*(X;A)\right] \leq \mathrm{TC}(X)$.

It is also interesting to mention two particular bounds of the Lusternik-Schnirelmann category of a map, as we will make use of them a couple of times later during this work.

Proposition 3.2.12. *Let $f\colon X \to Y$ a map between topological spaces. The category of f satisfies*

(a) $\mathrm{cat}(f) \leq \min\{\mathrm{cat}(X), \mathrm{cat}(Y)\}$.

(b) $\mathrm{cat}(f) \geq \mathrm{cl}_A(\mathrm{Im}\, f^*)$.

Furthermore, we will need the following characterization of the sectional category of certain pullback fibrations in terms of the category of the base changing map:

Proposition 3.2.13 (Proposition 9.18 of [34]). *Let $p\colon E \to B$ be a fibration arising as a pullback of a fibration $p'\colon E' \to B'$*

$$\begin{array}{ccc} E & \xrightarrow{f'} & E' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ B & \xrightarrow[f]{} & B' \end{array}$$

where E' is contractible. Then we have $\mathrm{secat}(p) = \mathrm{cat}(f)$.

3.2.2 Examples of computations

In what follows we provide some computations of basic cases for both LS-category and topological complexity, mostly using the properties stated above.

LS-category

We will start by adressing the computation of LS-category for some common topological spaces.

- For every $n \geq 1$ the computation of LS-category of the n-sphere S^n is pretty much straightforward from the definition: given that S^n is not contractible, we have $\text{cat}(S^n) > 0$. Now notice that every n-sphere can be covered by two contractible open subspaces, two discs each of them covering each of the hemispheres. Consequently, $\text{cat}(S^n) = 1$, regardless of the dimension.

- Now we quickly compute the LS-category of the n-torus, T^n. It is straightforward to see that $\text{cl}_{\mathbb{Z}}(T^n) = n$. Hence, by Corollaries 3.2.11 and 3.2.10 we have

$$n = \text{cl}_{\mathbb{Z}}(T^n) \leq \text{cat}(T^n) \leq \dim T^n = n$$

  hence $\text{cat}(T^n) = n$.

- The same strategy as in the torus case can be employed to find the LS-category of both real and complex projective spaces.

  It is well-known that $H^*(\mathbb{R}P^n; \mathbb{Z}_2) = \mathbb{Z}_2[\alpha]/\alpha^{n+1}$ with $\dim(\alpha) = 1$. Therefore $\text{cl}_{\mathbb{Z}_2}(\mathbb{R}P^n) = n$ and we have

$$n = \text{cl}_{\mathbb{Z}_2}(\mathbb{R}P^n) \leq \text{cat}(\mathbb{R}P^n) \leq \dim(\mathbb{R}P^n) = n$$

  so $\text{cat}(\mathbb{R}P^n)$.

  Similarly, we know that $H^*(\mathbb{C}P^n) = \mathbb{Z}[\alpha]/\alpha^{n+1}$ with $\dim(\alpha) = 2$, so $\text{cl}_{\mathbb{Z}}(\mathbb{C}P^n) = n$ and, in analogy with the real case, we obtain

$$n = \text{cl}_{\mathbb{Z}}(\mathbb{C}P^n) \leq \text{cat}(\mathbb{C}P^n) \leq \dim(\mathbb{C}P^n)/2 = 2n/2 = n$$

  (notice that here the denominator reflects the fact that $\mathbb{C}P^n$ is simply connected) thus $\text{cat}(\mathbb{C}P^n) = n$.

Topological Complexity

The situation for topological complexity, however, is in most cases far more complicated.

- Indeed, for the case of the spheres there is a noticeable contrast in that their topological complexity depends on whether the dimension is odd or even. Let us consider first the odd case. Define an open cover of S^{2n+1} formed by the open sets

$$U_0 := \{(x, y) | x, y \in S^{2n+1} \text{ with } x \neq -y\}.$$

$$U_1 := \{(x, y) | x, y \in S^{2n+1} \text{ such that } x = -y\}.$$

A local section of the path space fibration over U_0 can be defined by assigning to every pair of points $x, y \in U_0$ the shortest geodesic joining them. In order to define the local section over U_1, recall that S^{2n+1} has a non-vanishing continuous tangent vector field v. Then, for every $x \in U_1$ the local section is defined by joining x with $-x$ through the geodesic with tangent vector at x equal to $v(x)$.

For even-dimensional spheres, we will use the cohomological lower bound from Corollary 3.2.11. Start by taking a generator of the cohomology ring $u \in H^{2n}(S^{2n})$ and define the following cohomology class

$$v := u \otimes 1 - 1 \otimes u \in H^{4n}(S^{2n} \times S^{2n}).$$

It is easy to see that this class is a zero-divisor. Indeed, $\Delta^*(u \otimes 1) = u = \Delta^*(1 \otimes u)$ and, consequently, $\Delta^*(v) = 0$. Now observe that the product of v with itself is itself a non-trivial class.

$$\begin{aligned} v \cup v &= ((u \otimes 1) - (1 \otimes u)) \cup ((u \otimes 1) - (1 \otimes u)) \\ &= -(u \otimes 1) \cup (1 \otimes u) - (1 \otimes u) \cup (u \otimes 1) \\ &= -2u \otimes u \neq 0. \end{aligned}$$

Therefore, by the cohomological lower bound in Corollary 3.2.11, we have that $\mathrm{TC}(S^{2n}) \geq 2$. By the upper dimensional bound provided in Corollary 3.2.10, we know that $\mathrm{TC}(S^{2n}) \leq 2$, and thus we obtain the equality $\mathrm{TC}(S^{2n}) = 2$.

- Let us think about product of spheres

$$X = \underbrace{S^n \times \cdots \times S^n}_{k}$$

(which, in particular, encompass the case of the k-torus T^k). Using the product inequality in Theorem 3.2.8(g) and the computation of the TC of spheres above, we have that

$$\mathrm{TC}(X) \leq \begin{cases} k & \text{if } n \text{ is odd} \\ 2k & \text{if } n \text{ is even.} \end{cases} \tag{3.2.2}$$

We will see that it is, in fact, an equality. Let $u_i \in H^n(X; \mathbb{Q})$ be the cohomology class obtained as a pullback of the fundamental class of S^n via the projection $X \to S^n$ onto the i-th factor, for $1 \leq i \leq k$. Observe that the products

$$\prod_{i=1}^{k}(1 \otimes u_i - u_i \otimes 1) \neq 0 \text{ if n is odd} \qquad \prod_{i=1}^{k}(1 \otimes u_i - u_i \otimes 1)^2 \neq 0 \text{ if n is even}$$

which, by the cohomological lower bound, shows that we have the equality in 3.2.2.

- The values of the topological complexity of real and complex projective spaces were studied by Farber, Tabachnikov and Yuzvinsky in [59]. The complex case was computed through the more general calculation of the topological complexity of symplectic manifolds. We reproduce here the argument. Let X be a closed $2n$-dimensional simply connected symplectic manifold. The upper dimensional bound informs us that $\mathrm{TC}(X) \leq 2\dim(X)/2 = 2n$. Now, let ω be the symplectic 2-form on X, determining a cohomology class $[\omega] \in H^2(X;\mathbb{R})$. We know that $[\omega]^n \neq 0$ thus $[\omega] \otimes 1 - 1 \otimes [\omega]$ is a zero divisor of X whose $2n$-th power is non-trivial, since it contains the term

$$(-1)^n \binom{2n}{n} [\omega]^n \otimes [\omega]^n$$

so the cohomological lower bound shows that $\mathrm{TC}(X) = 2n$. Since $\mathbb{C}P^n$ are examples of simply connected symplectic manifolds, $\mathrm{TC}(\mathbb{C}P^n) = 2n$.

The real case is more interesting. Indeed, the cornerstone of [59] is the identification of the topological complexity of $\mathbb{R}P^n$ (save fringe cases also adressed in the article) with the immersion dimension of real projective spaces into real euclidean spaces.

Theorem 3.2.14 (Theorem 12, Proposition 18 and Corollary 2 of [59])**.** *For any $n \in \mathbb{N}$ different of $1, 3$ or 7, the value of $\mathrm{TC}(\mathbb{R}P^n)$ coincides with the smallest integer k such that there exists an immersion of $\mathbb{R}P^n$ into the euclidean space $\mathbb{R}^k$.*

For $n = 1, 3, 7$ we have $\mathrm{TC}(\mathbb{R}P^n) = n$.

- Let $X = \Sigma_g$ denote the compact orientable surface of genus g. We have to distinguish cases depending on possible values of g.

Both the cases of $g \in \{0, 1\}$ were already settled above: as $\Sigma_0 = S^2$ and $\Sigma_1 = T^2$, we know that

$$\mathrm{TC}(\Sigma_0) = \mathrm{TC}(\Sigma_1) = 2.$$

So let us consider the case $g \geq 2$. In this situation, we find 1-dimensional cohomology classes $u_1, u_2, v_1, v_2 \in H^1(\Sigma_g, \mathbb{Q})$ satisfying

$$u_1u_2 = v_1v_2 = u_1v_2 = u_2v_1 = u_1^2 = u_2^2 = v_1^2 = v_2^2 = 0$$

and $u_1v_1 = u_2v_2$ is non trivial in $H^2(\Sigma_g, \mathbb{Q})$. The product of zero divisors

$$\prod_{i=1}^{2} (u_i \otimes 1 - 1 \otimes u_i) \cup (v_i \otimes 1 - 1 \otimes v_i)$$

provides a non-zero cohomology class, so we obtain the lower bound $\mathrm{TC}(\Sigma_g) \geq 4$. By the dimension connectivity bound of Corollary 3.2.10, we see that $\mathrm{TC}(\Sigma_g) \leq 2\dim(\Sigma_g) = 4$, thus we conclude the identity $\mathrm{TC}(\Sigma_g) = 4$.

- The situation for non-orientable surfaces N_g is more complex. The non-orientable surface of genus $g = 1$ is the real projective plane $N_1 = \mathbb{R}P^2$, and so by Theorem 3.2.14

and the calculations of immersion dimension we know that $\mathrm{TC}(N_1) = 3$. For $g \geq 5$, A. Dranishnikov established in [42] that $\mathrm{TC}(N_g) = 4$ and, later on, in [40], he showed that $\mathrm{TC}(N_4) = 4$, and that the specific techniques used therein could not be extended to the cases $g \in \{2, 3\}$. These remaining cases were finally settled by D.C. Cohen and L. Vandembroucq in [32], where they proved through impressive explicit computations of zero-divisors cup lengths with local coefficients that $\mathrm{TC}(N_g) = 4$ for any $g \geq 2$.

- The topological complexity of ordered configuration spaces of euclidean spaces $\mathbb{R}^m$ was computed, for $m = 2$ and $m \geq 3$ odd, by M. Farber and S. Yuzvinsky in [60], while the remaining cases were calculated by M. Farber and M. Grant in [55].

$$\mathrm{TC}(F(\mathbb{R}^m, n)) = \begin{cases} 2n - 2 & \text{for all } m \text{ odd} \\ 2n - 3 & \text{for all } m \text{ even} \end{cases}$$

3.2.3 The sequential topological complexities

The notion of *sequential* or *higher topological complexities* was introduced by Y. Rudyak in [108] as a generalization of topological complexity which models the motion planning problem for robots that are supposed to make some pre-determined intermediate stops along their ways. We briefly recall their definition.

Definition 3.2.15. Let X be a path-connected topological space. For each $r \in \mathbb{N}$ with $r \geq 2$ the map

$$p_r : PX \to X^r, \qquad p_r(\gamma) = \left(\gamma(0), \gamma(\tfrac{1}{r-1}), \gamma(\tfrac{2}{r-1}), \ldots, \gamma(\tfrac{r-2}{r-1}), \gamma(1)\right),$$

is a fibration. The *r-th sequential topological complexity of* X is defined as

$$\mathrm{TC}_r(X) := \mathrm{secat}(p_r : PX \to X^r).$$

Notice that, by definition, the notion of sequential topological complexity includes that of the classic topological complexity, which occurs as $\mathrm{TC}_2(X) = \mathrm{TC}(X)$. As noted in [9] the r-th sequential topological complexity of X can be defined as the sectional category of the fibration

$$\begin{aligned} e_r^X \colon X^{J_r} &\longrightarrow X^r \\ \gamma &\mapsto (\gamma(1_1), \cdots, \gamma(1_r)). \end{aligned}$$

where J_r is the wedge of r unit intervals $[0, 1]$ (with 0 as the base point for each of them), and 1_i stands for 1 in the i^{th} interval, for every $1 \leq i \leq r$. Furthermore, as e_r^X is the standard fibrational substitute for the iterated diagonal map

$$\begin{aligned} \Delta_{X,r} \colon X &\longrightarrow X^r \\ x &\mapsto (\underbrace{x, \cdots, x}_{r}) \end{aligned}$$

one can equivalently define

$$\mathrm{TC}_r(X) = \mathrm{secat}(\Delta_{X,r}). \tag{3.2.3}$$

Other possible fibrations which are not necessarily fibrational substitutes of the iterated diagonal can be used to define $\mathrm{TC}_r(X)$. We will not make use of such approaches in this work, so we refer the reader to [9] for interesting discussion on that matter.

3.3 LS category and TC of Eilenberg-MacLane spaces

As we announced at the introduction, much of this work is focused at the investigation of the topological complexity, and related homotopy invariants, of Eilenberg-MacLane spaces of discrete groups. Consequently, we will close this chapter by giving the precise definitions of what we understand by LS-category or (sequential) topological complexity of an abstract group G. We will obviously restrict just to the case of torsion-free groups, as groups with torsion have infinite LS-category and topological complexity (see Remark 2.2.10).

By the homotopy invariance of Lusternik-Schnirelmann category, and given that the homotopy type of a $K(G,1)$-space is determined by the isomorphism class of G, it is natural to define the notion of LS-category of a discrete group simply by considering the LS-category of a space of type $K(G,1)$.

Definition 3.3.1. Let G be an abstract group, and $X = K(G,1)$. Define the LS-category of G by putting $\mathrm{cat}(G) := \mathrm{cat}(X)$.

In the very same celebrated paper where they elucidated the exact relationship between geometric and cohomological dimension of groups, Eilenberg and Ganea recorded as well the relationship between the cohomological dimension of a group and its LS-category save low dimensions, which were later complemented with the work of Stallings and Swann.

Theorem 3.3.2 (Eilenberg-Ganea theorem for category, [47]). *Let G be an abstract group, and $X = K(G,1)$. Then it holds* $\mathrm{cat}(X) = \mathrm{cd}(X)$.

The interested reader can consult [102] for an excelent survey of the topic, including a comprehensive proof of the Eilenberg-Ganea theorem for category.

A crucial ingredient for the study of the Lusternik-Schnirelmann category of Eilenberg-MacLane spaces is the *Berstein-Schwarz* class, defined as the one dimensional cohomology class $\beta \in H^1(\pi, K)$ represented by the augmentation short exact sequence

$$0 \to K \to \mathbb{Z}[\pi] \xrightarrow{\varepsilon} \mathbb{Z} \to 0. \tag{3.3.1}$$

This class was shown to be universal in [44], in the sense that for any other cohomology class $\alpha \in H^1(\pi, A)$ there exists a $\mathbb{Z}[\pi]$-homomorphism $f\colon K^n \to A$ such that $\alpha = f^*(\beta^k)$.

As the lector surely noticed, it is natural to define the topological complexity of a discrete group in complete analogy with the Lusternik-Schnirelmann category case.

Definition 3.3.3. Let π be an abstract group, and $X = K(\pi,1)$. Define the topological complexity of π by $\mathrm{TC}(G) := \mathrm{TC}(X)$.

In analogy to the Berstein-Schwarz class, A. Costa and M. Farber defined in [35] a cohomology class to play a similar role in the study of the topological complexity of groups. Observe that the group ring $\mathbb{Z}[\pi]$ can be seen as a $(\pi \times \pi)$-module with respect to the action $(g,h)a = gah^{-1}$ for every $g,h,a \in \pi$. Consequently, the augmentation map $\varepsilon\colon \mathbb{Z}[\pi] \to \mathbb{Z}$ is a $\mathbb{Z}[\pi \times \pi]$-homomorphism with $\mathbb{Z}$ seen as a trivial $\mathbb{Z}[\pi \times \pi]$-module, and the associated short exact sequence 3.3.1 becomes a sequence of $\mathbb{Z}[\pi \times \pi]$-modules. Therefore, we can define the *canonical class* as the one dimensional class $\mathbf{v} \in H^1(\pi \times \pi, K)$ represented by the sequence 3.3.1. However, in stark contrast from the Berstein-Schwarz class, this canonical class is not universal.

Motivated by the Eilenberg-Ganea theorem for category, Farber, in [54], naturally wondered whether it is possible to obtain some similar statement for the case of topological complexity.

Question 3.3.4 (Eilenberg-Ganea problem for TC)**.** For any torsion-free discrete group π, is it possible to characterize $\mathrm{TC}(\pi)$ purely in terms of algebraic properties of π?

Such question is not by any means a trivial one. Even though the homotopy type of a $K(\pi,1)$-space (and hence all of its homotopy invariants) is completely determined by the group π, the description of such invariants may involve some homotopy-theoretical constructions that can not be expressed in terms of classifying spaces. Indeed, as of today, Question 3.3.4 remains possibly as the single most important open problem for the TC-community. As such, the topological complexity of aspherical spaces has received a lot of attention, and has been studied from various perspectives, becoming one of the most fruithful lines of research in the field of topological robotics.

Up until recently, any progress in this context was mostly related to a specific choice of a family of groups: choose a family of groups, then use its characteristic features (e.g. a particularly well understood cohomology ring or a specific subgroup structure) to deduce, or at least estimate, topological complexity of its members, see among others [31], [72], [40] and [32]. Perhaps the most comprehensive result in this direction was obtained by M. Farber and S. Mescher [57], through the development of the notion of essential cohomology classes.

Definition 3.3.5. A cohomology class $\alpha \in H^n(\pi \times \pi, A)$ is *essential* if there exists a homomorphism of $\mathbb{Z}[\pi \times \pi]$-modules $f\colon K^n \to A$ such that $f^*(\mathbf{v}^n) = \alpha$.

By means of studying such essential classes, they proved that if a group π is hyperbolic in the sense of Gromov and it admits a compact model of a $K(\pi,1)$ space, then its topological complexity is equal to either $\mathrm{cd}\,(\pi \times \pi)$ or $\mathrm{cd}\,(\pi \times \pi) - 1$. Continuing on that line of thought, Dranishnikov in [39] improved that estimation by showing that, when π is hyperbolic

$$\mathrm{TC}(\pi) = \mathrm{cd}(\pi \times \pi) = 2\mathrm{cd}(\pi).$$

Further on, very recently K. Li [88] showed a generalization of this for certain toral relatively hyperbolic groups.

However, in a recent breakthrough, Farber, Grant, Lupton and Oprea [56] related $\mathrm{TC}(\pi)$ to invariants coming from equivariant Bredon cohomology. More specifically, they proved that

$$\mathrm{TC}(\pi) \leq \mathrm{cd}_{\langle \Delta_\pi \rangle}(\pi \times \pi),$$

where recall that $\mathrm{cd}_{\langle \Delta_\pi \rangle}(\pi \times \pi)$ denotes the cohomological dimension of $\pi \times \pi$ with respect to the family of subgroups of $\pi \times \pi$ generated by the diagonal subgroup Δ_π. By Theorem 2.3.12, this number can be seen as the smallest possible dimension of $E_{\langle \Delta_\pi \rangle}(\pi \times \pi)$. This result came as a consequence of a new characterization of $\mathrm{TC}(\pi)$ as follows

Theorem 3.3.6 (Theorem 3.3 of [56]). *Let X be a finite aspherical CW-complex with fundamental group $\pi = \pi_1(X, x_0)$. Then $\mathrm{TC}(X)$ coincides with the minimal integer k such that the canonical map*

$$E(G) \to E_{\langle \Delta_\pi \rangle}(G)$$

is G-equivariantly homotopic to a map taking values in the k-skeleton $(E_{\langle \Delta_\pi \rangle}(G))_k$.

The sequential topological complexities of aspherical spaces have been studied by M. Farber and J. Oprea in [58]. In particular, given a geometrically finite group π, it is shown in [58, Lemma 4.2 and Corollary 4.3] that $\mathrm{TC}_r(K(\pi, 1))$ coincides with the sectional category of the covering of $(K(\pi, 1))^r$ that is associated with the diagonal subgroup

$$\Delta_{\pi,r} := \{(g, g, \ldots, g) \in \pi^r \mid g \in \pi\}. \tag{3.3.2}$$

While the original proof relies on the identification of $\mathrm{TC}_r(K(\pi, 1))$ with the notion of $\mathcal{D}$-topological complexity of $K(\pi, 1)$, we will give a direct proof in Chapter 6, see the proof of Proposition 6.2.4 and also Remark 6.2.5.

So far little is known about the sequential TC of aspherical spaces. In [58] the Bredon cohomology approach from [56] is transferred to sequential TC yielding lower and upper bounds for sequential TC as well. There are also computations of sequential TCs of certain classes of aspherical spaces in the literature, see e.g. [66] for the case of a closed oriented surface.

Part II

Sectional category and topological complexity of $K(G, 1)$-spaces

CHAPTER 4

Sectional category of subgroup inclusions

Introduction

In this chapter we begin a systematic study of the sectional category of subgroup inclusions: briefly put, given a group G and its subgroup H, we define $\mathrm{secat}(H \hookrightarrow G)$ as the sectional category of the corresponding map between Eilenberg–MacLane spaces. This setting includes TC, as the topological complexity of X can be seen as the sectional category of the diagonal inclusion $X \to X \times X$, so that $\mathrm{TC}(\pi) = \mathrm{secat}(\Delta_\pi \hookrightarrow \pi \times \pi)$ and, as we will see in the next chapter, other variants of topological complexity. In fact, the cornerstone of [56], a characterization of $\mathrm{TC}(\pi)$ as the smallest integer $n \geq 0$ such that a certain canonical $(\pi \times \pi)$-equivariant map $E(\pi \times \pi) \to E_{\langle \Delta \pi \rangle}(\pi \times \pi)$ can be equivariantly deformed into the n-dimensional skeleton of $E_{\langle \Delta_\pi \rangle}(\pi \times \pi)$, has a generalization to this more general context. We also describe and develop a "relative canonical class" analogous to the one developed by Berstein and Schwarz for the study of Lusternik–Schnirelmann category theory, which particularizes to the Costa-Farber canonical class for the inclusion of the diagonal subgroup. In fact, we will prove a generalization of the Costa-Farber theorem, relating $\mathrm{secat}(H \hookrightarrow G)$ and powers of the Berstein-Schwarz relative class.

Moreover, we introduce the Adamson cohomology theory (first described in [1]) into the study of $\mathrm{secat}(H \hookrightarrow G)$, hence also into the study of $\mathrm{TC}(\pi)$. We also dig a bit into the Adamson cohomology theory itself, and as such we define a notion of cup products, and of a canonical class, which turns out to be universal. In particular, we exhibit a relationship between the "zero-divisors" of $H^*(G, M) \to H^*(H, M)$, which provide a lower bound for $\mathrm{secat}(H \hookrightarrow G)$, and the Adamson cohomology of the pair (G, H), and we provide an alternative proof of the relationship between Adamson and Bredon cohomology with respect to the family generated by the subgroup.

Most of the contents of this chapter appear in [14], though there are some additions which are not present in the aforementioned article.

Notation

- Throughout the next two chapters G is a discrete group and $H \subseteq G$ its fixed subgroup, and G/H denotes the set of left cosets of H in G equipped with a canonical G-action. Whenever we specialize to the setting of topological complexity, we take $G = \pi \times \pi$ and $H = \Delta_\pi$, the diagonal subgroup of $\pi \times \pi$.

- We let $1 \in G$ denote the unit element and $\mathbb{Z}[G]$ the integer group ring of G.

- We further let

$$\varepsilon : \mathbb{Z}[G] \to \mathbb{Z} \qquad \varepsilon(\sum_{g\in G} n_g \cdot g) = \sum_{g \in G} n_g$$

  be the augmentation and $K = \ker \varepsilon$ be the augmentation ideal, seen as a left $\mathbb{Z}[G]$-module.

- Given a subgroup $H \leq G$, we let $\mathbb{Z}[G/H]$ be the associated permutation module as a left $\mathbb{Z}[G]$-module. We further let

$$\sigma : \mathbb{Z}[G/H] \to \mathbb{Z} \qquad \sigma(\sum_{x \in G/H}) n_x \cdot x = \sum_{x \in G/H} n_x$$

  denote its augmentation.

- For any left $\mathbb{Z}[G]$-module M, we put $\widetilde{M} := \mathrm{Res}^G_H(M)$ for the left $\mathbb{Z}[H]$-module that is obtained via restriction of scalars from $\mathbb{Z}[G]$ to $\mathbb{Z}[H]$.

- We always let $\otimes$ without any subscript denote the tensor product $\otimes_{\mathbb{Z}}$ of abelian groups. Given a $\mathbb{Z}[G]$-module M and $p \in \mathbb{N}$ we denote the p-fold tensor power of M by

$$M^p := M \otimes M \otimes \cdots \otimes M$$

  and consider it as a $\mathbb{Z}[G]$-module with respect to the diagonal G-action on M^p.

- Given two left $\mathbb{Z}[G]$-modules M_1 and M_2 we consider $\mathrm{Hom}_{\mathbb{Z}}(M_1, M_2)$, the set of group homomorphisms from M_1 to M_2, as a left $\mathbb{Z}[G]$-module via the diagonal G-action given by

$$(g \cdot f)(m) := gf(g^{-1}m) \qquad \forall m \in M_1,\ g \in G,\ f \in \mathrm{Hom}_{\mathbb{Z}}(M_1, M_2).$$

- Given a short exact sequence of G-modules

$$0 \to A \xrightarrow{i} B \to C \to 0,$$

  and a G-module map $f\colon B \to M$ with $f \circ i = 0$, we will write $\hat{f}$ for the induced map

$$\begin{array}{ccccc} A & \xrightarrow{i} & B & \longrightarrow & C \\ & & {\scriptstyle f}\downarrow & \swarrow {\scriptstyle \hat{f}} & \\ & & M. & & \end{array}$$

4.1 Sectional category of subgroup inclusions

By classical results of algebraic topology, for any two groups G_1 and G_2 there is a one-to-one correspondence between based homotopy classes of continuous maps $K(G_1, 1) \to K(G_2, 1)$ and group homomorphisms $G_1 \to G_2$, induced by associating with any continuous map

$$f : K(G_1, 1) \to K(G_2, 1)$$

the induced homomorphism $\pi_1(f) \colon G_1 \to G_2$ between the fundamental groups. As such, we can consider the associated sectional category.

Definition 4.1.1. Let G_1 and G_2 be groups and let $\varphi : G_1 \to G_2$ be a group homomorphism. We define the sectional category of φ by

$$\mathrm{secat}(\varphi : G_1 \to G_2) := \mathrm{secat}(f_\varphi : K(G_1, 1) \to K(G_2, 1)),$$

or just $\mathrm{secat}(\varphi)$, where f_φ is a continuous map with $\pi_1(f_\varphi) = \varphi$. By Theorem 3.2.8.b), $\mathrm{secat}(\varphi)$ is well-defined.

In particular, given a group G and a subgroup $H \leq G$, the inclusion $i \colon H \hookrightarrow G$ induces a covering map

$$K(i, 1) \colon K(H, 1) \to K(G, 1)$$

between the corresponding Eilenberg–MacLane spaces, which satisfies $\pi_1\big(K(i, 1)\big) = i$.

Definition 4.1.2. Define the *sectional category* of the subgroup inclusion $H \hookrightarrow G$, denoted $\mathrm{secat}(H \hookrightarrow G)$, as

$$\mathrm{secat}(K(i, 1) \colon K(H, 1) \to K(G, 1))$$

Due to homotopy invariance of sectional category, $\mathrm{secat}(H \hookrightarrow G)$ depends only on the conjugacy class of H in G.

Notice that, for π a torsion free group, if X is a space of type $K(\pi, 1)$, then obviously $X \times X$ is of type $K(\pi \times \pi, 1)$, so we may view Δ_X as a continuous map $K(\pi, 1) \to K(\pi \times \pi, 1)$. One easily checks that the homomorphism between fundamental groups induced by this map is indeed given by

$$\Delta : \pi \to \pi \times \pi, \quad \Delta(g) := (g, g),$$

which is evidently a monomorphism. We denote $\Delta_\pi := \mathrm{Im}\,\Delta$. It then follows from the equality (3.2.1) and the definition that

$$\mathrm{TC}(K(\pi, 1)) = \mathrm{secat}(\Delta_\pi \hookrightarrow \pi \times \pi). \tag{4.1.1}$$

As such, we can visualize the whole investigation of the topological complexity of aspherical spaces as a particular case of the more general theory of sectional category of subgroup inclusions as defined above.

Both statements of the following theorem are obtained straightforwardly as special cases of Theorem 3.2.8 a) and e).

Theorem 4.1.3. *Let G be geometrically finite and let $\iota : H \hookrightarrow G$ be the inclusion of a subgroup.*

a) $\mathrm{secat}(\iota : H \hookrightarrow G) \leq \mathrm{cd}(G)$.

b) If there are reduced cohomology classes

$$u_i \in \ker \left[\iota^* : \widetilde{H}^*(G; A_i) \to \widetilde{H}^*(H, \mathrm{Res}^G_H(A_i))\right],$$

where A_i is a left $\mathbb{Z}[G]$-module for each $i \in \{1, 2, \ldots, k\}$, which satisfy $u_1 \cup u_2 \cdots \cup u_k \neq 0$, then

$$\mathrm{secat}(\iota : H \hookrightarrow G) \geq k.$$

4.1.1 A characterization of $\mathrm{secat}(H \hookrightarrow G)$

The aim of this subsection is to prove the following result.

Theorem 4.1.4. *The sectional category of $H \hookrightarrow G$ coincides with the minimal integer $n \geq 0$ such that the G-equivariant map $\rho \colon EG \to E_{\langle H \rangle}G$ can be G-equivariantly factored up to G-homotopy as*

$$\begin{array}{ccc} EG & \xrightarrow{\quad\rho\quad} & E_{\langle H \rangle}G \\ & \searrow \quad \nearrow & \\ & (E_{\langle H \rangle}G)_n, & \end{array}$$

where $(E_{\langle H \rangle}G)_n$ denotes the n-skeleton of $E_{\langle H \rangle}G$.

This is a generalization of Farber, Grant, Lupton and Oprea's [56, Theorem 3.3], (see Theorem 3.3.6) where $\mathrm{TC}(\pi)$ is described as the minimal integer $n \geq 0$ such that the $(\pi \times \pi)$-equivariant map $E(\pi \times \pi) \to E_{\langle \Delta_\pi \rangle}(\pi \times \pi)$ can be equivariantly deformed into the n-skeleton of $E_{\langle \Delta_\pi \rangle}(\pi \times \pi)$. Our approach closely follows theirs. This was very recently proved independently in [25], albeit by different means. The next lemma is an abstraction of an intermediate step in the proof of [56, Theorem 2.1].

Lemma 4.1.5. *We have $\mathrm{secat}(H \hookrightarrow G) \leq n$ if and only if the Borel fibration*

$$p_n \colon EG \times_G *^{n+1}(G/H) \to EG/G$$

has a section, where $*^{n+1}(G/H)$ denotes the $(n+1)$-fold join of G/H.

Proof. The map

$$EG \times_G (G/H) \to EG/H \qquad \text{given by} \qquad G(x, gH) \mapsto Hg^{-1}x$$

is easily seen to be a homeomorphism which commutes with projections onto EG/G. Consequently, p_0 is isomorphic to the fibration $EG/H \to EG/G$, which is a model for the map $K(H, 1) \to K(G, 1)$.

It follows that $\mathrm{secat}(H \hookrightarrow G) = \mathrm{secat}(p_0)$. By statement (f) of Theorem 3.2.8, $\mathrm{secat}(p_0) \leq n$ if and only if the $(n+1)$-fold fibrewise join of p_0 has a section. Thus in order to conclude the proof, it remains to verify that the $(n+1)$-fold fibrewise join of p_0 coincides with p_n. To this end, note that the map

$$EG \times_G *^{n+1}(G/H) \to *^{n+1}_{EG/G}(EG \times_G (G/H))$$

given by

$$G\left(x, \sum_{i=1}^{n+1} t_i g_i H\right) \mapsto \sum_{i=1}^{n+1} t_i G(x, g_i H)$$

is a homeomorphism which commutes with projections onto EG/G. □

Proof of Theorem 4.1.4. In view of Theorem 2.3.6 sections of the fibration

$$p_n \colon EG \times_G *^{n+1}(G/H) \to EG/G$$

introduced in Lemma 4.1.5 are in one-to-one correspondence with G-equivariant maps of the form

$$EG \to *^{n+1}(G/H).$$

Consequently, Lemma 4.1.5 can be restated as saying that $\mathrm{secat}(H \hookrightarrow G)$ coincides with the minimal integer $n \geq 0$ such that there exists a G-equivariant map $EG \to *^{n+1}(G/H)$.

Let $m \geq 0$ be the minimal integer such that the G-equivariant map $EG \to E_{\langle H\rangle}G$ can be deformed into the m-dimensional skeleton of $E_{\langle H\rangle}G$. We will now use the fact that the infinite join $*^{\infty}(G/H)$ is a model for $E_{\langle H\rangle}G$ (recall (2.3.3)). Given that $\dim *^{n+1}(G/H) = n$, the existence of a G-equivariant map $EG \to *^{n+1}(G/H)$ implies the existence of a G-equivariant map

$$EG \to *^{n+1}(G/H) \to (E_{\langle H\rangle}G)_n \to E_{\langle H\rangle}G$$

by the equivariant cellular approximation theorem. Since any two G-equivariant maps $EG \to E_{\langle H\rangle}G$ are G-equivariantly homotopic, this last composition is ρ and we see that $\mathrm{secat}(H \hookrightarrow G) \geq m$.

On the other hand, the G-equivariant map

$$(E_{\langle H\rangle}G)_m \to *^{\infty}(G/H)$$

yields an associated G-equivariant map to the $m+1$-join

$$(E_{\langle H\rangle}G)_m \to *^{m+1}(G/H)$$

by the equivariant Whitehead Theorem (see Theorem 2.3.3). This, in turn, implies the existence of a final G-equivariant map

$$EG \to (E_{\langle H\rangle}G)_m \to *^{m+1}(G/H)$$

which shows that $\mathrm{secat}(H \hookrightarrow G) \leq m$. □

As an immediate corollary, we obtain a generalization of [56, Corollary 3.5.1]:

Corollary 4.1.6. *Let $H \hookrightarrow G$ be a monomorphism of groups, then*

$$\mathrm{secat}(H \hookrightarrow G) \leq \dim E_{\langle H\rangle}G.$$

4.1.2 The Berstein-Schwarz class of G relative to H

We will now recall a construction from [44]. Take the usual augmentation associated to a group G

$$0 \hookrightarrow K \overset{i}{\hookrightarrow} \mathbb{Z}[G] \overset{\varepsilon}{\to} \mathbb{Z} \to 0$$

and K^n with G-module structure induced by the diagonal action of G on the tensor product of copies of K. Given that K^n is a free abelian group, and that there exists an isomorphism $g \otimes m \mapsto g \otimes gm$ from $\mathbb{Z}[G] \otimes K^n$ with action on the first factor to $\mathbb{Z}[G] \otimes K^n$ with the diagonal one, we have that $\mathbb{Z}[G] \otimes K^n$ is a free G-module. Splicing together short exact sequences of G-modules

$$0 \to K^{n+1} \to \mathbb{Z}[G] \otimes K^n \xrightarrow{\varepsilon \otimes \mathrm{id}} K^n \to 0$$

yields a G-module free resolution of $\mathbb{Z}$

$$\cdots \to \mathbb{Z}[G] \otimes K^s \overset{p_s}{\to} \mathbb{Z}[G] \otimes K^{s-1} \to \cdots \to \mathbb{Z}[G] \otimes K \overset{p_1}{\to} \mathbb{Z}[G] \overset{\varepsilon}{\to} \mathbb{Z} \to 0, \qquad (4.1.2)$$

where for each $s > 1$

$$p_s : \mathbb{Z}[G] \otimes K^s \to \mathbb{Z}[G] \otimes K^{s-1}, \quad p_s(x \otimes y \otimes z) = \varepsilon(x) \cdot i(y) \otimes z \quad \forall x \in \mathbb{Z}[G],\, y \in K,\, z \in K^{s-1}.$$

Such projective resolution will be denoted by $\mathcal{G}$.

Using $\mathcal{G}$ just defined, we can give an alternative and simple description of the cup product on the cohomology of G.

Proposition 4.1.7. *Let $[a] \in H^p(G, A)$ and $[b] \in H^q(G, B)$ be cohomology classes represented by cocycles*

$$a\colon \mathbb{Z}[G] \otimes K^p \to A \qquad \text{and} \qquad b\colon \mathbb{Z}[G] \otimes K^q \to B.$$

Then the cup product $[a][b] \in H^{p+q}(G, A \otimes B)$ is represented by the map

$$\mathbb{Z}[G] \otimes K^{p+q} \xrightarrow{\varepsilon \otimes \mathrm{id}} K^{p+q} \xrightarrow{\hat{a} \otimes \hat{b}} A \otimes B.$$

Proof. Denote by $\mathcal{F}$ the standard resolution of $\mathbb{Z}$ as a G-module and consider a map $\varphi\colon \mathcal{F} \to \mathcal{G}$ defined by $\varphi_p\colon \mathbb{Z}[G^{p+1}] \to \mathbb{Z}[G] \otimes K^p$ with

$$\varphi_p(x_0, x_1 \ldots, x_p) = x_0 \otimes (x_1 - x_0) \otimes \cdots \otimes (x_p - x_{p-1}).$$

This is a well defined chain map. The proof of this fact follows closely the combinatorics of the proof of [57, Lemma 3.1]. To show it, we need to see that

$$\varphi_{p-1}(d_p(x_0, x_1, \cdots, x_p)) = (x_1 - x_0) \otimes \cdots \otimes (x_p - x_{p-1}). \qquad (4.1.3)$$

The case $p = 1$ is immediate. To prove it for $p > 1$, start by putting

$$\prod_p (x_0, x_1, \cdots, x_{p-1}) := x_0 \otimes (x_1 - x_0) \otimes \cdots \otimes (x_{p-1} - x_{p-2})$$

so we can rewrite 4.1.3 in terms of these expresions as

$$\varphi_{p-1}(d_p(x_0, x_1, \cdots, x_p)) = \sum_{i=0}^{p} (-1)^i \prod_p (x_0, \cdots, \hat{x}_i, \cdots, x_{p-1}).$$

Observe that the last two terms in the sum above amount to

$$(-1)^{p-1} x_0 \otimes (x_1 - x_0) \otimes \cdots \otimes (x_{p-2} - x_{p-3}) \otimes (x_p - x_{p-1})$$

and thus the left hand side of 4.1.3 can be expressed as

$$\left[\sum_{i=0}^{p-1} (-1)^i \prod_{p-1} (x_0, \cdots, \hat{x}_i, \cdots, x_{p-1}) \right] \otimes (x_p - x_{p-1}).$$

By induction on p, it follows that this coincides with 4.1.3. Now, the result follows from the commutativity of the following G-module diagram

$$\begin{array}{ccccc} \mathcal{F}_{p+q} & \xrightarrow{\Delta} & (\mathcal{F} \otimes \mathcal{F})_{p+q} & \xrightarrow{\varphi \otimes \varphi} & (\mathcal{G} \otimes \mathcal{G})_{p+q} \\ \varphi \downarrow & & & & \downarrow a \otimes b \\ \mathcal{G}_{p+q} & \xrightarrow[\varepsilon \otimes \mathrm{id}]{} & K^{p+q} & \xrightarrow[\hat{a} \otimes \hat{b}]{} & A \otimes B, \end{array}$$

where Δ denotes the Alexander–Whitney diagonal map, see expression 2.2.5 in Chapter 2, and the action on tensor products is diagonal. □

Consider a permutation G-module $\mathbb{Z}[G/H]$ and write I for the kernel of the augmentation homomorphism $\mathbb{Z}[G/H] \to \mathbb{Z}$, given by $gH \mapsto 1$ for any $gH \in G/H$. Define a G-module homomorphism $\xi\colon \mathbb{Z}[G] \otimes K \to I$ as the composition of $\varepsilon \otimes \mathrm{id}$ and the map $\mu\colon K \to I$ induced by the canonical projection $G \to G/H$ in

$$\begin{array}{ccccc} K & \longrightarrow & \mathbb{Z}[G] & \xrightarrow{\varepsilon} & \mathbb{Z} \\ \downarrow & & \downarrow & & \downarrow = \\ I & \longrightarrow & \mathbb{Z}[G/H] & \longrightarrow & \mathbb{Z}. \end{array}$$

This is obviously a cocycle, and thus it represents a one-dimensional cohomology class $\omega \in H^1(G, I)$, which motivates the following definition:

Definition 4.1.8. Let $H \leq G$ be a subgroup. We define the *Berstein-Schwarz class of G relative to H* as the class $\omega \in H^1(G; I)$ represented by the cocycle

$$\xi \in \mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}[G] \otimes K, I), \qquad \xi = \mu \circ (\varepsilon \otimes \mathrm{id}_K),$$

where $\mu\colon K \to I$ is induced by the canonical projection $G \to G/H$.

By Proposition 4.1.7, its n-th power $\omega^n \in H^n(G, I^n)$ is represented by the map

$$\mathbb{Z}[G] \otimes K^n \xrightarrow{\varepsilon \otimes \mathrm{id}} K^n \xrightarrow{\mu^n} I^n.$$

Lemma 4.1.9. *The class ω defined as above is a zero-divisor, i.e.*

$$\omega \in \ker\big[H^1(G, I) \to H^1(H, I)\big].$$

Proof. The H-module homomorphism $\mathbb{Z}[G] \to I$ defined by $g \mapsto gH - H$ shows that ξ considered as an H-module homomorphism is a coboundary. □

The next proposition relates the powers of relative Berstein-Schwarz classes to sectional category.

Proposition 4.1.10. *Let $H \leq G$ be a subgroup and let $\omega \in H^1(G; I)$ be the Berstein-Schwarz class of G relative to H. Then*

$$\operatorname{secat}(H \hookrightarrow G) \geq \operatorname{height}(\omega) = \sup\{n \in \mathbb{N} \mid \omega^n \neq 0\}.$$

Proof. Let $i : H \hookrightarrow G$ be the inclusion and let $k := \operatorname{height}(\omega)$. By 4.1.9, it holds that $\omega \in \ker \iota^*$. Thus, it follows immediately from Theorem 4.1.3.b) by taking $u_i = \omega$ for each $i \in \{1, 2, \ldots, k\}$. □

In the particular case of $G = \pi \times \pi$ and $H = \Delta_\pi$ is the diagonal subgroup of $\pi \times \pi$, this class coincides with the canonical TC-class introduced by Costa and Farber in [35]. This follows from the fact that $(\pi \times \pi)/\Delta_\pi$ and π seen as a $(\pi \times \pi)$-set via the action $(g, h)x = gxh^{-1}$ are isomorphic as $(\pi \times \pi)$-sets.

Berstein in [11] showed that for a connected CW-complex X of dimension $n \geq 3$, $\operatorname{cat}(X) = n$ if and only if there exists a class $u \in H^1(X, K)$ such that $u^n \neq 0$ in $H^n(X, K^n)$, where K is the augmentation ideal of $\mathbb{Z}[\pi_1(X)]$. Costa and Farber gave a version of this result for topological complexity ([35, Theorem 7]). Here, we state an analogue of their result illustrating the relationship between $\operatorname{secat}(H \hookrightarrow G)$ and the Berstein-Schwarz class of G relative to H.

Theorem 4.1.11. *Put $n := \operatorname{cd} G \geq 3$. Then $\operatorname{secat}(H \hookrightarrow G) \leq n - 1$ if and only if $\omega^n = 0$.*

We postpone the proof to the end of this chapter.

4.2 Adamson cohomology and sectional category

In this section we briefly review Adamson cohomology, a theory first introduced by Adamson [1] for finite groups. Later Hochschild [75] generalized the ideas of Adamson to develop a homological algebra theory in the relative setting. Then we proceed to recast Adamson cohomology in terms of equivariant Bredon cohomology.

4.2.1 Review of the Adamson cohomology theory

Recall that an exact sequence of G-modules

$$\cdots \to M_i \xrightarrow{f_{i+1}} M_i \xrightarrow{f_i} M_{i-1} \to \cdots$$

is said to be (G,H)-*exact* provided that $\ker f_i$ is a direct summand of M_i as an H-module for each i. A G-module P is said to be (G,H)-*projective* provided that for every short (G,H)-exact sequence of modules $M \xrightarrow{f} N \to 0$ and every G-homomophism $g\colon P \to N$, there exists a G-homomorphism $h\colon P \to M$ making the diagram

$$\begin{array}{ccccc} & & P & & \\ & \overset{h}{\swarrow} & \big\downarrow g & & \\ M & \xrightarrow[f]{} & N & \longrightarrow & 0 \end{array}$$

commutative. Finally, given a G-module M, a (G,H)-*projective resolution* of M is an (G,H)-exact sequence of G-modules

$$\cdots \to P_n \to P_{n-1} \to \cdots \to P_0 \to M \to 0$$

such that P_i is (G,H)-projective for each $i \geq 0$.

Example 4.2.1. Given $n \geq 0$, define $C_n(G/H)$ to be the permutation module $\mathbb{Z}\big[(G/H)^{n+1}\big]$, where $(G/H)^{n+1}$ is equipped with the diagonal G-action, i.e

$$g(g_0H,\ldots,g_nH) = (gg_0H,\ldots,gg_nH)$$

Furthermore, let $d_n\colon C_n(G/H) \to C_{n-1}(G/H)$ be given by

$$d_n(g_0H,\ldots,g_nH) = \sum_{i=0}^{n}(-1)^i(g_0H,\ldots,\widehat{g_nH},\ldots,g_nH),$$

where $\widehat{g_nH}$ means that the element g_nH is removed from the tuple. Hochschild [75] proved that (C,d) forms a (G,H)-projective resolution of the trivial G-module $\mathbb{Z}$, with the augmentation map defined by sending every coset to 1. This resolution will be called the *standard resolution of G relative to H*.

In an analogous way to the non-relative case, Hochschild defined the relative extension functor as

$$\mathrm{Ext}^n_{(G,H)}(M,N) := H^n\big(\mathrm{Hom}_{\mathbb{Z}[G]}(\mathcal{P}_*,N)\big),$$

where M and N are G-modules, and $\mathcal{P}_*$ is a (G,H)-projective resolution of M. Then the *Adamson cohomology* of G with respect to H with coefficients in a G-module M is defined as

$$H^*\big([G:H],N\big) := \mathrm{Ext}^*_{(G,H)}(\mathbb{Z},N),$$

where $\mathbb{Z}$ is the trivial G-module. The *Adamson cohomological dimension* of G relative to H, defined as the length of the shortest possible (G,H)-projective resolution of $\mathbb{Z}$, will be denoted by $\mathrm{cd}\,[G:H]$. This number can be equivalently characterized as the maximal integer $n \geq 0$ such that $H^n\big([G:H],M\big) \neq 0$ for some G-module M, as in spirit of [116, Chapter 4, Lemma 4.1.6].

Note that the Adamson relative cohomology defined like this can be seen as a particular case of the cohomology of a permutation representation, with G/H as the base G-set, see Blowers [18].

4.2.2 Adamson canonical class and its universality

Consider the short exact sequence of G-modules,

$$0 \to I \hookrightarrow \mathbb{Z}[G/H] \xrightarrow{\varepsilon} \mathbb{Z} \to 0.$$

Tensoring it over $\mathbb{Z}$ with I^{k-1}, the $k-1$-fold tensor power of I over $\mathbb{Z}$ seen as a G-module via the diagonal G-action, yields another short exact sequence:

$$0 \to I^k \hookrightarrow \mathbb{Z}[G/H] \otimes I^{k-1} \xrightarrow{\varepsilon \otimes \mathrm{id}} I^{k-1} \to 0.$$

Splicing all those sequences together for varying k yields an exact sequence

$$\cdots \to \mathbb{Z}[G/H] \otimes I^k \to \mathbb{Z}[G/H] \otimes I^{k-1} \to \cdots \to \mathbb{Z}[G/H] \to \mathbb{Z} \to 0.$$

This is a (G,H)-projective resolution. To see (G,H)-exactness, note the decomposition as an H-module

$$\mathbb{Z}[G/H] \otimes I^k \cong (\mathbb{Z} \otimes I^{k-1}) \oplus (I \otimes I^{k-1}).$$

In order to see projectiveness, define the inverse maps

$$\alpha \colon \mathbb{Z}[G/H] \otimes I \to \mathbb{Z}[G] \otimes_H I \qquad \alpha(xH \otimes y) = (x \otimes x^{-1}y)$$

and

$$\beta \colon \mathbb{Z}[G] \otimes_H I \to \mathbb{Z}[G/H] \otimes I \qquad \beta(x \otimes y) = xH \otimes xy.$$

We define the G-action on $\mathbb{Z}[G] \otimes_H I$ such that is compatible with the diagonal one in $\mathbb{Z}[G/H] \otimes I$. As such, define the G-module structure on $\mathbb{Z}[G] \otimes_H I$ by the action

$$g(x \otimes y) = \alpha(g(xH \otimes xy)).$$

As a consequence, we have

$$g(x \otimes y) = \alpha(g(xH \otimes xy)) = \alpha(gxH \otimes gxy) = gx \otimes y,$$

and we see that the action restricts to the first component. Then we use [75, Lemma 2] and thus, generalizing this morphism to $\mathbb{Z}[G/H] \otimes I^n$ for every $n > 0$, we have that every term in the exact sequence constructed above is (G,H)-projective.

The (G,H)-projective resolution above lets us define a cup product on Adamson cohomology as in Proposition 4.1.7.

Definition 4.2.2. Let $[a] \in H^p([G:H],A)$ and $[b] \in H^q([G:H],B)$ be cohomology classes represented by cocycles

$$a \colon \mathbb{Z}[G/H] \otimes I^p \to A \qquad \text{and} \qquad b \colon \mathbb{Z}[G/H] \otimes I^q \to B.$$

Define the *cup product* $[a][b] \in H^{p+q}([G:H], A \otimes B)$ as the class represented by the map

$$\mathbb{Z}[G/H] \otimes I^{p+q} \xrightarrow{\varepsilon \otimes \mathrm{id}} I^{p+q} \xrightarrow{\hat{a} \otimes \hat{b}} A \otimes B.$$

It is easy to check that this product verifies the properties *dimension* 0, *naturality with respect to coefficient homomorphisms*, *compatibility with* δ, *associativity* and *commutativity* analogous to the ones listed in Theorem 2.2.15.

Definition 4.2.3. The *Adamson canonical class* $\phi \in H^1\big([G:H], I\big)$ is the class represented by the cocycle $\mathbb{Z}[G/H] \otimes I \xrightarrow{\varepsilon \otimes id} I$. Also, height$(\phi)$ is the largest $n \geq 0$ such that

$$\phi^n \in H^n([G:H], I^n)$$

is nonzero.

The Adamson canonical class is universal in the sense that every other Adamson cohomology class can be recovered from a corresponding power of the canonical class through a change of coefficient system, as the next proposition shows.

Proposition 4.2.4. *For any G-module A and any class $\lambda \in H^n\big([G:H], A\big)$ there exists a G-homomorphism $h\colon I^n \to A$ such that $h^*(\phi^n) = \lambda$.*

Proof. Let $f\colon \mathbb{Z}[G/H] \otimes I^n \to A$ be a cocycle representing the class $\lambda \in H^n([G:H], A)$. By the definition of the cup product in Adamson cohomology, the class ϕ^n is represented by

$$\mathbb{Z}[G/H] \otimes I^n \xrightarrow{\varepsilon \otimes \mathrm{id}} I^n.$$

Taking $h = \hat{f}$ we see that $h^*(\phi^n) = \lambda$. □

Corollary 4.2.5. *Let $\phi \in H^1([G:H], I)$ be the Adamson canonical class, we have that*

$$\mathrm{cd}\,[G:H] = \mathrm{height}(\phi).$$

4.2.3 Adamson cohomology and zero-divisors

One of the consequences of the universality of the canonical class defined in the subsection above is the possibility of characterizing Adamson cohomology groups in terms of zero divisors sets of usual group cohomology for a suitable choice of coefficient systems. Before we proceed to show it, however, we need to introduce first two auxiliary lemmas.

Lemma 4.2.6. *Let M and N be left $\mathbb{Z}[G]$-modules. Let $\mathbb{Z}[G] \otimes M$ be equipped with the diagonal G-action. Then*

$$\Phi : \mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}[G/H] \otimes M, N) \to \mathrm{Hom}_{\mathbb{Z}[H]}(\widetilde{M}, \widetilde{N}), \quad (\Phi(f))(x) := f(H \otimes x) \quad \forall x \in M,$$

is an isomorphism of abelian groups.

Proof. It is easy to see that Φ is a well-defined group homomorphism. Consider the map

$$\Psi : \mathrm{Hom}_{\mathbb{Z}[H]}(\widetilde{M}, \widetilde{N}) \to \mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}[G/H] \otimes M, N), \quad (\Psi(f))(H \otimes x) = gf(g^{-1}x).$$

For each $f \in \mathrm{Hom}_{\mathbb{Z}[H]}(\widetilde{M}, \widetilde{N})$ and $h \in H$ we obtain that

$$ghf(h^{-1}g^{-1}x) = gf(g^{-1}x) \quad \forall g \in G, x \in M,$$

since f is a $\mathbb{Z}[H]$-homomorphism. This shows that $\Psi(f)(gH \otimes m)$ is independent of the chosen representative of gH, thus $\Psi(f) : \mathbb{Z}[G/H] \otimes M \to N$ well-defined.

For all $f \in \mathrm{Hom}_{\mathbb{Z}[H]}(\widetilde{M}, \widetilde{N})$, $g_1, g_2 \in G$ and $x \in M$ we further compute that

$$\begin{aligned}(\Psi(f))(g_1 \cdot g_2 H \otimes x) &= \Psi(f)(g_1 g_2 H \otimes g_1 x) = g_1 g_2 f(g_2^{-1} g_1^{-1} g_1 x) \\ &= g_1 \cdot g_2 f(g_2^{-1} x) = g_1 \cdot (\Psi(f))(g_2 H \otimes x),\end{aligned}$$

so $\Psi(f) \in \mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}[G/H] \otimes M, N)$. Hence, Ψ is well-defined and it is apparent that Ψ is a group homomorphism. A simple computation shows that Ψ is a two-sided inverse of Φ. □

This isomorphism at the level of Hom groups induces, in turn, another isomorphism between Ext groups.

Lemma 4.2.7. *Let M and N be left $\mathbb{Z}[G]$-modules. Let $\mathbb{Z}[G/H] \otimes M$ be equipped with the diagonal G-action. Then there are isomorphisms*

$$\mathrm{Ext}^r_{\mathbb{Z}[G]}(\mathbb{Z}[G/H] \otimes M, N) \cong \mathrm{Ext}^r_{\mathbb{Z}[H]}(\widetilde{M}, \widetilde{N}) \qquad \forall r \in \mathbb{N}_0.$$

Proof. Let

$$0 \to N \hookrightarrow J_0 \xrightarrow{j_0} J_1 \xrightarrow{j_1} J_2 \xrightarrow{j_2} \dots$$

be an injective resolution of N over $\mathbb{Z}[G]$. By Lemma 4.2.6, there is an isomorphism

$$\Phi_i : \mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}[G/H] \otimes M, J_i) \to \mathrm{Hom}_{\mathbb{Z}[H]}(\widetilde{M}, \tilde{J}_i)$$

for each $i \in \mathbb{N}_0$ and one checks without difficulties that Φ_i is compatible with the maps induced by the j_i. At this point, it suffices to show that each of the $\tilde{J}_i$ is an injective $\mathbb{Z}[H]$-module as passing to cohomology then shows the claim.

Let J be an injective $\mathbb{Z}[G]$-module, X and Y be $\mathbb{Z}[H]$-modules and $i : X \hookrightarrow Y$ be a monomorphism of $\mathbb{Z}[H]$-modules and let $f \in \mathrm{Hom}_{\mathbb{Z}[H]}(X, \tilde{J})$. We consider the induced $\mathbb{Z}[G]$-modules $\mathrm{Ind}_H^G(X)$ and $\mathrm{Ind}_H^G(Y)$. One checks from the universal property of induced modules (see diagram 2.1.1 for the particular case of induced modules), that i induces a $\mathbb{Z}[G]$-homomorphism $\widetilde{i} : \mathrm{Ind}_H^G(X) \hookrightarrow \mathrm{Ind}_H^G(Y)$, which is again injective since $\mathbb{Z}[G]$ is free as a right $\mathbb{Z}[H]$-module, and f induces $\widetilde{f} \in \mathrm{Hom}_{\mathbb{Z}[G]}(\mathrm{Ind}_H^G(X), J)$. Since J is injective over $\mathbb{Z}[G]$, it follows that there exists

$$\widetilde{\varphi} \in \mathrm{Hom}_{\mathbb{Z}[G]}(\mathrm{Ind}_H^G(Y), J) \qquad \text{with } \widetilde{f} = \widetilde{\varphi} \circ \widetilde{i}.$$

Define the map

$$\varphi : Y \to J \qquad \text{by } \varphi(y) := \widetilde{\varphi}(1 \otimes_{\mathbb{Z}[H]} y)$$

for each $y \in Y$. One checks without difficulties that φ is a $\mathbb{Z}[H]$-homomorphism with

$$(\varphi \circ i)(y) = \widetilde{\varphi}(1 \otimes_{\mathbb{Z}[H]} i(y)) = \widetilde{\varphi}(\widetilde{i}(1 \otimes_{\mathbb{Z}[H]} y)) = \widetilde{f}(1 \otimes_{\mathbb{Z}[H]} y) = f(y)$$

for all $y \in J$. This shows that $\tilde{J}$ is injective over $\mathbb{Z}[H]$ and thereby completes the proof. □

Using the previous two corollaries, we obtain the following characterization of Adamson cohomology groups in terms of zero divisors of usual group cohomology.

Theorem 4.2.8. *For any G-module A and $n \geq 1$, we have*

$$H^n([G:H],A) = \ker\left[H^1(G,\mathrm{Hom}_{\mathbb{Z}}(I^{n-1},A)) \to H^1(H,\mathrm{Hom}_{\mathbb{Z}}(I^{n-1},A))\right].$$

In particular,

$$H^1([G:H],A) = \ker\left[H^1(G,A) \to H^1(H,\widetilde{A})\right].$$

Proof. Consider the short exact sequence

$$0 \to I^n \to \mathbb{Z}[G/H] \otimes I^{n-1} \to I^{n-1} \to 0.$$

Applying the Ext functor with coefficients on A, we obtain the associated long exact sequence

$$0 \to \mathrm{Hom}_{\mathbb{Z}[G]}(I^{n-1},A) \to \mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}[G/H] \otimes I^{n-1},A) \xrightarrow{\varkappa} \mathrm{Hom}_{\mathbb{Z}[G]}(I^n,A) \xrightarrow{\nu}$$

$$\mathrm{Ext}^1_{\mathbb{Z}[G]}(I^{n-1},A) \xrightarrow{\gamma} \mathrm{Ext}^1_{\mathbb{Z}[G]}(\mathbb{Z}[G/H] \otimes I^{n-1},A) \to \mathrm{Ext}^1_{\mathbb{Z}[G]}(I^n,A) \to \cdots$$

By the universality of the Adamson canonical class given in Proposition 4.2.4, through the correspondence $f \mapsto \hat{f}$, we get

$$H^n([G:H],A) \cong \mathrm{Hom}_{\mathbb{Z}[G]}(I^n,A)/\mathrm{Im}(\varkappa).$$

We also have, by exactness, the chain of isomorphisms

$$\mathrm{Hom}_{\mathbb{Z}[G]}(I^n,A)/\mathrm{Im}(\varkappa) \cong \mathrm{Hom}_{\mathbb{Z}[G]}(I^n,A)/\ker(\nu) \cong \mathrm{Im}(\nu) \cong \ker(\gamma).$$

To continue the proof, observe that Lemma 4.2.7 gives us the isomorphism

$$\mathrm{Ext}^1_{\mathbb{Z}[G]}(\mathbb{Z}[G/H] \otimes I^{n-1},A) \cong \mathrm{Ext}^1_{\mathbb{Z}[H]}(\widetilde{I}^{n-1},\widetilde{A}),$$

which is induced by the map

$$\mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}[G/H] \otimes I^{n-1},A) \to \mathrm{Hom}_{\mathbb{Z}[H]}(\widetilde{I}^{n-1},\widetilde{A})$$

defined by associating to any G-homomorphism $f : \mathbb{Z}[G/H] \otimes I^{n-1} \to A$ the restriction $f_{|_{1\otimes I^{n-1}}}$ to $1 \otimes I^{n-1} \subset \mathbb{Z}[G/H] \otimes I^{n-1}$ (this, in turn, is a particular case of the isomorphism defined in Lemma 4.2.6). Finally, by the isomorphism provided in Proposition 2.2.6, we have that

$$\mathrm{Ext}^1_{\mathbb{Z}[G]}(I^{n-1},A) \cong H^1(G,\mathrm{Hom}_{\mathbb{Z}}(I^{n-1},A))$$

and

$$\mathrm{Ext}^1_{\mathbb{Z}[H]}(\widetilde{I}^{n-1},\widetilde{A}) \cong H^1(H,\mathrm{Hom}_{\mathbb{Z}}(I^{n-1},A)).$$

The action on $\mathrm{Hom}_{\mathbb{Z}}(I^{n-1},A)$ is defined by

$$(gf)(x) = gf(g^{-1}x) \qquad \forall g \in G, f \in \mathrm{Hom}_{\mathbb{Z}}(I^{n-1},A) \text{ and } x \in I^{n-1}.$$

Then γ becomes the restriction homomorphism, which finishes the proof. □

Consider the canonical map

$$\rho\colon EG \to E_{\langle H\rangle}G$$

and the chain homotopy homomorphism between the cellular chain complex of EG and the relative standard resolution of G with respect to H which corresponds to sending $g \mapsto gH$, its class in the coset space G/H. Now, after applying the functor $\mathrm{Hom}_{\mathbb{Z}[G]}(\cdot, A)$, consider the induced maps on cohomology, which gives a map between the Adamson cohomology and the usual cohomology of the group

$$\rho^*\colon H^*([G:H], A) \to H^*(G, A). \tag{4.2.1}$$

Observe that, at the chain level, this map is induced by the projection $\mathbb{Z}[G] \to \mathbb{Z}[G/H]$ and therefore it respects product structures. The following result arises immediately from the definitions involved, if we take I as the coefficient module in the previous homomorphism. Nonetheless, it is relevant enough to be highlighted on its own:

Proposition 4.2.9. *With ρ^* defined as before, let ϕ and ω be the Adamson canonical class and the Berstein-Schwarz class relative to H, respectively. We have*

$$\rho^*(\phi) = \omega.$$

Remark 4.2.10. It is interesting to note that the naturality of ρ^* with respect to change of coefficient system implies that Im ρ^* corresponds to the *essential classes* in the sense of [57], recall Definition 3.3.5. The notion of essential cohomology classes will play a crucial role in the next chapter, when we will generalize them to arbitrary group monomorphisms.

4.2.4 Relative crossed and principal homomorphisms

Before proceeding with further considerations about Adamson cohomology and sectional category, let us take a moment to consider an alternative description of the Adamson cohomology groups at dimension one, in light of the discussion of the previous subsection. We will introduce relative analogues to the notion of crossed and principal homomorphisms, and will see that the Adamson cohomology groups at dimension one are expressible, in full analogy to the non-relative case, in terms of those types of homomorphisms.

Let A be a G-module, and consider the short exact sequence of G-modules associated to the augmentation

$$0 \to I \to \mathbb{Z}[G/H] \to \mathbb{Z} \to 0$$

and their associated relative Ext-sequence

$$\begin{aligned} 0 &\to \mathrm{Ext}^0_{(G,H)}(\mathbb{Z}, A) \to \mathrm{Ext}^0_{(G,H)}(\mathbb{Z}[G/H], A) \to \mathrm{Ext}^0_{(G,H)}(I, A) \\ &\to \mathrm{Ext}^1_{(G,H)}(\mathbb{Z}, A) \to \mathrm{Ext}^1_{(G,H)}(\mathbb{Z}[G/H], A) \to \ldots \end{aligned}$$

Since $\mathbb{Z}[G/H]$ is a (G, H)-projective module, we know that

$$\mathrm{Ext}^1_{(G,H)}(\mathbb{Z}[G/H], A) = \{0\}.$$

Moreover, as described in [75, Pg 253], we can identify the Ext^0-terms as Hom-groups in the following manner

$$0 \to \mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, A) \to \mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}[G/H], A) \xrightarrow{i^*} \mathrm{Hom}_{\mathbb{Z}[G]}(I, A) \to H^1([G:H], A) \to 0.$$

In particular, this shows that the Adamson cohomology group of dimension one can be seen as

$$H^1([G:H], A) \cong \mathrm{coker}\left[i^* : \mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}[G/H], A) \to \mathrm{Hom}_{\mathbb{Z}[G]}(I, A)\right].$$

By Lemma 4.2.6 we have the isomorphism

$$\Phi : \mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}[G/H], A) \to \mathrm{Hom}_{\mathbb{Z}[H]}(\mathbb{Z}, A) \cong A^H, \quad \Phi(f) = f(H).$$

Such isomorphism fits in a diagram of the form

$$\begin{array}{ccc} \mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}[G/H], A) & \xrightarrow{\quad i^* \quad} & \mathrm{Hom}_{\mathbb{Z}[G]}(I, A) \\ & {\scriptstyle \Phi}\searrow \quad \nearrow{\scriptstyle \lambda} & \\ & A^H & \end{array} \tag{4.2.2}$$

where the map λ is defined as follows

$$\lambda : A^H \to \mathrm{Hom}_{\mathbb{Z}[G]}(I, A), \qquad (\lambda(a))(gH - H) = g \cdot a - a.$$

It is straightforward to see that, indeed, $\lambda(a)$ is a $\mathbb{Z}[G]$-homomorphism for each element $a \in A^H$. If we compute now the composition $(\lambda \circ \Phi)$ we observe

$$\begin{aligned} ((\lambda \circ \Phi)(f))(gH - H) &= (\lambda(f(H)))(gH - H) = g \cdot f(H) - f(H) \\ &= f(gH) - f(H) = (i^* f)(gH - H), \end{aligned}$$

and thus the diagram 4.2.2 commutes. Hence, we can alternatively express

$$H^1([G:H], A) \cong \mathrm{coker}[\lambda : A^H \to \mathrm{Hom}_{\mathbb{Z}[G]}(I, A)]. \tag{4.2.3}$$

Putting

$$P_{(G,H)}(A) := \{f \in \mathrm{Hom}_{\mathbb{Z}[G]}(I, A) \mid \exists a \in A^H \,:\, f(gH - H) = g \cdot a - a \ \forall gH - H \in I\} = \mathrm{Im}\,\lambda,$$

we obtain from the isomorphism 4.2.3 and the previous definition yet another possible characterization of the one-dimensional Adamson cohomology group, in the form

$$H^1([G:H], A) \cong \mathrm{Hom}_{\mathbb{Z}[G]}(I, A) \big/ P_{(G,H)}(A)\,. \tag{4.2.4}$$

Most readers that are already aware of the notion of principal homomorphisms (or those who previously read Remark 2.2.5) have probably noticed that the set $P_{(G,H)}(A)$ essentially constitutes a relative notion of the set of principal homomorphisms with respect to the $\mathbb{Z}[G]$-module A. As such, it is natural to introduce a definition of relative crossed and principal homomorphisms.

Definition 4.2.11. Let A be a left G-module.

a) A (G, H)-*relative crossed homomorphism* with coefficients in A is a map $\varphi : G/H \to A$, which satisfies

$$\varphi(g \cdot xH) = \varphi(gH) + g \cdot \varphi(xH) \qquad \forall g, x \in G.$$

The set of all (G, H)-relative crossed homomorphisms with coefficients in A is denoted by $C_{(G,H)}(A)$.

b) A (G, H)-*relative principal homomorphism* with coefficients in A is a map $\varphi : G/H \to A$, for which there exists an $a \in A^H$, such that

$$\varphi(gH) = g \cdot a - a \qquad \forall g \in G.$$

The set of all (G, H)-relative principal homomorphisms with coefficients in A is denoted by $P_{(G,H)}(A)$.

Same as the non-relative case, it is straightforward to check that every (G, H)-relative principal homomorphism is a (G, H)-relative crossed homomorphism. One further checks that $C_{(G,H)}(A)$ and $P_{(G,H)}(A)$ are groups with respect to pointwise addition.

Proposition 4.2.12. *Let A be a left G-module. Then*

$$H^1([G:H]; A) \cong C_{(G,H)}(A) \Big/ P_{(G,H)}(A) \, .$$

Proof. One checks that

$$\Psi : \mathrm{Hom}_{\mathbb{Z}[G]}(I, A) \to C_{(G,H)}(A), \qquad (\Psi(f))(gH) = f(gH - H) \quad \forall g \in G,$$

is a group isomorphism and that

$$\mathrm{Im}\,(\Psi \circ \lambda) = P_{(G,H)}(A).$$

The claim then follows from the above computation. □

4.2.5 A spectral sequence

We will make a brief introduction to the existence of a spectral sequence which contains information about both Adamson and usual cohomology. This sequence is derived from a much more general theory of relative homological algebra developed in [48]. We will restrict here to our case of interest.

Take the (G, H)-projective resolution of $\mathbb{Z}$

$$\cdots \to \mathbb{Z}[G/H] \otimes I^n \to \cdots \to \mathbb{Z}[G/H] \otimes I \to \mathbb{Z}[G/H] \to \mathbb{Z} \to 0.$$

Looking at it as an object in the category of sequences of G-modules consider a G-projective resolution of it, which gives us a double complex

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & P_{i,j} & \longrightarrow & \cdots & \longrightarrow & P_{i,0} & \longrightarrow & Q_i & \longrightarrow & 0 \\
 & & \downarrow & & & & \downarrow & & \downarrow & & \\
 & & \vdots & & & & \vdots & & \vdots & & \\
 & & \downarrow & & & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & P_{0,j} & \longrightarrow & \cdots & \longrightarrow & P_{0,0} & \longrightarrow & Q_0 & \longrightarrow & 0 \\
 & & \downarrow & & & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & \mathbb{Z}[G/H] \otimes I^j & \longrightarrow & \cdots & \longrightarrow & \mathbb{Z}[G/H] & \longrightarrow & \mathbb{Z} & \longrightarrow & 0
\end{array}$$

such that every $P_{i,j}$ is G-projective, every column is a G-projective resolution, and each row (except the first one) is split exact. Now, applying the functor $\operatorname{Hom}_{\mathbb{Z}[G]}(-, A)$ for some choice of coefficient system A, we obtain another double complex and its associated spectral sequence. Let us have a glance at the horizontal filtration. Given that every row above the first one is split exact we have that

$$\mathrm{E}_0^{p,q} = \operatorname{Hom}_{\mathbb{Z}[G]}(P_{p,q}, A) \qquad \text{and} \qquad \mathrm{E}_1^{p,q} = 0$$

for every $q > 0$. Moreover, one observes that

$$\mathrm{E}_1^{p,0} \cong \operatorname{Hom}_{\mathbb{Z}[G]}(Q_p, A).$$

As we can see, the spectral sequence collapses and, given that Q_* is a projective resolution of $\mathbb{Z}$ as a trivial G-module, as stated before, it converges to

$$\operatorname{Ext}^*_{\mathbb{Z}[G]}(\mathbb{Z}, A) = H^*(G, A).$$

The vertical filtration provides more information. Every column is a projective resolution, so the first page of the spectral sequence has the form

$$\mathrm{E}_1^{p,q} = \operatorname{Ext}^q_{\mathbb{Z}[G]}(\mathbb{Z}[G/H] \otimes I^p, A).$$

The differential on this page is the map $\mathrm{E}_1^{p,q} \xrightarrow{\overline{d_1}} \mathrm{E}_1^{p+1,q}$ induced by the original differential on the (G, H)-projective resolution,

$$\mathbb{Z}[G/H] \otimes I^{p+1} \xrightarrow{\iota \circ \varepsilon \otimes \mathrm{id}} \mathbb{Z}[G/H] \otimes I^p.$$

Therefore, the second page of the spectral sequence corresponds to

$$\mathrm{E}_2^{p,q} = H^p(\operatorname{Ext}^q_{\mathbb{Z}[G]}(\mathbb{Z}[G/H] \otimes I^*, A)).$$

It is in this second page where, if we restrict to $q = 0$, Adamson cohomology appears. Indeed

$$\mathrm{E}_2^{p,0} = H^p([G : H], A)$$

and we have the following proposition.

Proposition 4.2.13. *There exists a spectral sequence*

$$\mathrm{E}_2^{p,q} = H^p(\operatorname{Ext}^q_{\mathbb{Z}[G]}(\mathbb{Z}[G/H] \otimes I^*, A)) \Rightarrow H^{p+q}(G, A)$$

such that $\mathrm{E}_2^{p,0} = H^p([G : H], A)$.

4.2.6 Adamson vs Bredon cohomology

We will now recast Adamson cohomology in terms of equivariant Bredon cohomology in order to reconcile our approach with that of Farber, Grant, Lupton and Oprea [56].

Theorem 4.2.14. *Given a G-module A, let $\underline{A}$ be the* $\mathrm{Or}_{\langle H\rangle}G$*-module defined by setting* $\underline{A}(G/K) = A^K$*. Then*

$$H^*\big([G:H],A\big) \cong H^*_{\langle H\rangle}(E_{\langle H\rangle}G,\underline{A}).$$

In particular, $\mathrm{cd}\,[G:H] \leq \mathrm{cd}_{\langle H\rangle}G$*.*

We note that this result has been recently derived with different methods in [5], and also in [103] when G is a finite group.

Proof. In what follows, we take as a model for $E_{\langle H\rangle}(G)$ the geometric realization of a suitable Δ-complex such that its cellular chain complex coincides with the standard resolution of G relative to H (for details on the construction see [4, Proposition 4.16], recalled in Chapter 2 as Proposition 2.3.8). In order to compare Adamson and Bredon cohomologies, first evaluate the cellular $\mathrm{Or}_{\langle H\rangle}(G)$-chain complex on the principal component, wich gives us, as previously described (see Chapter 2 chain of isomorphisms 2.3.5)

$$\underline{C_n}(E_{\langle H\rangle}(G))(G/\{1\}) = H_n(E_{\langle H\rangle}(G)_{n+1}, E_{\langle H\rangle}(G)_n).$$

By excision, we have that

$$\underline{C_n}(E_{\langle H\rangle}(G))(G/\{1\}) = \mathbb{Z}[(G/H)^{n+1}].$$

For every $n \geq 0$ define a homomorphism

$$\Phi\colon \mathrm{Hom}_{\mathbb{Z}[G]}\big(\mathbb{Z}\big[(G/H)^n\big],A\big) \to \mathrm{Hom}_{\mathrm{Or}_{\langle H\rangle}G}\big(\underline{C}_n(E_{\langle H\rangle}G),\underline{A}\big)$$

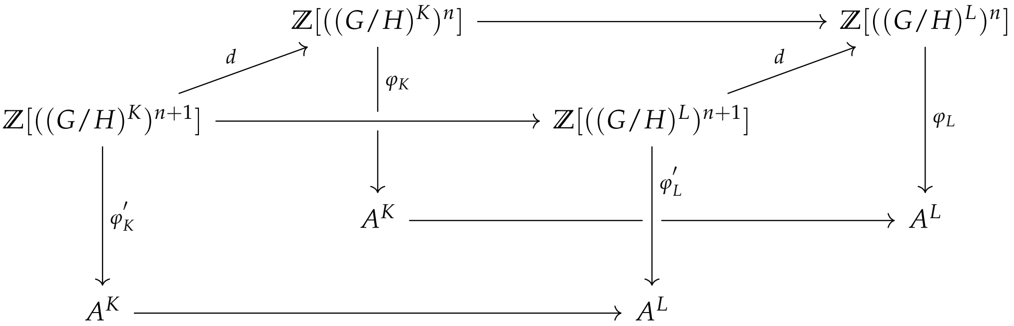
by assigning to every $\varphi \in \mathrm{Hom}_{\mathbb{Z}[G]}\big(\mathbb{Z}\big[(G/H)^n\big],A\big)$ a map φ_K for every subgroup $K \in \langle H\rangle$, defined as the composition

$$\mathbb{Z}[((G/H)^K)^n] \hookrightarrow \mathbb{Z}[(G/H)^n] \xrightarrow{\varphi} A$$

where the first map is the inclusion (that is, the one induced by the trivial element). If we consider, given $H, K \in \langle H\rangle$, an equivariant map $G/L \to G/K$, which can be identified as a $g \in G$ such that $gLg^{-1} \leq K$, we have the following diagram

$$\begin{array}{ccc}
\mathbb{Z}[((G/H)^K)^n] & \longrightarrow & \mathbb{Z}[((G/H)^L)^n] \\
\downarrow & & \downarrow \\
\mathbb{Z}[(G/H)^n] & & \mathbb{Z}[(G/H)^n] \\
\varphi\downarrow & & \downarrow\varphi \\
A^K & \longrightarrow & A^L
\end{array}$$

where both the top and bottom horizontal morphisms denote action by g. Due to the fact that φ is a G-module homomorphism, the diagram above is commutative. Moreover, Φ commutes with the differential. Indeed, if we consider $\Phi(\delta\varphi)$ with $\varphi \in \mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}[(G/H)^n], A)$ (and δ the corresponding differential) we obtain, for every $g\colon G/L \to G/K$, a diagram analogous to the one above with the top and bottom horizontal arrows being the action by g and the vertical ones the maps which assigns to every tuple $(x_0, \cdots, x_n)$ the element $\sum_i (-1)^i \varphi(x_0, \cdots, \widehat{x_i}, \cdots, x_n)$. Now, considering ∂ as the differential in the Bredon complex, for every $g\colon G/L \to G/K$ the map $\partial\Phi(\varphi)$ gives us a diagram



with the diagonal arrows the respective differentials in the corresponding complexes and φ'_* defined as the composition $\varphi_* \circ d$. Now, given that every φ_* is defined as the composition of an inclusion followed by φ, φ'_* assigns to every tuple $(x_0, \cdots, x_n)$ the element $\sum_i (-1)^i \varphi(x_0, \cdots, \widehat{x_i}, \cdots, x_n)$. And so Φ is a well-defined homomorphism of cochain complexes.

Finally, the map Φ is surjective and injective. In order to see surjectiveness, construct for any map

$$\alpha \in \mathrm{Hom}_{\mathrm{Or}_{\langle H\rangle}(G)}(\underline{C_*}(E_{\langle \mathcal{H}\rangle}(G)), \underline{A})$$

and for every $K \in \langle H \rangle$ a diagram

$$\begin{array}{ccc} \mathbb{Z}[((G/H)^K)^n] & \hookrightarrow & \mathbb{Z}[G/H] \\ {\scriptstyle \alpha_K}\downarrow & & \downarrow{\scriptstyle \alpha_1} \\ A^K & \longrightarrow & A \end{array}$$

where the top vertical arrow is the inclusion induced by $1 \in G$. Then such a map α can be seen as the image of α_1 via Φ. The injectivity is immediate from the definition of Φ.

Given that Φ is a bijective map for every n, there exists a map

$$\Psi\colon \mathrm{Hom}_{\mathrm{Or}_{\langle H\rangle} G}(\underline{C}_n(E_{\langle H\rangle} G), \underline{A}) \to \mathrm{Hom}_G(\mathbb{Z}[(G/H)^n], A)$$

such that $\Phi_n \circ \Psi_n$ and $\Psi_n \circ \Phi_n$ are the respective identities for every $n \geq 0$. The map Ψ is easily seen as a chain homomorphism, given that

$$\Psi_{n+1} \circ \partial_n = \Psi_{n+1}\partial_n(\Phi_n\Psi_n) = \Psi_{n+1}(\Phi_{n+1}\delta_n)\Psi_n = \delta_n \circ \Psi_n.$$

Finally, we have that

$$\Psi_n \circ \Phi_n - \mathrm{id} = \delta_{n-1} h_n + h_{n+1}\delta_n \qquad \wedge \qquad \Phi_n \circ \Psi_n - \mathrm{id} = \partial_{n-1} h'_n + h'_{n+1}\partial_n$$

where the chain homotopies are defined as

$$h_n \colon \mathrm{Hom}_G\big(\mathbb{Z}\big[(G/H)^n\big], A\big) \to \mathrm{Hom}_G\big(\mathbb{Z}\big[(G/H)^{n-1}\big], A\big)$$

and

$$h'_n \colon \mathrm{Hom}_{\mathrm{Or}_{\langle H\rangle} G}\big(\underline{C}_n(E_{\langle H\rangle}G), \underline{A}\big) \to \mathrm{Hom}_{\mathrm{Or}_{\langle H\rangle} G}\big(\underline{C}_{n-1}(E_{\langle H\rangle}G), \underline{A}\big)$$

diagonal maps corresponding with sending every element in their respective domains to 0. Thus Φ defines a chain homotopy equivalence between the Adamson and Bredon cochain complexes, which gives us the desired isomorphism

$$H^*\big([G:H], A\big) \cong H^*_{\langle H\rangle}(E_{\langle H\rangle}G, \underline{A}).$$

□

Even though Bredon cohomology theory has raised an extensive amount of research since its very inception, the main setback is still the high difficulty of making not only explicit computations, but also of obtaining good bounds for cohomological dimension in most cases. In the face of this structural difficulty Adamson cohomology offers a simpler tool, both theoretically and computationally that, as we just showed, allows to bound Bredon cohomological dimension from below. The natural question that arises is when does Adamson cohomological dimension detect Bredon cohomological dimension?

The most natural example of coincidence of both dimensions happens, as expected, when the subgroup is normal. Indeed, consider $H \leq G$ a subgroup and $K \triangleleft G$ is a normal subgroup contained in H. The group G acts on $E_{\langle H/K\rangle}(G/K)$ through the natural projection to the lateral classes by K, and if we take an H/K-fixed point $p \in E_{\langle H/K\rangle}(G/K)$, p is also an H-fixed point, given that $hp = hKp = p$ for any $h \in H$. Since the projection onto the lateral classes by K sends ghg^{-1} to $gK(hK)g^{-1}K$, the definition of the family $\langle H\rangle$ and the universal property of $E_{\langle H\rangle}(G)$ gives us a way of relating the model of the classifying space of G with respect to $\langle H\rangle$ and the classifying space modulo K:

Proposition 4.2.15 ([4] Proposition 4.21 and Corollary 4.22)**.** *Let $H \leq G$, and $K \triangleleft G$ a normal subgroup of G contained in H. Then, a model for $E_{\langle H/K\rangle}(G/K)$ is also a model for $E_{\langle H\rangle}(G)$. In particular, if H is normal in G, $E(G/H)$ is a model for $E_{\langle H\rangle}(G)$.*

The natural future line of work here is to investigate in which other cases Adamson cohomological dimension is enough to detect Bredon cohomological dimension, and to study how to control and bound the differences between them when they differ.

Additionally, despite the fact that Adamson cohomology is easier to approach than Bredon cohomology, in general it is not a simple task to make explicit calculations. As such, there is ample room for future investigation of ways of computing Adamson cohomological dimension. In particular, we trust the naturality of the Adamson canonical class will prove fruitful in this matter.

4.3 Further remarks on secat$(H \hookrightarrow G)$

In view of [56, Corollary 3.5.1], we know that $\mathrm{TC}(\pi) \leq \mathrm{cd}_{\langle \Delta_\pi \rangle}(\pi \times \pi)$ under certain mild assumptions on π. By our generalization of this result in Corollary 4.1.6, and the definition of Bredon cohomological dimension, we know that $\mathrm{secat}(H \hookrightarrow G) \leq \mathrm{cd}_{\langle H \rangle} G$. It is therefore hard not to ask whether $\mathrm{TC}(\pi) = \mathrm{cd}_{\langle \Delta_\pi \rangle}(\pi \times \pi)$ or, more generally, whether $\mathrm{secat}(H \hookrightarrow G) = \mathrm{cd}_{\langle H \rangle} G$ or $\mathrm{secat}(H \hookrightarrow G) = \mathrm{cd}[G : H]$. The two latter cannot possibly be true, as the following examples show.

Example 4.3.1. (1) Consider the inclusion $2\mathbb{Z} \hookrightarrow \mathbb{Z}$. By Theorem 3.2.8, $\mathrm{secat}(2\mathbb{Z} \hookrightarrow \mathbb{Z}) = 1$. On the other hand, given that the subgroup is normal, the Adamson cohomology coincides with the usual cohomology of the quotient (see [1, Theorem 3.2]) and then $H^*([\mathbb{Z} : 2\mathbb{Z}], A) = H^*(\mathbb{Z}/2\mathbb{Z}, A)$. Therefore we observe

$$\mathrm{cd}[G : H] = \mathrm{cd}\, \mathbb{Z}_2 = \mathrm{cd}_{\langle 2\mathbb{Z} \rangle} \mathbb{Z} = \infty.$$

(2) It is perhaps interesting to note that this phenomenon is not torsion-related. Consider the inclusion $[F_n, F_n] \hookrightarrow F_n$, where F_n denotes the free group of n generators, and $[F_n, F_n]$ its commutator subgroup. Similarly as above, $\mathrm{secat}\big([F_n, F_n] \hookrightarrow F_n\big) = 1$, but

$$\mathrm{cd}[F_n : [F_n, F_n]] = \mathrm{cd}_{\langle [F_n, F_n] \rangle} F_n = \mathrm{cd}\, \mathbb{Z}^n = n.$$

Nevertheless, it is possible to find cases where sectional category and Adamson cohomological dimension coincide, as in the next example.

Example 4.3.2. (1) Recall that a group G is said to be nilpotent of order n if there exists a series of normal subgroups $\{1\} = G_0 \lhd G_1 \lhd \cdots \lhd G_n = G$ where $G_{i+1}/G_i \leq Z(G/G_i)$ (equivalently $[G, G_{i+1}] \leq G_i$). Consider the group

$$H_3 = \langle a_1, a_2, b \,|\, [a_1, b] = 1, [a_2, b] = 1, [a_2, a_1] = b \rangle$$

known as the *three dimensional Heisenberg group*. This is one of the most paradigmatic torsion-free nilpotent groups. The infinite cyclic group N generated by b is a central subgroup of H_3 and H_3 fits in a central group extension

$$\{1\} \to N \to H_3 \to F \to \{1\}$$

where F is a free abelian group with basis in one-to-one correspondence with the generators of H_3. Given that N is central (and so is normal) in H_3, by [1, Theorem 3.2] we have

$$H^*([H_3 : N]) = H^*(F) \qquad \text{with } H^n(F) \cong \mathbb{Z}^{\binom{2}{n}}.$$

The details on how to obtain the cohomology ring structure for integer coefficients of H_3 can be consulted in [76]. It can be seen that the nilpotence of the kernel of the homomorphism in cohomology induced by the inclusion $N \hookrightarrow H_3$ is 2 and so, by Theorem 3.2.8, we obtain

$$2 \leq \mathrm{secat}(N \hookrightarrow H_3) \leq 3 = \mathrm{cd}(H_3).$$

Proposition 4.2.15 gives us that a model for $E_{\langle N\rangle}H_3$ is homotopically equivalent to EH_3/N, and then

$$\dim E_{\langle N\rangle}H_3 = \mathrm{cd}[H_3 : N] = \mathrm{cd}(H_3/N) = \mathrm{cd}(F) = 2.$$

By Theorem 4.1.4 and its Corollary 4.1.6 we know that $\mathrm{secat}(N \hookrightarrow H_3) \leq \dim E_{\langle N\rangle}H_3$. Therefore $\mathrm{secat}(N \hookrightarrow H_3) \leq \mathrm{cd}[H_3 : N] = 2$ and we conclude that

$$\mathrm{secat}(N \hookrightarrow H_3) = \mathrm{cd}[H_3 : N] = 2.$$

(2) Consider the subgroup inclusion of $\mathbb{Z} \hookrightarrow \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ taken as the inclusion in the first factor. This can be represented by the fibration

$$S^1 \times \mathbb{R} \xrightarrow{\mathrm{id}\times\exp} T^2.$$

where exp denotes the exponential map $\exp\colon \mathbb{R} \to S^1$ defined by $\exp(\theta) = e^{i\theta}$. Looking at T^2 as $S^1 \times S^1$, take as an open cover the one defined by

$$U_0 = S^1 \times S^1 \setminus \{1\} \qquad U_1 = S^1 \times S^1 \setminus \{-1\}$$

(or, equivalently, any choice of two antipodal points). For each U_i, we have a local section of the fibration, defined by taking the identity on the S^1 factor, and lifting the other factor to $\mathbb{R}$ by considering the argument of the exponential map, i.e.

$$s_i(e^{i\theta_1}, e^{i\theta_2}) = (e^{i\theta_1}, \theta_2).$$

This informs us that $\mathrm{secat}(\mathbb{Z} \hookrightarrow \mathbb{Z} \times \mathbb{Z}) = 1$. By the normality of the subgroup, we have that

$$\mathrm{cd}[\mathbb{Z} \times \mathbb{Z} : \mathbb{Z}] = \mathrm{cd}(\mathbb{Z}) = 1$$

and consequently

$$\mathrm{secat}(\mathbb{Z} \hookrightarrow \mathbb{Z} \times \mathbb{Z}) = \mathrm{cd}[\mathbb{Z} \times \mathbb{Z} : \mathbb{Z}].$$

As an open question, it remains to elucidate the full relationship between sectional category of subgroup inclusions and Adamson cohomological dimension with respect to the subgroup, and in which cases the former can be represented, or bounded in some direction, by the latter. This line of work, of course, is strongly related to the investigation of how good Adamson cohomological dimension detects Bredon cohomological dimension, and how big the difference can be in interesting cases, as discussed at the end of section 3. It is also important to note that these cases do not provide much information in the context of topological complexity, given that the diagonal subgroup is normal in the product group only under the assumption that the group is abelian.

In [56] an analogue of the Costa–Farber canonical class is defined in the context of Bredon cohomology, $\mathbf{u} \in H^1_{\langle\Delta_\pi\rangle}(\pi \times \pi, \underline{I})$. This class is shown to be universal in loc. cit. Moreover, the image of this class via the homomorphism ρ^* is precisely the usual Costa-Farber class. If instead of the family $\langle\Delta_\pi\rangle$ generated by the diagonal subgroup, we would take the family of subgroups $\mathcal{F}$, generated by a subgroup $H \leq G$, it is possible to define a cohomology class represented by the short exact sequence of Bredon modules

$$0 \to \underline{I} \to \mathbb{Z}[\cdot, G/H] \xrightarrow{\varepsilon} \underline{\mathbb{Z}} \to 0$$

where $\underline{I}$ is the kernel of the augmentation ε. Let us denote it also, abusing notation, by $\mathbf{u}$. This class is the canonical class associated to the family $\mathcal{F}$, and arguments analogues to the case of the diagonal family show that it is also universal, and, evaluating in the principal component, it is immediate that its image by the principal component evaluation homomorphism

$$\rho^1 \colon H^1_{\langle H \rangle}(G, \underline{I}) \to H^1(G, \underline{I}(G/\{1\}))$$

is the Berstein-Schwarz class of G relative to H introduced in Subsection 4.1.2.

Remark 4.3.3. To our knowledge, despite Theorem 4.2.14, universality of the Bredon class does not imply in a straightforward manner universality of the Adamson class. This is due to the fact that I^H need not coincide with $\underline{I}(G/H)$.

Put $\rho_{\langle H \rangle}$ for the greatest integer $n \geq 0$ such that the principal component evaluation homomorphism

$$\rho^n \colon H^n_{\langle H \rangle}(G, \underline{A}) \to H^n(G, \underline{A}(G/\{1\}))$$

is non-trivial for some $\mathrm{Or}_{\langle H \rangle}G$-module $\underline{A}$. A straightforward generalization of [56, Theorem 4.1], using Theorem 4.1.4, immediately shows that

$$\rho_{\langle H \rangle} \leq \mathrm{secat}(H \hookrightarrow G).$$

The next result shows that this lower bound for sectional category is never better than the standard cohomological lower bound.

Proposition 4.3.4. *With the notation above,* $\mathrm{height}(\omega) \geq \rho_{\langle H \rangle}$.

Proof. Suppose there exists $\alpha \in H^n_{\langle H \rangle}(G, \underline{A})$ such that $\rho^*(\alpha) \neq 0$. Universality of $\mathbf{u}$ implies that there exists an $\mathrm{Or}_{\langle F \rangle}G$-homomorphism $f \colon \underline{I}^n \to \underline{A}$ such that $f^*(\mathbf{u}^n) = \alpha$. But then f induces also a G-module homomorphism between the principal components of Bredon modules and, thus, it gives a commutative diagram of group cohomologies

$$\begin{array}{ccc} H^n_{\langle H \rangle}(G, \underline{I}^n) & \xrightarrow{\rho^*} & H^n(G, I^n) \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle f^*} \\ H^n_{\langle H \rangle}(G, \underline{A}) & \xrightarrow[\rho^*]{} & H^n(G, A). \end{array}$$

By hypothesis $\rho^*(\alpha) \neq 0$ so $\rho^*(f^*(\mathbf{u}^n)) \neq 0$ and, by commutativity, $\rho^*(\mathbf{u}^n) = \omega^n$, the Berstein-Schwarz class of G relative to H, is nonzero. □

Observe that one can also define the *essential dimension* of a subgroup inclusion $\iota \colon H \hookrightarrow G$, denoted by $\rho^*_{[G:H]}$ as the greatest integer $n \geq 0$ such that the canonical homomorphism in cohomology defined in (4.2.1) is non trivial for some G-module A (the nomenclature is inherited from Remark 4.2.10). The fact that ρ^* respects product structures allows to reinterpretate this definition as the greatest dimension of $\mathrm{Im}(\rho^*) \subset H^*(G, M)$ as a subalgebra of the usual group cohomology for some suitable choice of coefficients M (probably twisted). We have a clear lower bound for secat($H \hookrightarrow G$), as well as a straightforward analogue of Proposition 4.3.4 for the essential dimension.

Proposition 4.3.5. *Let $\iota\colon H \hookrightarrow G$ be a group monomorphism, we have that*

$$\mathrm{secat}(H \hookrightarrow G) \geq \mathrm{height}(\omega) \geq \rho^*_{[G:H]}.$$

Proof. The proof is completely analogous to that of Proposition 4.3.4: consider $\rho^*_{[G:H]} = n$. Then there exists a G-module A and a non-trivial class $\alpha \in H^n([G:H], A)$ with $\rho^*(\alpha) = \eta \neq 0$ in $H^*(G, M)$. By Proposition 4.2.4, we know that there exists a G-module homomorphim $\mu\colon I^n \to A$ such that $\alpha = \mu^*(\phi^n)$. By naturality, we have a commutative diagram of group cohomologies

$$\begin{array}{ccc} H^n([G:H], I^n) & \xrightarrow{\rho^*} & H^n(G, I^n) \\ {\scriptstyle\mu^*}\downarrow & & \downarrow{\scriptstyle\mu^*} \\ H^n([G:H], A) & \xrightarrow[\rho^*]{} & H^n(G, A). \end{array}$$

Given that ρ^* respects products, $\rho(\phi^n) = \omega^n$. As a consequence, we get that $\omega^n \neq 0$, and the rest of the claim follows from Proposition 4.1.10. □

Proposition 4.3.4 also allows for a particularly simple proof of Theorem 4.1.11.

Proof of Theorem 4.1.11. Put $n := \mathrm{cd}G \leq 3$. The "only if" part is an immediate consequence of the kernel-nilpotency lower bound for sectional category, see Theorem 3.2.8. For the converse statement, recall that the extension problem

$$\begin{array}{ccc} EG_{n-1} & \xrightarrow{\rho} & (E_{\langle H\rangle}G)_{n-1} \\ & \searrow \quad \nearrow & \\ & EG_n & \end{array}$$

has a solution provided that the cocycle $c^n(\rho)$ representing the extension is cohomologous to zero in $H^n\big(G, \pi_{n-1}(E_{\langle H\rangle}G)_{n-1}\big)$. Let us take a closer look at how the obstruction cocycle arises; further details can be found in [37, Chapter II.3]. Write $[\rho]$ for the G-homotopy class of

$$\rho\colon EG_{n-1} \to (E_{\langle H\rangle}G)_{n-1}.$$

Note that both EG_{n-1} and $(E_{\langle H\rangle}G)_{n-1}$ are $(n-2)$-connected spaces, and the pair (EG_n, EG_{n-1}) is $(n-1)$-connected, hence the (relative) Hurewicz homomorphism gives isomorphisms

$$\pi_n(EG_n, EG_{n-1}) \to H_n(EG_n, EG_{n-1}),$$
$$\pi_{n-1}(EG_{n-1}) \to H_{n-1}(EG_{n-1}),$$
$$\pi_{n-1}\big((E_{\langle H\rangle}G)_{n-1}\big) \to H_{n-1}\big((E_{\langle H\rangle}G)_{n-1}\big).$$

Consequently, we have a diagram

$$\begin{array}{ccccc} \pi_n(EG_n, EG_{n-1}) & \xrightarrow{\partial} & \pi_{n-1}(EG_{n-1}) & \xrightarrow{\rho_*} & \pi_{n-1}\big((E_{\langle H\rangle}G)_{n-1}\big) \\ \downarrow{\scriptstyle \varrho} & & \downarrow{\scriptstyle \varrho} & & \downarrow{\scriptstyle \varrho} \\ H_n(EG_n, EG_{n-1}) & & H_{n-1}(EG_{n-1}) & & H_{n-1}\big((E_{\langle H\rangle}G)_{n-1}\big), \end{array}$$

where ∂ is the boundary operator of the long exact sequence of homotopy groups of the pair (EG_n, EG_{n-1}) and can be identified as the corresponding epimorphism over the kernel of the n-differential in the cellular chain complex via the Hurewicz isomorphisms. The obstruction cocycle associated to ρ is defined as

$$c^n(\rho) = \rho_* \partial \varrho^{-1}.$$

By hypothesis $\omega^n = 0$ so $\rho^*(\mathbf{u}^n) = \omega^n = 0$ and we conclude that ρ^* is trivial in degree n. Now select an $\mathrm{Or}_{\langle H\rangle}$-module having the coefficient system $\pi_{n-1}((E_{\langle H\rangle})_{n-1})$ as its principal component, such as

$$\underline{\pi_{n-1}((E_{\langle H\rangle})_{n-1})}(G/K) = \pi_{n-1}((E_{\langle H\rangle})_{n-1})^K$$

By the universality of the Bredon class $\mathbf{u}$, we know that there exists an $\mathrm{Or}_{\langle H\rangle}$-module homomorphism

$$f^*\colon \underline{I}^n \to \underline{\pi_{n-1}((E_{\langle H\rangle})_{n-1})}$$

which, in turns, after evaluation on the principal components, induces a commutative diagram

$$\begin{array}{ccc} H^n_{\langle H\rangle}(G, \underline{I}^n) & \xrightarrow{\rho^*} & H^n(G, I^n) \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle f^*} \\ H^n_{\langle H\rangle}(G, \underline{\pi_{n-1}((E_{\langle H\rangle})_{n-1})}) & \xrightarrow[\rho^*]{} & H^n(G, \pi_{n-1}((E_{\langle H\rangle})_{n-1})). \end{array}$$

We observe that the obstruction class lives in the image of ρ^n and therefore it must be zero. The Eilenberg-Ganea theorem (see Theorem 2.2.11) states that $\mathrm{cd}(G) = n$ implies the existence of a n-dimensional EG thus, by dimensional reasons, the map ρ can be then extended to the whole space EG. Consequently, by Theorem 4.1.4

$$\mathrm{secat}(H \hookrightarrow G) \leq n - 1.$$

□

Remark 4.3.6. Notice that, in general, the obstruction class should be taken in the Bredon cohomology of the corresponding G-CW complex. However, as EG is a free G-CW complex, as discussed in Chapter 2 Bredon cohomology reduces to the cellular cohomology of the quotient space by the group action (and hence the usual group cohomology of G) see isomorphism 2.3.7.

4.3.1 Addendum: on the case of topological complexity

After the publication of [14], the article that constitutes the ground material for this chapter, M. Grant, K. Li, E. Meir and I. Patchkoria further investigated the relationship between some equivariant cohomological dimensions in [70], and explored the connections of some of them with the topological complexity of groups. In particular, they found explicit examples of groups for which the topological complexity is strictly less than the Adamson (and, consequently, also Bredon) cohomological dimension with respect to the diagonal subgroup. To finish this chapter we will describe the main of such examples.

Definition 4.3.7. Let Γ be a discrete group. We say that a Γ-*group* is another discrete group G equipped with an action of Γ by automorphisms (i.e. a homomorphism $\varphi\colon \Gamma \to \mathrm{Aut}(G)$).

For an arbitrary group G there is a natural G-group structure by considering the action of G on itself by conjugation. In this situation, we have an isomorphism of pairs of the form

$$(G \rtimes G, G) \xrightarrow{\cong} (G \times G, \Delta_G), \qquad (g,h) \mapsto (gh,h). \tag{4.3.1}$$

Example 4.3.8 (Example 5.9 of [70])**.** Let $G = \langle a, b \mid c := a^2 = b^2 \rangle$ be the fundamental group of the Klein bottle. The group G decomposes as the amalgamated group $G \cong \langle a \rangle *_{\langle c \rangle} \langle b \rangle$. In fact, if two powers a^m and b^n are conjugated in G for some $m, n \in \mathbb{Z}$, then m and n satisfies $m = n = 2k$ for some $k \in \mathbb{Z}$, thus $a^m = b^n = c^k$.

Consider the structure of G-group by conjugation on G. It is possible to proof the equalities

$$\mathrm{cd}[G \rtimes G : G] = \mathrm{cd}_{\langle G \rangle}(G \rtimes G) = \infty.$$

By the pair isomorphism 4.3.1 we observe that this is equivalent to show that

$$\mathrm{cd}[G \times G : \Delta_G] = \mathrm{cd}_{\langle \Delta_G \rangle}(G \times G) = \infty.$$

Take the $(G \times G)$-subgroup generated by (a, b), i.e.

$$\delta(G) := \{(a^m, b^n) \mid m \in \mathbb{Z}\}.$$

Note that $\langle (c,c) \rangle$ is in the intersection family $\langle \Delta_G \rangle \cap \delta(G)$. Moreover, due to the amalgam structure of G, and since if a^m is conjugate to b^n then $m = 2k$ and $(a^m, b^n) = (c^k, c^k)$, we see that any element of $\delta(G)$ contained in a subgroup in $\langle \Delta_G \rangle$ is contained in the subgroup $\langle (c,c) \rangle$, and therefore $\langle \Delta_G \rangle \cap \delta(G)$ coincides with the family generated by $\langle (c,c) \rangle$.

As $\langle (c,c) \rangle$ is normal in $\delta(G)$ with quotient $\mathbb{Z}_2$, Shapiro's Lemma for Bredon cohomology (see [63, Chapter 3 Proposition 3.31]) gives

$$\infty = \mathrm{cd}(\mathbb{Z}_2) \overset{4.2.15}{=} \mathrm{cd}_{\langle \Delta_G \rangle \cap \delta(G)}(\delta(G)) \leq \mathrm{cd}_{\langle \Delta_G \rangle}(G \times G).$$

The proof for Adamson cohomology is essentially analogous, applying Shapiro's Lemma for relative cohomology with respect of group families (see for example [70, Lemma 2.14]).

But remember that $\mathrm{TC}(G) = 4$, as shown by Cohen and Vandembroucq in [32], thereby proving that $\mathrm{TC}(G)$ cannot coincide neither with Bredon, nor with Adamson cohomological dimensions.

CHAPTER 5

Lower bounds of sectional category of subgroup inclusions

Introduction

In the previous chapter we laid the foundation of the study of sectional category of subgroup inclusions. In particular, we introduced the notion of relative Berstein-Schwarz class with respect to subgroup monomorphisms, in strong analogy to the usual Berstein-Schwarz class for Lusternik-Schnirelmann category, and which becomes the Farber-Costa class for the particular case of the diagonal subgroup inclusion. In this chapter we will make use of these tools to derive new lower bounds for sectional category of fibrations inducing subgroup inclusions.

We will commence by providing an alternative description of the relative Berstein-Schwarz class in terms of the Bockstein homomorphism associated to the augmentation short exact sequence, and then we will derive several of its implications. The most important for our purposes is the generalization of the *essential cohomology classes* defined by M. Farber and S. Mescher in [57] for the case of topological complexity to our more general setting of sectional categories of subgroup inclusions. We will study in detail the case of essential classes relative to normal subgroups, providing new bounds.

Later on, we turn our attention to the spectral sequence constructions developed by Farber and Mescher in [57], containing the information about essential cohomology classes. While such spectral sequence constructed therein can be used to derive lower bounds for the topological complexity of aspherical spaces, we show how to generalize the construction to sectional categories of fibrations inducing subgroup inclusions. By means of said spectral sequence, we derive a new lower bound for such sectional categories, depending on the cohomological dimension of isotropy subgroups of the left subgroup action action induced on the left cosets space. In the last part of the chapter we specialize such lower bound and study its applications to the case of sequential topological complexity of aspherical spaces (which subsumes the) and of parametrized topological complexity of group epimorphisms, as studied by M. Grant in [67].

The contents of this chapter constitute a large part of [49].

5.1 Essential classes relative to subgroups

As we were discussing in the previous chapter, the role of relative Berstein-Schwarz classes for sectional categories of subgroup inclusions is analogous to the role of Berstein-Schwarz classes for Lusternik-Schnirelmann category and the role of the so-called canonical classes for topological complexity. Throughout this section we shall discuss more extensively how canonical classes are related to relative Berstein-Schwarz classes, and we will generalize the notion of essential classes to arbitrary subgroup inclusions.

The following lemma provides an alternative characterization of relative Berstein-Schwarz classes and is an analogue of [35, Lemma 5].

Lemma 5.1.1. *Consider the short exact sequence of G-modules*

$$0 \to I \xrightarrow{i} \mathbb{Z}[G/H] \xrightarrow{\sigma} \mathbb{Z} \to 0 \tag{5.1.1}$$

and let $\delta : H^0(G,\mathbb{Z}) \to H^1(G,I)$ denote the Bockstein homomorphism associated with that sequence. Then

$$\omega = \delta(1),$$

where $1 \in H^0(G,\mathbb{Z})$ is a generator.

Proof. Let $\rho : \mathbb{Z}[G] \to \mathbb{Z}[G/H]$ denote the homomorphism induced by the orbit space projection. As we defined in Subsection 4.1.2 of the previous chapter we know that we can represent $\omega \in H^1(G,I)$ by a cocycle

$$f : \mathbb{Z}[G] \otimes K \to I \qquad f = \rho \circ (\varepsilon \otimes \mathrm{id}_K).$$

We consider the long exact sequence associated with

$$0 \to C^0(G,I) \to C^0(G,\mathbb{Z}[G/H]) \to C^0(G,\mathbb{Z}) \to 0.$$

In terms of our resolution, its connecting homomorphism, i.e. the Bockstein homomorphism, is obtained via diagram chasing in

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}[G],I) & \xrightarrow{i_*} & \mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}[G],\mathbb{Z}[G/H]) & \xrightarrow{\sigma_*} & \mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}[G],\mathbb{Z}) & \longrightarrow & 0 \\ & & \downarrow d_1^* & & \downarrow d_1^* & & \downarrow d_1^* & & \\ 0 & \longrightarrow & \mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}[G]\otimes K,I) & \xrightarrow{i_*} & \mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}[G]\otimes K,\mathbb{Z}[G/H]) & \xrightarrow{\sigma_*} & \mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}[G]\otimes K,\mathbb{Z}) & \longrightarrow & 0 \end{array}$$

The augmentation $\varepsilon \in \mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}[G],\mathbb{Z})$ is a cocycle. By definition of the maps involved, it holds that $\varepsilon = \sigma \circ \rho$, i.e. $\sigma_*(\rho) = \varepsilon$. Diagram chasing shows that

$$i_*(f) = d_1^*(\rho),$$

so by definition of the Bockstein homomorphism, we obtain that

$$\delta(1) = \delta([\varepsilon]) = [f] = \omega.$$

Here, one sees that $1 = [\varepsilon]$ generates $H^0(G, \mathbb{Z})$ as $\varepsilon(g) = 1$ for each $g \in G$. □

By elementary homological algebra, tensoring the short exact sequence (5.1.1) with a left $\mathbb{Z}[G]$-module M that is $\mathbb{Z}$-free yields a short exact sequence of $\mathbb{Z}[G]$-modules

$$0 \to I \otimes M \to \mathbb{Z}[G/H] \otimes M \to M \to 0 \tag{5.1.2}$$

with respect to the diagonal G-actions. In complete analogy with a statement for the canonical class observed in Section 3 of [57], we derive the following statement.

Corollary 5.1.2. *Let M be a $\mathbb{Z}$-free left $\mathbb{Z}[G]$-module and let $u \in H^i(G, M)$, where $i \in \mathbb{N}_0$. Consider the Bockstein homomorphism δ of the coefficient sequence*

$$0 \to I \otimes M \to \mathbb{Z}[G/H] \otimes M \to M \to 0.$$

Then we have the equality

$$\delta(u) = \omega \cup u \in H^{i+1}(G, I \otimes M).$$

Proof. It follows straight from Theorem 2.2.15 by the compatibility with the connecting homomorphism and the graded commutativity of the cup product that

$$\delta(u) = \delta(u \cup 1) = (-1)^i u \cup \delta(1) = (-1)^i u \cup \omega = \omega \cup u.$$

□

We let $\mathrm{Hom}_{\mathbb{Z}}(I^s, M)$ be equipped with the diagonal G-action and consider

$$\mathrm{ev}_s : I \otimes \mathrm{Hom}_{\mathbb{Z}}(I^{s+1}, M) \to \mathrm{Hom}_{\mathbb{Z}}(I^s, M), \qquad \mathrm{ev}(x \otimes f) = f(x \otimes \cdot),$$

which is seen to be a $\mathbb{Z}[G]$-homomorphism. The following statement and its proof are analogues and carried out along the lines of [57, Proposition 7.3].

Proposition 5.1.3. *Let A be a left $\mathbb{Z}[G]$-module. For any cohomology class $u \in H^r(G, \mathrm{Hom}_{\mathbb{Z}}(I^{s+1}, A))$ one has*

$$\delta(u) = -(\mathrm{ev}_s)_*(\omega \cup u),$$

where δ is the Bockstein homomorphism associated with the short exact coefficient sequence

$$0 \to \mathrm{Hom}_{\mathbb{Z}}(I^s, A) \xrightarrow{\sigma^*} \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[G/H] \otimes I^s, A) \xrightarrow{i^*} \mathrm{Hom}_{\mathbb{Z}}(I^{s+1}, A) \to 0 \tag{5.1.3}$$

obtained by applying $\mathrm{Hom}_{\mathbb{Z}}(\cdot, A)$ to (5.1.2) in the case of $M = I^s$.

Proof. We first observe that (5.1.3) is indeed short exact, since I^s, $\mathbb{Z}[G/H] \otimes I^s$ and I^{s+1} are all $\mathbb{Z}$-free. Let

$$\beta : H^r(G, \mathrm{Hom}_{\mathbb{Z}}(I^{s+1}, A)) \to H^{r+1}(G, I \otimes \mathrm{Hom}_{\mathbb{Z}}(I^{s+1}, A))$$

denote the Bockstein homomorphism of the short exact coefficient sequence

$$0 \to I \otimes \mathrm{Hom}_{\mathbb{Z}}(I^{s+1}, A) \hookrightarrow \mathbb{Z}[G/H] \otimes \mathrm{Hom}_{\mathbb{Z}}(I^{s+1}, A) \overset{\sigma\otimes\mathrm{id}}{\to} \mathrm{Hom}_{\mathbb{Z}}(I^{s+1}, S) \to 0$$

obtained by letting $M = \mathrm{Hom}_{\mathbb{Z}}(I^{s+1}, A)$ in (5.1.2). By Corollary 5.1.2, it holds that

$$\beta(u) = \omega \cup u.$$

Thus, the claim immediately follows if we can show that $\delta = -(\mathrm{ev}_s)_* \circ \beta$. We first consider the homomorphism

$$F : \mathbb{Z}[G/H] \otimes \mathrm{Hom}_{\mathbb{Z}}(I^{s+1}, A) \to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[G/H] \otimes I^s, A)$$

given by $\mathbb{Z}$-linearly extending

$$(F(xH \otimes f))(zH \otimes y) = f((zH - xH) \otimes y) \qquad \forall x, z \in G,\ y \in I^s,\ f \in \mathrm{Hom}_{\mathbb{Z}}(I^{s+1}, A).$$

We compute that

$$\begin{aligned}(F(g \cdot (xH \cdot f)))(zH \otimes y) &= F(gxH \otimes (g \cdot f))(zH \otimes y)\\ &= (g \cdot f)((zH - gxH) \otimes y) = gf((g^{-1}zH - xH) \otimes g^{-1}y)\\ &= g(F(x \otimes f))(g^{-1}zH \otimes g^{-1}y) = (g \cdot F(x \otimes f))(zH \otimes y)\end{aligned}$$

for all $g, x, z \in G$, $y \in I^s$ and $f \in \mathrm{Hom}_{\mathbb{Z}}(I^{s+1}, M)$. Hence, F is a $\mathbb{Z}[G]$-homomorphism. Consider the following diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I \otimes \mathrm{Hom}_{\mathbb{Z}}(I^{s+1}, M) & \xrightarrow{i\otimes\mathrm{id}} & \mathbb{Z}[G/H] \otimes \mathrm{Hom}_{\mathbb{Z}}(I^{s+1}, M) & \xrightarrow{\sigma\otimes\mathrm{id}} & \mathrm{Hom}_{\mathbb{Z}}(I^{s+1}, M) & \longrightarrow & 0\\ & & \downarrow{\scriptstyle \mathrm{ev}_s} & & \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle \mathrm{id}} & & \\ 0 & \longrightarrow & \mathrm{Hom}_{\mathbb{Z}}(I^s, M) & \xrightarrow{-\sigma^*} & \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[G/H] \otimes I^s, M) & \xrightarrow{i^*} & \mathrm{Hom}_{\mathbb{Z}}(I^{s+1}, M) & \longrightarrow & 0 \end{array}$$

To show that the left-hand square of this diagram commutes, we compute for all $x, z \in G$, $y \in I^s$ and $f \in \mathrm{Hom}_{\mathbb{Z}}(I^{s+1}, M)$ that

$$((-\sigma^* \circ \mathrm{ev}_s)((xH - H) \otimes f))(zH \otimes y) = -\sigma(zH) \cdot f((xH - H) \otimes y) = -f((xH - H) \otimes y).$$

and moreover

$$\begin{aligned}&((F \circ (i \otimes \mathrm{id}))((xH - H) \otimes f))(zH \otimes y) = (F(xH \otimes f))(zH \otimes y) - (F(H \otimes f))(zH \otimes y)\\ &= f((zH - xH) \otimes y) - f((zH - H) \otimes y) = -f((xH - H) \otimes y).\end{aligned}$$

Comparing the results shows the commutativity of the left-hand square. Concerning the right-hand square, we derive that

$$\begin{aligned}&((i^* \circ F)(xH \otimes f))((zH - H) \otimes y) = (F(xH \otimes f))(zH \otimes y) - (F(xH \otimes f))(H \otimes y)\\&= f((zH - xH) \otimes y) - f((H - xH) \otimes y)\\&= f((zH - H) \otimes y) = \sigma(xH) \cdot f((zH - H) \otimes y) = ((\sigma \otimes \mathrm{id})(x \otimes f))((zH - H) \otimes y).\end{aligned}$$

Thus, the above diagram commutes. Considering the long exact cohomology sequences associated with the coefficient groups of the above diagram, the naturality of Bockstein homomorphisms shows that

$$-(\mathrm{ev}_s)_* \circ \beta = \delta \circ \mathrm{id}_* = \delta,$$

which we wanted to show. The claim immediately follows. Here, the additional sign stems from the fact that we have considered $-\sigma^*$ instead of σ^* in the bottom row of the diagram. □

We introduce some additional terminology which generalizes the notion of essential classes introduced in [57].

Definition 5.1.4. Let $n \in \mathbb{N}$ and let $\alpha \in H^n(G, A)$ with $\alpha \neq 0$. We say that α is *essential relative to H* if there exists a homomorphism of $\mathbb{Z}[G]$-modules $\varphi\colon I^n \to A$, such that

$$\varphi_*(\omega^n) = \alpha.$$

Remark 5.1.5. Cohomology classes which are essential relative to subgroups are used to derive lower bounds on the sectional category of the corresponding subgroup inclusion: Assume that for some $n \in \mathbb{N}$, there exists a class $u \in H^n(G, A)$ with $u \neq 0$ that is essential relative to H. By definition of essential classes, this requires that $\omega^n \neq 0 \in H^n(G, I^n)$, which in turn yields that $\mathrm{secat}(H \hookrightarrow G) \geq n$ by Proposition 4.1.10.

5.1.1 Essential classes relative to normal subgroups

To close this section, let us consider the case of the inclusion of a normal subgroup. In this setting we can characterize essential classes relative to that subgroup as pullbacks of non-trivial classes in the cohomology of the quotient group through the homomorphism in cohomology induced by the quotient map. This is, in certain measure, a generalization of the ideas present in the case of the inclusion of the diagonal subgroup in abelian groups, as considered in [57, Section 6].

Proposition 5.1.6. *Let $N \triangleleft G$ be a normal subgroup, put $Q := G/N$ for the quotient group and let $\pi : G \to Q$ denote the projection.*

a) Let $\omega \in H^1(G, I)$ be the Berstein-Schwarz class of G relative to N and let $\beta \in H^1(Q, I_Q)$ be the Berstein-Schwarz class of Q, where $I_Q \subset \mathbb{Z}[Q]$ denotes the augmentation ideal of Q. Then

$$\pi^*\beta = \omega.$$

b) Let A be a left $\mathbb{Z}[Q]$-module and let $n \in \mathbb{N}$. A cohomology class $u \in H^n(G, \pi^* A)$ with $u \neq 0$ is essential relative to N if and only if there exists $v \in H^n(Q, A)$ with $\pi^* v = u$.

Proof. Throughout the proof, we will use the projective resolution $(\mathbb{Z}[G] \otimes K^*, p_*)$ of $\mathbb{Z}$ over $\mathbb{Z}[G]$ and the projective resolution $(\mathbb{Z}[Q] \otimes I_Q^*, p_*)$ of $\mathbb{Z}$ over $\mathbb{Z}[Q]$, both defined as in 4.1.2 of Chapter 4, to compute the cohomology groups of G and Q, respectively. By abuse of notation, we further denote the ring homomorphism induced by π by $\pi : \mathbb{Z}[G] \to \mathbb{Z}[Q]$ as well.

a) By definition of the augmentation ideals and of π, it holds that $\pi(K) \subset I_Q$ and in the notation of Definition 4.1.8, we write $\mu := \pi|_K : K \to I_Q$. By factorwise applying π, one obtains a chain map
$$\pi_\# : \mathbb{Z}[G] \otimes K^* \to \mathbb{Z}[Q] \otimes I_Q^*,$$
This map induces a cochain map
$$(\pi_\#)^* : \mathrm{Hom}_{\mathbb{Z}[Q]}(\mathbb{Z}[Q] \otimes I_Q^*, A) \to \mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}[G] \otimes K^*, \pi^* A),$$
which in turn induces the pullback map
$$\pi^* : H^*(Q, A) \to H^*(G, \pi^* A).$$
Denote the augmentations of G and Q by ε_G and ε_Q, respectively. As shown in [44], the Berstein-Schwarz class $\beta \in H^1(Q, I_Q)$ is then represented by the cocycle
$$f_\beta : \mathbb{Z}[Q] \otimes I_Q \to I_Q, \qquad f_\beta = \varepsilon_Q \otimes \mathrm{id}_{I_Q}.$$
One easily checks that $\varepsilon_Q \circ \pi = \varepsilon_G$, so that
$$(\pi_\#)^*(f_\beta)(x)(y) = (\varepsilon_Q \circ \pi)(x) \cdot \pi_\#(y) = \varepsilon_G(x) \cdot \mu(y) = (\mu \circ (\varepsilon_G \otimes \mathrm{id}_K))(x \otimes y).$$
for all $x \in \mathbb{Z}[G]$ and $y \in K$. By definition of ω, it is represented by this latter cocycle. Passing to cohomology then shows that $\pi^* \beta = \omega$.

b) Assume that u is essential relative to N, such that there exists a $\mathbb{Z}[G]$-homomorphism $\varphi : I^n \to \pi^* A$ with
$$u = \varphi_*(\omega^n) = \varphi_*((\pi^* \beta)^n) = (\varphi_* \circ \pi^*)(\beta^n).$$
One easily checks that $\pi^* I_Q = I$ as $\mathbb{Z}[G]$-modules. Moreover, since we can view $\varphi : I_Q^n \to A$ as a $\mathbb{Z}[Q]$-homomorphism and since the diagram
$$\begin{array}{ccc} H^n(Q, I_Q^n) & \xrightarrow{\varphi_*} & H^n(Q, A) \\ {\scriptstyle \pi^*}\downarrow & & {\scriptstyle \pi^*}\downarrow \\ H^n(G, I^n) & \xrightarrow{\varphi_*} & H^n(G, \pi^* A) \end{array} \tag{5.1.4}$$
obviously commutes, we obtain that $u = \pi^* v$, where $v := \varphi_*(\beta^n) \in H^n(Q, A)$.

Conversely, assume that there exists a class $v \in H^n(Q, A)$, for which $u = \pi^* v$. By the universality of Berstein-Schwarz classes, see [44], there exists a $\mathbb{Z}[Q]$-homomorphism

$$\psi : I_Q^n \to A,$$

such that $v = \psi_*(\beta^n)$. In fact, by definition of pullback modules, we can view ψ as a $\mathbb{Z}[G]$-homomorphism

$$\psi : I^n = \pi^* I_Q^n \to \pi^* A.$$

Replacing φ_* by ψ_*, the diagram corresponding to (5.1.4) commutes as well, so we obtain that

$$u = \pi^*(\psi_*(\beta^n)) = \psi_*(\pi^*(\beta^n)) = \psi_*(\omega^n),$$

hence u is essential.

□

We want to derive an estimate for $\mathrm{secat}(N \hookrightarrow G)$ from the previous proposition, for which we need to introduce another notion. Recently, Mark Grant defined the cohomological dimension $\mathrm{cd}(\phi)$ of a group homomorphism $\phi\colon G \to H$ to be the maximum k for which there exists some H-module A so that the induced homomorphism at cohomology level

$$\phi^* \colon H^k(H, A) \to H^k(G, \phi^* A)$$

is non-trivial. The first published account of the study of this new dimension is even more recent, see [43].

For the proof of the following theorem we recall that the LS-category of a map $f\colon X \to Y$ is the smallest integer m for which there are $m+1$ open sets $U_0, \ldots, U_m$ which cover X and such that each of the restrictions $f|_{U_j}$ is nullhomotopic. For a group homomorphism $\phi\colon G \to H$, we write $\mathrm{cat}(\phi)$ for the category of the associated map of aspherical spaces $K(G,1) \to K(H,1)$.

Theorem 5.1.7. *Let $N \lhd G$ be a normal subgroup, put $Q := G/N$ for the quotient group and let $\pi : G \to Q$ denote the projection. Then*

$$\mathrm{cd}(\pi : G \to Q) \le \mathrm{secat}(N \hookrightarrow G) \le \mathrm{cd}(Q).$$

In particular, if $\pi^* : H^{\mathrm{cd}(Q)}(Q, A) \to H^{\mathrm{cd}(Q)}(G, \pi^* A)$ is non-zero for some $\mathbb{Z}[Q]$-module A, then $\mathrm{secat}(N \hookrightarrow G) = \mathrm{cd}(Q)$.

Proof. Put $k := \mathrm{cd}(\pi : G \to Q)$ and let A be a left $\mathbb{Z}[Q]$-module and $u \in H^k(Q, A)$ with $\pi^* u \neq 0$. Then, by Proposition 5.1.6, $\pi^* u \in H^k(G, \pi^* A)$ is essential relative to N. This in particular yields that $\omega^k \neq 0$, where $\omega \in H^1(G, I)$ denotes the Berstein-Schwarz class of G relative to N and it follows from Proposition 4.1.10 that $\mathrm{secat}(N \hookrightarrow G) \ge k$.

To show the other inequality we can argue through properties of Lusternik-Schnirelmann category as found in [34]. First, let's note that the exact sequence

$$\{1\} \to N \xrightarrow{i} G \xrightarrow{\pi} Q \to \{1\}$$

gives a fibre sequence

$$K(N,1) \xrightarrow{i} K(G,1) \xrightarrow{\pi} K(Q,1)$$

where we have used the same notation for the space maps. A fibre sequence arises as a homotopy pullback

$$\begin{array}{ccc} K(N,1) & \longrightarrow & P_*(K(Q,1)) \\ {\scriptstyle i}\downarrow & & \downarrow \\ K(G,1) & \xrightarrow[\pi]{} & K(Q,1) \end{array}$$

where $P_*(K(Q,1)) \to K(Q,1)$ is the based path space fibration. By Proposition 3.2.13, because $P_*(K(Q,1))$ is contractible, we have $\mathrm{secat}(i\colon N \hookrightarrow G) = \mathrm{cat}(\pi)$. But a standard property of the category of a map is that it is bounded above by both the category of its domain and the category of its codomain (see Proposition 3.2.12). Hence,

$$\mathrm{secat}(i\colon N \hookrightarrow G) = \mathrm{cat}(\pi) \leq \mathrm{cat}(K(Q,1)) = \mathrm{cd}(Q). \tag{5.1.5}$$

Assume now the hypothesis that $\pi^* : H^{\mathrm{cd}(Q)}(Q,A) \to H^{\mathrm{cd}(Q)}(G,\pi^*A)$ is non-zero for some $\mathbb{Z}[Q]$-module A. Then, by definition, $\mathrm{cd}(\pi) \geq \mathrm{cd}(Q)$. Combining this with the lower bound by $\mathrm{cd}(\pi)$ and with inequality 5.1.5 shows that

$$\mathrm{secat}(N \hookrightarrow G) = \mathrm{cd}(\pi) = \mathrm{cd}(Q).$$

□

Remark 5.1.8. (1) Notice that if we assume $\mathrm{cd}(Q) \neq 2$ (and thus we remove the pathological case prescribed by the Eilenberg-Ganea conjecture) one could also argue in the proof of the upper bound of Theorem 5.1.7 as follows: observe first that, by Corollary 4.1.6, it holds that

$$\mathrm{secat}(N \hookrightarrow G) \leq \dim E_{\langle N \rangle}G$$

where $E_{\langle N \rangle}G$ is the classifying space of the family of groups generated by N. Since N is a normal subgroup of G, we derive from Proposition 4.2.15 that $\dim(E_{\langle N \rangle}G) = \dim(K(Q,1))$, where $K(Q,1)$ is a classifying space of Q. Using the Eilenberg-Ganea theorem and Theorem 4.1.3.a) we derive that

$$\mathrm{secat}(N \hookrightarrow G) \leq \dim(K(Q,1)) = \mathrm{cd}(Q).$$

Combining this with the first inequality of Theorem 5.1.7 shows the claim.

(2) Since $\mathrm{cat}(K(Q,1)) = \mathrm{cd}(Q)$ for any Q by the Eilenberg-Ganea theorem, there arose the natural conjecture that $\mathrm{cat}(\phi) = \mathrm{cd}(\phi)$ for any homomorphism $\phi\colon G \to H$. This was disproved by T. Goodwillie using an infinitely generated group G. In [43, Theorem 5.4] a finitely generated example was derived and we shall use this in Example 5.1.9.

Example 5.1.9. While the hypothesis that $\mathrm{cd}(\pi : G \to Q) = \mathrm{cd}(Q)$ on cohomology in Theorem 5.1.7 is sufficient to derive $\mathrm{secat}(N \hookrightarrow G) = \mathrm{cd}(Q)$, it is not necessary. We can see this using [43, Theorem 5.4] as follows. We recall that in [19], D. Bolotov defined a closed manifold M^4 with fundamental group $\pi_1(M) = \mathbb{Z} * \mathbb{Z}^3$ for which, as shown in [43], the pullback map

$$\mu^* \colon H^3(K(\mathbb{Z} * \mathbb{Z}^3, 1), A) \to H^3(M, A)$$

is the zero homomorphism for all $\mathbb{Z} * \mathbb{Z}^3$-modules A, where $\mu \colon M \to K(\pi, 1)$ is a classifying map of the universal cover. The hyperbolization procedure of [27] gives a closed aspherical manifold W^4 and a degree one map $\alpha \colon W \to M$ which induces a surjection of fundamental groups. The surjective group homomorphism $G = \pi_1(W) \to \mathbb{Z} * \mathbb{Z}^3$ is then induced by the composition

$$W \xrightarrow{\alpha} M \xrightarrow{\mu} K(\mathbb{Z} * \mathbb{Z}^3, 1) = S^1 \vee T^3$$

and we have a map $\theta \colon W \to T^3$ given by the composition

$$W \xrightarrow{\alpha} M \xrightarrow{\mu} S^1 \vee T^3 \xrightarrow{c} T^3$$

where $c \colon S^1 \vee T^3 \to T^3$ collapses S^1. Abusing notation, the induced homomorphism of fundamental groups $\theta \colon G \to \mathbb{Z}^3$ is also a surjection. Letting $N = \ker \theta$, we have an exact sequence

$$\{1\} \to N \xrightarrow{i} G \xrightarrow{\theta} \mathbb{Z}^3 \to \{1\}.$$

In [43, Theorem 5.4] it is shown that

$$\mathrm{cat}(\theta) = \mathrm{cd}(\mathbb{Z}^3) = 3.$$

As in Remark 5.1.8, this means that $\mathrm{secat}(N \hookrightarrow G) = 3$ as well. However, the fact that $\mu^* = 0$ and that $\theta = c \circ \mu \circ \alpha$ shows that the map $\theta^* \colon H^3(\mathbb{Z}^3, A) \to H^3(G, A)$ is trivial for any coefficient module, hence $\mathrm{cd}(\theta) < 3$.

5.2 Forming the spectral sequence

In this section we will proceed to generalize the construction of a spectral sequence to sectional categories of subgroup inclusions that has been carried out for the topological complexity of aspherical spaces by M. Farber and S. Mescher in [57, Section 7]. In our setting, the spectral sequence fom [57] corresponds to the choice of

$$G = \pi \times \pi \qquad \text{and} \qquad H = \Delta_\pi,$$

for a given group π. The steps of the construction are carried out in analogy with the corresponding parts of [57] and instead of giving individual references for each statement, we view this as a general reference to [57, Section 7]. The interested reader will have no difficulties in finding the analogous statements therein.

5.2.1 The construction of the spectral sequence

Let G be a group, let $H \leq G$ be a subgroup and let $\omega \in H^1(G, I)$ be the Berstein-Schwarz class of G relative to H. Let A be a left $\mathbb{Z}[G]$-module. Define the groups

$$E_0^{r,s} = \operatorname{Ext}^r_{\mathbb{Z}[G]}(\mathbb{Z}[G/H] \otimes I^s, A), \qquad D_0^{r,s} = \operatorname{Ext}^r_{\mathbb{Z}[G]}(I^s, A) \qquad \forall r, s \in \mathbb{N}_0.$$

Let $i : I \hookrightarrow \mathbb{Z}[G/H]$ denote the inclusion. For each $s \in \mathbb{N}$ the short exact sequence from (5.1.2) with $M = I^s$ yields a short exact sequence of $\mathbb{Z}[G]$-modules

$$0 \to I^{s+1} \xrightarrow{f_s} \mathbb{Z}[G/H] \otimes I^s \xrightarrow{g_s} I^s \to 0, \tag{5.2.1}$$

where we have

$$f_s : I^{s+1} \to \mathbb{Z}[G/H] \otimes I^s \qquad f_s := i \otimes \mathrm{id}_{I^s}$$

and

$$g_s : \mathbb{Z}[G/H] \otimes I^s \to I^s \qquad g_s(x \otimes y) = \sigma(x) \cdot y.$$

For each s, the sequence in (5.2.1) induces a long exact Ext-sequence with coefficients in A, which is in the above notation given as

$$\cdots \to E_0^{r,s} \xrightarrow{k_0} D_0^{r,s+1} \xrightarrow{i_0} D_0^{r+1,s} \xrightarrow{j_0} E_0^{r+1,s} \to \cdots \tag{5.2.2}$$

where

- $i_0 : D_0^{r,s+1} \to D_0^{r+1,s}$ denotes the connecting homomorphism,
- $j_0 : D_0^{r,s} \to E_0^{r,s}$ is induced by $(g_s)^* : \operatorname{Hom}_{\mathbb{Z}[G]}(I^s, A) \to \operatorname{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}[G/H] \otimes I^s, A)$,
- $k_0 : E_0^{r,s} \to D_0^{r,s+1}$ is induced by $(f_s)^* : \operatorname{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}[G/H] \otimes I^s, A) \to \operatorname{Hom}_{\mathbb{Z}[G]}(I^{s+1}, A)$.

We put

$$E_0 := \bigoplus_{r,s \in \mathbb{N}_0} E_0^{r,s} = \bigoplus_{r,s \in \mathbb{N}_0} \operatorname{Ext}^r_{\mathbb{Z}[G]}(\mathbb{Z}[G/H] \otimes I^s, A)$$

and

$$D_0 := \bigoplus_{r,s \in \mathbb{N}_0} D_0^{r,s} = \bigoplus_{r,s \in \mathbb{N}_0} \operatorname{Ext}^r_{\mathbb{Z}[G]}(I^s, A)$$

and consider the summandwise defined maps

$$i_0 : D_0 \to D_0, \qquad j_0 : D_0 \to E_0, \qquad k_0 : E_0 \to D_0.$$

Together with these maps the groups D_0 and E_0 form an exact couple

$$\begin{array}{ccc} D_0 & \xrightarrow{\quad i_0 \quad} & D_0. \\ {\scriptstyle k_0} \nwarrow & & \swarrow {\scriptstyle j_0} \\ & E_0 & \end{array}$$

For each $p \in \mathbb{N}$ we denote its p-th derived exact couple as

$$\begin{array}{ccc} D_p & \xrightarrow{\quad i_p \quad} & D_p \\ & {\scriptstyle k_p}\nwarrow \qquad \swarrow{\scriptstyle j_p} & \\ & E_p & \end{array}$$

where, for each $p \in \mathbb{N}$ the module $D_p^{r,s}$ is defined as the image of p compositions of the map i_0, i.e.

$$D_p^{r,s} = \operatorname{Im}\left(i_{p-1}\colon D_{p-1}^{r-1,s+1} \to D_{p-1}^{r,s}\right) = \operatorname{Im}\left(\underbrace{i_0 \circ \cdots \circ i_0}_{p}\colon D_0^{r-p,s+p} \to D_0^{r,s}\right)$$

and, naturally, the module $E_p^{*,*}$ is defined by taking cohomology with respect to the differential defined by the exact couple, that is

$$E_p^{*,*} = H^*(E_{p-1}^{*,*}, d_{p-1}).$$

The zeroth page of the first-quadrant cohomological spectral sequence obtained thereby is formed by the groups $E_0^{r,s}$ and the differential

$$d_0 : E_0^{r,s} \to E_0^{r,s+1}, \qquad d_0 := j_0 \circ k_0 \qquad \forall r, s \in \mathbb{N}_0.$$

Note that we can view $D_p^{r,s} \subset D_0^{r,s}$ as subsets for each $p \in \mathbb{N}$ and we will occasionally do so without further mention.

Let $n, p \in \mathbb{N}$ with $p \leq n$. Taking a class $\alpha \in D_p^{n,0}$ we know by definition that $\alpha = i_0^p(\gamma)$ for some $\gamma \in D_0^{n-p,p}$. By Proposition 2.2.6, we can identify

$$D_0^{n-p,p} = \operatorname{Ext}^{n-p}_{\mathbb{Z}[G]}(I^p, A) \cong H^{n-p}(G, \operatorname{Hom}_{\mathbb{Z}}(I^p, A)).$$

Following an iterated use of the identification provided by Proposition 5.1.3, we obtain the following characterization of $D_p^{n,0}$, which is a generalization of [57, Corollary 7.4].

Proposition 5.2.1. *Let $n, p \in \mathbb{N}$ with $p \leq n$ and let $\alpha \in D_0^{n,0}$. Then $\alpha \in D_p^{n,0}$ if and only if there exists $\gamma \in H^{n-p}(G, \operatorname{Hom}_{\mathbb{Z}}(I^p, A))$ with*

$$\alpha = \psi_*(\omega^p \cup \gamma),$$

where $\psi\colon I^p \otimes \operatorname{Hom}_{\mathbb{Z}}(I^p, A) \to A$ is the $\mathbb{Z}[G]$-homomorphism given by

$$\psi(x_1 \otimes \cdots \otimes x_p \otimes f) = f(x_p \otimes x_{p-1} \otimes \cdots \otimes x_1).$$

This proposition has an immediate consequence for sectional categories.

Theorem 5.2.2. *Let $n, p \in \mathbb{N}$ with $p \leq n$. If $D_p^{n,0} \neq \{0\}$, then $\omega^p \neq 0$ and thus*

$$\operatorname{secat}(H \hookrightarrow G) \geq p.$$

Proof. By Proposition 5.2.1, every class in $D_p^{n,0}$ is obtained as a pushforward of a cup product of ω^p with another class. So if there is a non-trivial class in $D_p^{n,0}$, then it necessarily holds that $\omega^p \neq 0$ and the claim follows from Proposition 4.1.10. $\square$

5.2.2 Essential classes and the spectral sequence

To extract further consequences for $\mathrm{secat}(H \hookrightarrow G)$ from the spectral sequence, we need to find a more manageable description of our groups $E_0^{r,s}$. This is immediately achieved through the isomorphism defined in 4.2.7, as the following corollary shows:

Corollary 5.2.3. *Let $r \in \mathbb{N}$ and $s \in \mathbb{N}_0$. Then*

$$E_0^{r,s} \cong \mathrm{Ext}^r_{\mathbb{Z}[H]}(\widetilde{I}^s, \widetilde{A}).$$

Proof. This is the special case of Lemma 4.2.7 obtained by letting $M = I^s$ and $N = A$. $\square$

The following theorem summarizes the most important properties of the spectral sequence. In particular, it provides an equivalent condition for a cohomology class to be essential in terms of its pertenence to a group $D_n^{n,0}$.

Theorem 5.2.4. *Let $n \in \mathbb{N}$ and let $u \in H^n(G, A)$ with $u \neq 0$.*

a) The class u is essential relative to H if and only if $u \in D_n^{n,0}$.

b) $D_1^{n,0} = \ker[\iota^* : H^n(G, A) \to H^n(H, \widetilde{A})]$, where ι^* is induced by the inclusion $\iota : H \hookrightarrow G$.

c) Let $s \in \{0, 1, \ldots, n-1\}$. Then $u \in D_{s+1}^{n,0}$ if and only if

$$u \in D_s^{n,0} \qquad \text{and} \qquad u \in \ker\left[j_s : D_s^{n,0} \to E_s^{n-s,s}\right].$$

Proof. a) By Proposition 5.2.1, $u \in D_n^{n,0}$ if and only if there is a class $\mu \in H^0(G, \mathrm{Hom}_{\mathbb{Z}}(I^n, A))$, such that $u = \psi_*(\omega^n \cup \mu)$, where ψ is described in the statement of said proposition. But

$$H^0(G, \mathrm{Hom}_{\mathbb{Z}}(I^n, A)) = (\mathrm{Hom}_{\mathbb{Z}}(I^n, A))^G = \mathrm{Hom}_{\mathbb{Z}[G]}(I^n, A)$$

and one checks without difficulties that, seeing μ as a $\mathbb{Z}[G]$-homomorphism, it holds that

$$u = \psi_*(\omega^n \cup \mu) = \mu_*(\omega^n).$$

The claim immediately follows.

b) By definition and exactness of the exact couple,

$$\begin{aligned} D_1^{n,0} &= \mathrm{Im}\left[i_0 : D_0^{n-1,1} \to D_0^{n,0}\right] = \ker\left[j_0 : D_0^{n,0} \to E_0^{n,0}\right] \\ &= \ker\left[j_0 : \mathrm{Ext}^n_{\mathbb{Z}[G]}(\mathbb{Z}, A) \to \mathrm{Ext}^n_{\mathbb{Z}[G]}(\mathbb{Z}[G/H], A)\right] \\ &= \ker\left[j_0 : \mathrm{Ext}^n_{\mathbb{Z}[G]}(\mathbb{Z}, A) \to \mathrm{Ext}^n_{\mathbb{Z}[H]}(\mathbb{Z}, \widetilde{A})\right] \\ &= \ker\left[\iota^* : H^n(G, A) \to H^n(H, \widetilde{A})\right], \end{aligned}$$

where we used Corollary 5.2.3.

c) This is an immediate consequence of the inner workings of exact couples (see Subsection 2.4.1).

$\square$

5.2.3 Computing the zero-th page

In [57], the authors proceeded from the construction of the spectral sequence by introducing certain decompositions of terms of the form $E_0^{r,s}$ for $r > 0$ and $s > 0$ into products of cohomology groups of centralizers of elements of the groups involved. We will show next that this can be generalized as well and derive decompositions of parts of our spectral sequence as products of cohomology groups of certain isotropy groups of H-actions that will be introduced momentarily.

We consider the left H-action on the left cosets G/H given by

$$H \times G/H \to G/H, \qquad h \cdot gH = (hg)H. \tag{5.2.3}$$

For each $s \in \mathbb{N}$ we further consider the diagonal H-action

$$H \times (G/H)^s \to (G/H)^s, \quad h \cdot (g_1 H, \ldots, g_s H) = (hg_1 H, hg_2 H, \ldots, hg_s H).$$

We denote the set of orbits of this action for each $s \in \mathbb{N}$ by

$$\mathcal{C}_s(G/H) := \{H \cdot (g_1 H, g_2 H, \ldots, g_s H) \mid g_1 H, \ldots, g_s H \in G/H\}.$$

We put $(G/H)^* := (G/H) \setminus \{H\}$ and

$$\mathcal{C}'_s(G/H) := \{H \cdot (g_1 H, g_2 H, \ldots, g_s H) \mid g_1 H, \ldots, g_s H \in (G/H)^*\} \subset \mathcal{C}_s(G/H).$$

The above action equips $\mathbb{Z}[G/H]^{\otimes s}$ with the structure of a left $\mathbb{Z}[H]$-module and we consider $I^s \subset \mathbb{Z}[G/H]^{\otimes s}$ as a $\mathbb{Z}[H]$-submodule. This submodule structure obviously coincides with the one obtained by $\widetilde{I}^s = (\mathrm{Res}_H^G(I))^s$ that we previously considered. One checks that as free abelian groups

$$I^s = \bigoplus_{g_1 H, \ldots, g_s H \in (G/H)^*} \mathbb{Z} \cdot (g_1 H - H) \otimes (g_2 H - H) \otimes \cdots \otimes (g_s H - H) \qquad \forall s \in \mathbb{N}$$

and note that for all $s \in \mathbb{N}$, $g_1 H, \ldots, g_s H \in G/H$ and $h \in H$ it holds that

$$h \cdot (g_1 H - H) \otimes (g_2 H - H) \otimes \cdots \otimes (g_s H - H) = (hg_1 H - H) \otimes (hg_2 H - H) \otimes \cdots \otimes (hg_s H - H).$$

From this, one observes that for each $C \in \mathcal{C}'_s(G/H)$, we obtain a $\mathbb{Z}[H]$-submodule

$$J_C \subset I^s, \qquad J_C := \bigoplus_{(g_1 H, \ldots, g_s H) \in C} \mathbb{Z} \cdot (g_1 H - H) \otimes (g_2 H - H) \otimes \cdots \otimes (g_s H - H),$$

and that

$$\widetilde{I}^s = \bigoplus_{C \in \mathcal{C}'_s(G/H)} J_C \tag{5.2.4}$$

is a decomposition of $\mathbb{Z}[H]$-modules. Moreover, for each $C \in \mathcal{C}'_s(G/H)$, we let $\mathbb{Z}[C]$ denote the free abelian group generated by the elements of C. One checks without difficulties that for each C the map $\varphi_C : \mathbb{Z}[C] \to J_C$ that is obtained by $\mathbb{Z}$-linearly extending

$$\varphi(g_1 H, g_2 H \ldots, g_s H) = (g_1 H - H) \otimes (g_2 H - H) \otimes \cdots \otimes (g_s H - H), \tag{5.2.5}$$

is an isomorphism of $\mathbb{Z}[H]$-modules.

Theorem 5.2.5. *Let $s \in \mathbb{N}$. For each $C \in \mathcal{C}'_s(G/H)$ fix a representative $x_C \in C$ and let $N_C := H_{x_C}$ be the isotropy group of x_C. Then*

$$E_0^{r,s} \cong \prod_{C \in \mathcal{C}'_s(G/H)} H^r(N_C, \mathrm{Res}^G_{N_C}(A)) \quad \forall r \in \mathbb{N}.$$

Proof. Fix $r \in \mathbb{N}$. By Corollary 5.2.3, it holds that $E_0^{r,s} \cong \mathrm{Ext}^r_{\mathbb{Z}[H]}(\widetilde{I}^s, \widetilde{A})$. From this, using (5.2.4), the isomorphism $\phi_C\colon \mathbb{Z}[C] \to J_C$ given by (5.2.5) and the addivity of Ext-functors (recall (f) of Proposition 2.1.11) we derive that

$$E_0^{r,s} \cong \prod_{C \in \mathcal{C}'_s(G/H)} \mathrm{Ext}^r_{\mathbb{Z}[H]}(J_C, \widetilde{A}) \cong \prod_{C \in \mathcal{C}'_s(G/H)} \mathrm{Ext}^r_{\mathbb{Z}[H]}(\mathbb{Z}[C], \widetilde{A}).$$

Let $C \in \mathcal{C}'_s(G/H)$. For any left $\mathbb{Z}[H]$-module M we observe that, since H acts transitively on C, the map

$$\mathrm{Hom}_{\mathbb{Z}[H]}(\mathbb{Z}[C], M) \xrightarrow{\cong} \mathrm{Hom}_{\mathbb{Z}[N_C]}(\mathbb{Z}, \mathrm{Res}^H_{N_C}(M)) \overset{(2.2.1)}{=} (\mathrm{Res}^H_{N_C}(M))^{N_C}, \quad f \mapsto f(x_C), \tag{5.2.6}$$

is a group isomorphism. Let

$$\cdots \longrightarrow P_r \xrightarrow{p_r} P_{r-1} \xrightarrow{p_{r-1}} \ldots \xrightarrow{p_2} P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} \mathbb{Z} \longrightarrow 0$$

be a free resolution of $\mathbb{Z}$ over $\mathbb{Z}[H]$. Since $\mathbb{Z}[C]$ is a free abelian group, it follows from [22, Corollary III.5.7] that

$$\cdots \longrightarrow \mathbb{Z}[C] \otimes P_r \xrightarrow{\mathrm{id}_{\mathbb{Z}[C]} \otimes p_r} \mathbb{Z}[C] \otimes P_{r-1} \longrightarrow \cdots \longrightarrow \mathbb{Z}[C] \otimes P_1 \xrightarrow{\mathrm{id}_{\mathbb{Z}[C]} \otimes p_1} \mathbb{Z}[C] \otimes P_0 \longrightarrow \mathbb{Z}[C] \longrightarrow 0$$

is a free resolution of $\mathbb{Z}[C]$ over $\mathbb{Z}[H]$. Consequently, we can compute the above Ext-groups as

$$\mathrm{Ext}^r_{\mathbb{Z}[H]}(\mathbb{Z}[C], \widetilde{A}) = H^r(\mathrm{Hom}_{\mathbb{Z}[H]}(\mathbb{Z}[C] \otimes P_*, \widetilde{A}), (\mathrm{id}_{\mathbb{Z}[C]} \otimes p_*)^*)$$

Let $r \in \mathbb{N}_0$. If we consider $\mathrm{Hom}_{\mathbb{Z}}(P_r, \widetilde{A})$ as a left $\mathbb{Z}[H]$-module with respect to the diagonal H-action then we obtain

$$\begin{aligned} \mathrm{Hom}_{\mathbb{Z}[H]}(\mathbb{Z}[C] \otimes P_r, \widetilde{A}) &\overset{(2.2.3)}{\cong} \mathrm{Hom}_{\mathbb{Z}[H]}(\mathbb{Z}[C], \mathrm{Hom}_{\mathbb{Z}}(P_r, \widetilde{A})) \overset{(5.2.6)}{\cong} (\mathrm{Hom}_{\mathbb{Z}}(P_r, \widetilde{A}))^{N_C} \\ &\overset{(2.2.2)}{=} \mathrm{Hom}_{\mathbb{Z}[N_C]}(P_r, \mathrm{Res}^G_{N_C}(A)) \end{aligned}$$

and one checks that an explicit isomorphism is given by

$$F_r : \mathrm{Hom}_{\mathbb{Z}[H]}(\mathbb{Z}[C] \otimes P_r, \widetilde{A}) \to \mathrm{Hom}_{\mathbb{Z}[N_C]}(P_r, \mathrm{Res}^G_{N_C}(A)), \quad (F_r(f))(q) = f(x_C \otimes q) \quad \forall q \in P_r.$$

One checks that the F_r are compatible with the differentials, thus induce isomorphisms

$$(F_r)_* : \mathrm{Ext}^r_{\mathbb{Z}[H]}(\mathbb{Z}[C], \widetilde{A}) \to H^r(\mathrm{Hom}_{\mathbb{Z}[N_C]}(P_*, \mathrm{Res}^G_{N_C}(A)), p_r^*) \qquad \forall r \in \mathbb{N}_0,$$

where we used the obvious fact that

$$\mathrm{Res}^H_{N_C}(\widetilde{A}) = \mathrm{Res}^H_{N_C}(\mathrm{Res}^G_H(A)) = \mathrm{Res}^G_{N_C}(A).$$

Since each P_r is free as a left $\mathbb{Z}[H]$-module, it is free as a left $\mathbb{Z}[N_C]$-module as well. Hence, P_* is a free resolution of $\mathbb{Z}$ over $\mathbb{Z}[N_C]$, such that

$$H^r(\mathrm{Hom}_{\mathbb{Z}[N_C]}(P_*, \mathrm{Res}^G_{N_C}(A)), p_r^*) = H^r(N_C, \mathrm{Res}^G_{N_C}(A)) \qquad \forall r \in \mathbb{N}_0.$$

Combining the previous observations shows the claim. ☐

5.3 Consequences for sectional categories of subgroup inclusions

In this section we derive a lower bound on sectional categories of subgroup inclusions from Theorem 5.2.5. We adopt all of the spectral sequence notation from the previous section.

Definition 5.3.1. Let G be a group and let $H \leq G$ be a subgroup. For each $x \in G$, we let H_x denote the isotropy group of the left H-action on G/H in xH and put

$$\kappa_{G,H} := \sup\{\mathrm{cd}(H_x) \mid x \in G \setminus H\}.$$

The following result is a consequence of the previous properties of the spectral sequence:

Proposition 5.3.2. *Let G be a geometrically finite group and let $H \leq G$ be a subgroup. Let A be a left $\mathbb{Z}[G]$-module, let $n \in \mathbb{N}$ and let $u \in H^n(G,A) = D_0^{n,0}$. If $n \geq \kappa_{G,H}$, then*

$$u \in D^{n,0}_{n-\kappa_{G,H}}.$$

Proof. Since G is geometrically finite, it follows that $\kappa_{G,H} < +\infty$. Let $s \in \mathbb{N}$. By definition of the H-actions, for each $C \in \mathcal{C}'_s(G/H)$ there is some $x \in G \setminus H$, such that $N_C \leq H_x$. By [22, Proposition VIII.2.4], this yields that

$$\mathrm{cd}(N_C) \leq \mathrm{cd}(H_x) \leq \kappa_{G,H}$$

for each $C \in \mathcal{C}'_s(G/H)$. In particular, $H^r(N_C, \mathrm{Res}^G_{N_C}(A)) = 0$ whenever $r > \kappa_{G,H}$, so we derive from Theorem 5.2.5 that

$$E_0^{r,s} = \{0\} \qquad \forall r > \kappa_{G,H},\ s \in \mathbb{N}. \tag{5.3.1}$$

In particular,

$$E_0^{n-s,s} = \{0\} \qquad \forall s \in \{1,2,\dots,n-\kappa_{G,H}-1\}.$$

In terms of Theorem 5.2.4.c), this yields that

$$u \in \ker\left[j_s : D_s^{n,0} \to E_s^{n-s,s}\right] \qquad \text{for } 1 \leq s \leq n-\kappa_{G,H}-1$$

so it follows directly from Theorem 5.2.4.c) that $u \in D^{n,0}_{n-\kappa_{G,H}}$. □

This has an immediate consequence for sectional categories of subgroup inclusions.

Theorem 5.3.3. *Let G be a geometrically finite group and $H \leq G$ be a subgroup. Then*

$$\mathrm{secat}(H \hookrightarrow G) \geq \mathrm{cd}(G) - \kappa_{G,H}.$$

Proof. Put $n := \mathrm{cd}(G)$, let A be a left $\mathbb{Z}[G]$-module with $H^n(G,A) \neq \{0\}$ and let $u \in H^n(G,A)$ with $u \neq 0$. It follows from Proposition 5.3.2 that $u \in D^{n,0}_{n-\kappa_{G,H}}$. Thus, we obtain from Theorem 5.2.2 that $\mathrm{secat}(H \hookrightarrow G) \geq n - \kappa_{G,H}$. □

We want to describe the number $\kappa_{G,H}$ in a more tangible way.

Lemma 5.3.4. *Let G be a group, let $H \leq G$ be a subgroup, let $x \in G$ and let H_x denote the isotropy group of the left G-action on G/H in xH. Then*

$$H_x = H \cap xHx^{-1}.$$

Proof. Let $h \in H$ with $xhx^{-1} \in H$. Then $xhx^{-1} \cdot xH = xhH = xH$, hence $xhx^{-1} \in H_x$.

Conversely, let $g \in H_x$ and let $h_1 \in H$ be arbitrary. Then there exists an $h_2 \in H$ with

$$g \cdot xh_1 = xh_2 \quad \Leftrightarrow \quad g = xh_2h_1^{-1}x^{-1} \quad \Rightarrow g \in xHx^{-1}.$$

□

Corollary 5.3.5. *Let G be a geometrically finite group and $H \leq G$ be a subgroup. Then*

$$\kappa_{G,H} = \max\{\mathrm{cd}(H \cap xHx^{-1}) \mid x \in G \setminus H\}.$$

Proof. This follows immediately by applying Lemma 5.3.4 to the definition of $\kappa_{G,H}$. □

Remark 5.3.6. If H is not a self-normalizing subgroup of G, then there exists some non-trivial element $x \in N_G(H) \setminus H$ such that $H \cap xHx^{-1} = H$. Thus, $\kappa_{G,H} = \mathrm{cd}(H)$ in this case and Theorem 5.3.3 yields

$$\mathrm{secat}(H \hookrightarrow G) \geq \mathrm{cd}(G) - \mathrm{cd}(H).$$

This is, of course, the case if we consider a normal subgroup $N \lhd G$. In this situation, with $Q = G/N$, and under the necessary assumption that $\mathrm{cd}(Q) < \infty$, we obtain

$$\mathrm{cd}(G) - \mathrm{cd}(N) \leq \mathrm{secat}(N \hookrightarrow G) \leq \mathrm{cd}(Q).$$

In Theorem 5.1.7 we have seen a condition for the sectional category to reach the top dimension. It is interesting to remark as well that if $H^{\mathrm{cd}(N)}(N, \mathbb{Z}[N])$ is free abelian then, by [13, Theorem 5.5], we have that $\mathrm{cd}(Q) = \mathrm{cd}(G) - \mathrm{cd}(N)$ and thus

$$\mathrm{secat}(N \hookrightarrow G) = \mathrm{cd}(G) - \mathrm{cd}(N)$$

under such assumption.

5.4 Applications to topological complexity

We next want to check explicitly that our Theorem 5.3.3 indeed recovers the corresponding result from [57]. However, we will do it through an application of it to the more general version of sequential topological complexity (which, as seen previously, subsumes the original notion as one of its concrete cases).

Once that goal is accomplished, we will close this section (and indeed this chapter) with a specialization of Theorem 5.3.3 to the case of parametrized topological complexity of group epimorphisms.

5.4.1 Sequential topological complexity of aspherical spaces

Let π be a geometrically finite group. As referenced in Chapter 3 at 3.3.2, the r^{th} sequential topological complexity $\mathrm{TC}_r(K(\pi,1))$ coincides with the sectional category of the covering of $(K(\pi,1))^r$ that is associated with the diagonal subgroup $\Delta_{\pi,r}$. By the definition of $\mathrm{secat}(H \hookrightarrow G)$, this shows that $\mathrm{TC}_r(K(\pi,1))$ is given as the sectional category of the inclusion of $\Delta_{\pi,r}$ into the product π^r, i.e.

$$\mathrm{TC}_r(K(\pi,1)) = \mathrm{secat}(\Delta_{\pi,r} \hookrightarrow \pi^r). \tag{5.4.1}$$

Our aim in this subsection is to put to use Theorem 5.3.3 in order to find a new lower bound for sequential TC of Eilenberg-MacLane spaces (of whom not much is known so far). However, to obtain a tangible lower bound, first we need to determine the value of $\kappa_{\pi^r,\Delta_{\pi,r}}$ more explicitly.

Lemma 5.4.1. *For each $r \in \mathbb{N}$ with $r \geq 2$, it holds that*

$$\kappa_{\pi^r,\Delta_{\pi,r}} = k(\pi) := \max\{\mathrm{cd}(C(g)) \mid g \in \pi \setminus \{1\}\},$$

where $C(g)$ denotes the centralizer of $g \in \pi$.

Proof. Let $x = (x_1, \dots, x_r) \in \pi^r \setminus \Delta_{\pi,r}$ and let $(h, h, \dots, h) \in \Delta_{\pi,r}$. Then

$$\begin{aligned} x(h,h,\dots,h)x^{-1} \in \Delta_{\pi,r} \quad &\Leftrightarrow \quad (x_1hx_1^{-1}, x_2hx_2^{-1}, \dots, x_rhx_r^{-1}) \in \Delta_{\pi,r} \\ &\Leftrightarrow \quad x_1hx_1^{-1} = x_2hx_2^{-1} = \dots = x_rhx_r^{-1}. \end{aligned}$$

For all $i, j \in \{1, 2, \dots, r\}$ we compute that

$$x_ihx_i^{-1} = x_jhx_j^{-1} \quad \Leftrightarrow \quad x_j^{-1}x_ih = hx_j^{-1}x_i \quad \Leftrightarrow \quad h \in C(x_j^{-1}x_i).$$

One derives from this observation that

$$(h,h,\dots,h) \in \Delta_{\pi,r} \cap x\Delta_{\pi,r}x^{-1} \quad \Leftrightarrow \quad h \in \bigcap_{i \neq j} C(x_j^{-1}x_i).$$

This shows in particular that any subgroup of π^r of the form $\Delta_{\pi,r} \cap x\Delta_{\pi,r}x^{-1}$, where $x \notin \Delta_{\pi,r}$, is isomorphic to a subgroup of the centralizer of an element of $\pi \setminus \{1\}$, so we derive that $\kappa_{\pi^r,\Delta_{\pi,r}} \leq k(\pi)$. On the other hand, given an arbitrary $g \in \pi$ with $g \neq 1$, if we put $x_0 := (g, 1, \dots, 1) \in \pi^r$, then it follows from the above that

$$(h,h,\dots,h) \in \Delta_{\pi,r} \cap x_0\Delta_{\pi,r}x_0^{-1} \quad \Leftrightarrow \quad h \in C(g),$$

so $C(g)$ is indeed isomorphic to a group of the form $\Delta_{\pi,r} \cap x\Delta_{\pi,r}x^{-1}$. This shows that $\kappa_{\pi,r} \geq k(\pi)$ and the two inequalities together show the claim. □

Thus, we obtain the following consequence of our main lower bound.

Theorem 5.4.2. *Let π be a geometrically finite group and let $r \in \mathbb{N}$ with $r \geq 2$. Then*

$$\mathrm{TC}_r(K(\pi,1)) \geq r \cdot \mathrm{cd}(\pi) - k(\pi),$$

where $k(\pi) = \max\{\mathrm{cd}(C(g)) \mid g \in \pi \setminus \{1\}\}$.

Proof. We derive from Theorem 5.3.3 and from (5.4.1) that

$$\begin{aligned}\mathrm{TC}_r(K(\pi,1)) &\geq \mathrm{cd}(\pi^r) - \kappa_{\pi^r,\Delta_{\pi,r}} \\ &= r\cdot \mathrm{cd}(\pi) - k(\pi),\end{aligned}$$

where in the last step we used Lemma 5.4.1 and the main result of [38] on cohomological dimensions of products of geometrically finite groups. □

Remark 5.4.3. (1) In the case of $r = 2$, the previous theorem yields the lower bound of

$$\mathrm{TC}(K(\pi,1)) \geq 2\mathrm{cd}(\pi) - k(\pi)$$

for topological complexity. Although not explicitly stated therein, this inequality is an immediate consequence of the main results of [57].

(2) If π is a torsion-free hyperbolic group, then $C(g)$ is infinite cyclic for each $g \in \pi^*$, so Theorem 5.4.2 and Theorem 4.1.3.a) imply that

$$r\cdot \mathrm{cd}(\pi) - 1 \leq \mathrm{TC}_r(K(\pi,1)) \leq r\cdot \mathrm{cd}(\pi) \qquad \forall r \geq 2$$

in this case. It has in fact been shown by S. Hughes and K. Li in [77] that indeed $\mathrm{TC}_r(\pi) = r\,\mathrm{cd}(\pi)$ for all torsion-free hyperbolic groups with π not isomorphic to $\mathbb{Z}$ and all $r \geq 2$. However, the methods of [77] only generalize slightly beyond the hyperbolic case and do not yield a general lower bound for geometrically finite groups.

(3) If π is a free abelian group, then $\Delta_{\pi,r}$ is a *normal* subgroup of π^r with $\pi^r \big/ \Delta_{\pi,r} \cong \pi^{r-1}$. In this case, we derive from Theorem 5.1.7 that

$$\mathrm{TC}_r(\pi) = \mathrm{cd}(\pi^{r-1}) = (r-1)\cdot \mathrm{cd}(\pi) \qquad \forall r \geq 2.$$

This has already been observed in [9, Corollary 3.13].

(4) Suppose that $x = (x_1, \cdots, x_r) \in \pi^r$ satisfies $x\Delta_{\pi,r}x^{-1} = \Delta_{\pi,r}$. From the proof of Lemma 5.4.1 we can infer that this implies $\pi \subset C(x_j x_i^{-1})$ for every $i \neq j$. But this means, in turn, that $x_j x_i^{-1} \in Z(\pi)$. Therefore

$$N_{\pi^r}(\Delta_{\pi,r}) = \{(x_1, \cdots, x_r) \in \pi^r \mid x_j x_i^{-1} \in Z(\pi), \forall i \neq j\}.$$

Consequently, if the group π satisfies $Z(\pi) = \{1\}$, the diagonal subgroup $\Delta_{\pi,r}$ is self-normalizing.

We want to apply Theorem 5.4.2 to a certain class of free amalgamated products whose centralizers were studied by T. Lewin. For this purpose, we need to introduce a notion from group theory.

Definition 5.4.4. Let G be a group. A subgroup $H \leq G$ is *malnormal* if

$$xHx^{-1} \cap H = \{1\} \qquad \forall x \in G \setminus H.$$

In the following, given a group G and $g \in G$ we let $C_G(g)$ denote its centralizer whenever it is ambiguous which group we are referring to.

Corollary 5.4.5. *Let π_1 and π_2 be geometrically finite groups and consider a free product with amalgamation $\pi_1 *_H \pi_2$, such that H is malnormal in π_1 or malnormal in π_2. Then for each $r \geq 2$*

$$\mathrm{TC}_r(\pi_1 *_H \pi_2) \geq r \cdot \mathrm{cd}(\pi_1 *_H \pi_2) - \max\{k(\pi_1), k(\pi_2)\}.$$

Proof. Put $\pi := \pi_1 *_H \pi_2$ and let $g \in \pi$, $g \neq 1$. By [87, Theorem 2], the centralizer $C_\pi(g)$ is infinite cyclic or isomorphic to $C_{\pi_1}(g)$ or $C_{\pi_2}(g)$. In the first case, it holds that $\mathrm{cd}(C_\pi(g)) = 1$, while in the other two cases it holds that $\mathrm{cd}(C_\pi(g)) \leq k(\pi_1)$ or $\mathrm{cd}(C_\pi(g)) \leq k(\pi_2)$, respectively. Since g was chosen arbitrarily, this yields that

$$k(\pi) \leq \max\{1, k(\pi_1), k(\pi_2)\} = \max\{k(\pi_1), k(\pi_2)\},$$

so the claim follows immediately from Theorem 5.4.2. □

Remark 5.4.6. For explicit computations using Corollary 5.4.5, there are some general results about cohomological dimensions of free amalgamated products that come in handy. More precisely, let π_1 and π_2 be groups of finite cohomological dimension and consider a free product with amalgamation $\pi_1 *_H \pi_2$. It is shown in [13, Proposition 6.1] that

$$\max\{\mathrm{cd}(\pi_1), \mathrm{cd}(\pi_2)\} \leq \mathrm{cd}(\pi_1 *_H \pi_2) \leq \max\{\mathrm{cd}(\pi_1), \mathrm{cd}(\pi_2)\} + 1.$$

and that a necessary condition for $\mathrm{cd}(\pi_1 *_H \pi_2) = \max\{\mathrm{cd}(\pi_1), \mathrm{cd}(\pi_2)\} + 1$ to hold is that $\mathrm{cd}(\pi_1) = \mathrm{cd}(\pi_2)$. It is further shown in [13, Corollary 6.5] that a sufficient condition for this equality is that both π_1 and π_2 are of type FP_∞ and that H is of finite index both in π_1 and in π_2.

5.4.2 Parametrized topological complexity of epimorphisms

The parametrized topological complexity of a fibration has been introduced by D. Cohen, M. Farber and S. Weinberger in [30]. Given a fibration $p : E \to B$, one considers E^I_B as the space of all continuous paths $\gamma \colon I := [0,1] \to E$ in a single fibre of p, i.e. such that the path $p \circ \gamma$ is constant. Define the space

$$E \times_B E = \{(e, e') \in E \times E \mid p(e) = p(e')\}$$

of all possible pairs of configurations lying in the same fibre of p. Then, the map

$$\Pi : E^I_B \to E \times_B E \qquad \Pi(\gamma) = (\gamma(0), \gamma(1))$$

is a fibration with fibre ΩX. The parametrized topological complexity of p is defined as

$$\mathrm{TC}[p : E \to B] = \mathrm{secat}(\Pi : E^I_B \to E \times_B E).$$

We want to apply the results previously obtained to the parametrized topological complexity of group epimorphisms. This algebraic variant of parametrized topological complexity was defined and investigated by M. Grant in [67]. Given two groups G and Q and an epimorphism $\rho : G \twoheadrightarrow Q$ there exists a fibration $f_\rho : K(G,1) \to K(Q,1)$, whose fibre is path-connected and which induces ρ on the level of fundamental groups. Moreover, it is shown in loc. cit. that $\mathrm{TC}[f_\rho : K(G,1) \to K(Q,1)]$ is independent of the choice of f_ρ and that

$$\mathrm{TC}[f_\rho : K(G,1) \to K(Q,1)] = \mathrm{secat}(\Delta_G \hookrightarrow G \times_Q G) =: \mathrm{TC}[\rho : G \twoheadrightarrow Q].$$

Here, $\Delta_G = \{(g,g) \in G \times G \mid g \in G\}$ denotes the diagonal subgroup and

$$G \times_Q G := \{(x,y) \in G \times G \mid \rho(x) = \rho(y)\}.$$

We discuss an alternative description of these pullback groups in the following lemma.

Lemma 5.4.7. *Let G, Q be groups, let $\rho : G \to Q$ be an epimorphism. Then*

$$G \times_Q G = ((\ker \rho) \times 1) \cdot \Delta_G.$$

Proof. Let $k \in \ker \rho$ and $g \in G$. Then, since ρ is a homomorphism, $\rho(kg) = \rho(k) \cdot \rho(g) = \rho(g)$, so that $(kg, g) \in G \times_Q G$. Conversely, let $(g_1, g_2) \in G \times_Q G$. Then

$$\rho(x) = \rho(y) \quad \Leftrightarrow \rho(x)(\rho(y))^{-1} = 1 \quad \Leftrightarrow \quad \rho(xy^{-1}) = 1 \quad \Leftrightarrow \quad xy^{-1} \in \ker \rho.$$

Thus $(x,y) = (xy^{-1}, 1) \cdot (y,y) \in ((\ker \rho) \times 1) \cdot \Delta_G$. □

We now want to apply our results on sectional categories to this setting. The following statement is a straightforward application of Theorem 5.3.3.

Theorem 5.4.8. *Let G and Q be geometrically finite groups and let $\rho : G \twoheadrightarrow Q$ be an epimorphism. Then*

$$\mathrm{TC}[\rho : G \to Q] \geq \mathrm{cd}(G \times_Q G) - k(\rho),$$

where

$$k(\rho) = \max\{\mathrm{cd}(C(g)) \mid g \in \ker \rho,\ g \neq 1\}.$$

Proof. It follows from Theorem 5.3.3 that $\mathrm{TC}[\rho : G \to Q] \geq \mathrm{cd}(G \times_Q G) - \ell$, where

$$\ell := \kappa_{G \times_Q G, \Delta_G} = \max\{\mathrm{cd}(\Delta_G \cap z\Delta_G z^{-1}) \mid z \in (G \times_Q G) \setminus \Delta_G\}.$$

It only remains to show that $k(\rho) = \ell$. It follows from Lemma 5.4.7 that

$$\begin{aligned}\ell &= \max\{\mathrm{cd}(\Delta_G \cap (xh,h)\Delta_G(xh,h)^{-1}) \mid x \in \ker \rho \setminus \{1\},\ h \in G\} \\ &= \max\{\mathrm{cd}(\Delta_G \cap (x,1)\Delta_G(x,1)^{-1}) \mid x \in \ker \rho \setminus \{1\}\},\end{aligned}$$

since, evidently, $(h,h)\Delta_G(h,h)^{-1} = \Delta_G$ for all $h \in G$.

Let $x \in \ker \rho$ with $x \neq 1$ and let $g \in G$. Then

$$\begin{aligned}&(g,g) \in (x,1)\Delta_G(x,1)^{-1} \quad \Leftrightarrow \quad \exists h \in G : \ (g,g) = (x,1)(h,h)(x,1)^{-1} = (xhx^{-1}, h) \\ &\Leftrightarrow \quad \exists h \in G : \ g = h \wedge g = xhx^{-1} \\ &\Leftrightarrow \quad g = xgx^{-1} \quad \Leftrightarrow \quad x = g^{-1}xg \quad \Leftrightarrow \quad g^{-1} \in C(x) \quad \Leftrightarrow \quad g \in C(x).\end{aligned}$$

Moreover, the map

$$C(x) \to \Delta_G \cap (x,1)\Delta_G(x,1)^{-1} \qquad g \mapsto (g,g)$$

is easily seen to be a group isomorphism. We immediately derive that $k(\rho) = \ell$. □

To study the cohomological dimension of $G \times_Q G$, we can characterize such pullback groups as semidirect products.

Lemma 5.4.9. *Let G, Q be groups and let $\rho : G \twoheadrightarrow Q$ be an epimorphism. Then*

$$\Phi : G \times_Q G \to (\ker \rho) \rtimes_\varphi G, \quad \Phi(g,h) = (gh^{-1}, h),$$

is a group isomorphism, where $\varphi : G \to \mathrm{Aut}(\ker \rho)$, $(\varphi(g))(x) = gxg^{-1}$.

Proof. One checks without difficulties that Φ is injective. Moreover, for each $(x,y) \in (\ker \rho) \rtimes G$ it holds that $\Phi(xy, y) = (x,y)$, so Φ is surjective as well. For all $(g_1,h_1),(g_2,h_2) \in G \times_Q G$ we further compute that

$$\begin{aligned}\Phi((g_1,h_1)\cdot(g_2,h_2)) &= \Phi(g_1g_2, h_1h_2) = (g_1g_2h_2^{-1}h_1^{-1}, h_1h_2) \\ &= (g_1h_1^{-1}\cdot h_1g_2h_2^{-1}h_1^{-1}, h_1h_2) = (g_1h_1^{-1}(\varphi(h_1))(g_2h_2^{-1}), h_1h_2) \\ &= (g_1h_1^{-1}, h_1) \bullet (g_2h_2^{-1}, h_2) = \Phi(g_1,h_1) \bullet \Phi(g_2,h_2).\end{aligned}$$

where $\bullet$ denotes multiplication in $(\ker \rho) \rtimes_\phi G$. Thus, Φ is an isomorphism. □

Corollary 5.4.10. *Let G and Q be geometrically finite groups and let $\rho : G \twoheadrightarrow Q$ be an epimorphism. Then*

$$TC[\rho : G \twoheadrightarrow Q] \le \mathrm{cd}(G) + \mathrm{cd}(\ker \rho).$$

Proof. It is well known that the cohomological dimension of a semidirect product is at most the sum of those of its factors. Thus, it follows from Lemma 5.4.9 and the lower bound from Theorem 4.1.3.a) that

$$\mathrm{TC}[\rho : G \twoheadrightarrow Q] \le \mathrm{cd}(G \times_Q G) \le \mathrm{cd}(G) + \mathrm{cd}(\ker \rho).$$

□

Corollary 5.4.11. *Let G and Q be geometrically finite groups and let $\rho : G \twoheadrightarrow Q$ be an epimorphism. Assume that $H^n(\ker \rho, \mathbb{Z}[\ker \rho])$ is $\mathbb{Z}$-free for $n = \mathrm{cd}(\ker \rho)$. Then*

$$2\mathrm{cd}(G) - \mathrm{cd}(Q) - k(\rho) \le \mathrm{TC}[\rho : G \twoheadrightarrow Q] \le 2\mathrm{cd}(G) - \mathrm{cd}(Q),$$

where $k(\rho) = \max\{\mathrm{cd}(C(g)) \mid g \in \ker \rho,\ g \ne 1\}$.

Proof. Since G is geometrically finite and $H^n(\ker \rho, \mathbb{Z}[\ker \rho])$ is $\mathbb{Z}$-free, it follows from [13, Theorem 5.5], that

$$\mathrm{cd}(\ker \rho) = \mathrm{cd}(G) - \mathrm{cd}(Q). \tag{5.4.2}$$

The upper bound on $\mathrm{TC}[\rho : G \twoheadrightarrow Q]$ thus follows directly from Corollary 5.4.11. Regarding the lower bound, we derive from Lemma 5.4.9 and again [13, Theorem 5.5] that

$$\mathrm{cd}(G \times_Q G) = \mathrm{cd}(G) + \mathrm{cd}(\ker \rho) \overset{(5.4.2)}{=} 2\mathrm{cd}(G) - \mathrm{cd}(Q).$$

The lower bound is then an immediate consequence of Theorem 5.4.8. □

If we want to consider the case of the inclusion of a normal subgroup notice that, for G and Q groups and $\rho : G \twoheadrightarrow Q$ an epimorphism, then Δ_G is a normal subgroup of $G \times_Q G$ if and only if $\ker \rho \subset Z(G)$, where $Z(G)$ denotes the center of G. Indeed, since $(g, h) \in G \times_Q G$, it holds that $\rho(g) = \rho(h)$ and thus $g^{-1}h \in \ker \rho$. So, if $\ker \rho \subset Z(G)$, this condition is satisfied for all $(g, h) \in G \times_Q G$ and $x \in G$. Conversely, if Δ_G is normal $G \times_Q G$, then we derive by taking $(g, h) = (a^{-1}, 1)$ for $a \in \ker \rho$ that indeed $\ker \rho \subset Z(G)$. As such, we are in the situation that the associated group extension

$$\{1\} \to \ker(\rho) \to G \xrightarrow{\rho} Q \to \{1\}$$

is central. Therefore, by [67, Corollary 5.2] we know that

$$\mathrm{TC}[\rho : G \twoheadrightarrow Q] = \mathrm{cd}(\ker(\rho)).$$

But we can also derive an approach to this case as a consequence of the more general computation provided by Theorem 5.1.7.

Proposition 5.4.12. *Let G and Q be geometrically finite groups and let $\rho : G \twoheadrightarrow Q$ be an epimorphism. Assume that* $\ker \rho$ *lies in the center of G and consider the homomorphism*

$$\phi : G \times_Q G \to \ker \rho, \qquad \phi(g, h) = gh^{-1}.$$

Then

$$\mathrm{cd}(\phi : G \times_Q G \to \ker \rho) \leq \mathrm{TC}[\rho : G \twoheadrightarrow Q] = \mathrm{cd}(\ker \rho).$$

Proof. We observe using Lemma 5.4.7 that the map

$$G \times_Q G \to \ker \rho \qquad (g, h) \mapsto gh^{-1}$$

in fact induces a group isomorphism

$$\psi : {(G \times_Q G)}\big/{\Delta_G} \xrightarrow{\cong} \ker \rho$$

by the assumption on $\ker \rho$. Since the projection $p : G \times_Q G \to (G \times_Q G)/\Delta_G$ is easily seen to satisfy $\psi \circ p = \phi$, we derive from the assumptions, Theorem 5.1.7 and [67, Corollary 5.2] that

$$\mathrm{TC}[\rho : G \twoheadrightarrow Q] = \mathrm{cd}\left({(G \times_Q G)}\big/{\Delta_G}\right) = \mathrm{cd}(\ker \rho)$$

$$\mathrm{TC}[\rho : G \twoheadrightarrow Q] \geq \mathrm{cd}(p) = \mathrm{cd}(\phi).$$

□

CHAPTER 6

Sectional category and topological complexity of groups as $\mathcal{A}$-genus

Introduction

We will conclude our study of the topological complexities of aspherical spaces and sectional category of subgroup inclusions by developing some of the ideas implicit in our characterization of $\mathrm{secat}(H \hookrightarrow G)$ given in Theorem 4.1.4. In this short chapter, we will see that the sectional category of a subgroup inclusion $H \hookrightarrow G$ and, consequently, also the (sequential) topological complexity of aspherical spaces can be characterized by means of another category-like homotopy invariant, known as the $\mathcal{A}$-genus, for $\mathcal{A}$ a suitable family of G-spaces. Even though the proof of such result is relatively straightforward, the characterization has some interesting consequences, and allows to find in a rather direct manner new bounds for both sectional category and sequential topological complexities, and to generalize and reformulate some of the already well-known bounds. We will close the chapter with some thoughts about further uses of $\mathcal{A}$-genus for proper actions of groups.

6.1 The notion of $\mathcal{A}$-genus

The main notion of interest for this chapter, as we just indicated above, is that of $\mathcal{A}$-genus of a G-space X with respect to a family $\mathcal{A}$ of G-spaces. Such concept was gradually developed by several authors in different degrees of generality. The first accounts, that of $\mathcal{A}$-genus for $\mathcal{A} = \{G\}$, are due to C.T. Yang [117] for $G = \mathbb{Z}_2$, M.A. Krasnoselski [83] for $\mathbb{Z}_p$, to Schwarz [110] for free actions of discrete groups and finally to E. Fadell [50] for free actions of arbitrary groups. However, the more general notion of $\mathcal{A}$-genus that we will employ during this chapter is due to T. Bartsch in [7], where he developed the notion for arbitrary choices of groups and families of G-spaces. Indeed, Bartsch will be our main reference throughout this chapter, specifically his classic book on the matter, see [8].

In particular, the notion of $\mathcal{A}$-genus can be seen as a special case of the more general concept of $\mathcal{A}$-category, which in the equivariant context was due to M. Clapp and D. Puppe in [29], as a way of generalizing the notion of Lusternik-Schnirelmann category and stuying critical point theory with respect to group actions, see [28]. Let G be a group, and fix a set of G-spaces $\mathcal{A}$.

Definition 6.1.1. Let X and Y be G-spaces. We define the $\mathcal{A}$-*category* of a G-equivariant map $f : X \to Y$, denoted by $\mathcal{A}\text{-cat}(f)$, as the smallest integer $k \geq 0$ such that there exists an open cover $\{U_0, \cdots, U_k\}$ of X such that for every $i \in \{0, \cdots, k\}$ there is some $A_i \in \mathcal{A}$ and G-equivariant maps $\alpha_i : X_i \to A_i$ and $\beta_i : A_i \to Y$ satisfying that the restriction $f_{|X_i}$ is G-homotopically equivalent to $\beta_i \circ \alpha_i$. If no such integer exists, we set $\mathcal{A}\text{-cat}(f) = \infty$.

We define the $\mathcal{A}$-category of a G space X as $\mathcal{A}\text{-cat}(X) := \mathcal{A}\text{-cat}(\mathrm{id}_X)$.

The relationship between different versions of topological complexity and the $\mathcal{A}$-category of the configuration space has been studied before. Indeed, N. Iwase and M. Sakai in [80] first recasted $\mathrm{TC}(X)$ in terms of the $\mathcal{A}$-category of X for a suitable family $\mathcal{A}$, while W. Lubawski and W. Marzantowicz in [89] and A. Ángel, H. Colman, M. Grant and J. Oprea in [3] studied the situation for equivariant versions of topological complexity. It is important to mention as well the work of P. Capovilla, C. Loeh and M. Moraschini, [25], where the ideas of Clapp and Puppe are employed to give new variants of $\mathcal{A}$-category and to study its relationship with TC. However, to our knowledge, all these approaches have been only from the perspective of the $\mathcal{A}$-category of a space in the sense of the above definition, and no attempt has been made to exploit the specificities of the properties of $\mathcal{A}$-genus in order to study its applications to the setting of topological complexity. This is precisely our goal in this chapter. It is time now to recall the definition of $\mathcal{A}$-genus.

Definition 6.1.2. The $\mathcal{A}$-genus of a G-space X is defined as the $\mathcal{A}$-category of the constant map, i.e. $\mathcal{A}\text{-genus}(X) := \mathcal{A}\text{-cat}(X \to *)$. Equivalently, $\mathcal{A}\text{-genus}(X)$ is the smallest integer $k \geq 0$ such that there exists an open cover $\{U_0, \cdots, U_k\}$ of X satisfying that, for every $0 \leq i \leq k$ there exists $A_i \in \mathcal{A}$ and a G-equivariant map $U_i \to A_i$.

It becomes immediately apparent from the definitions that $\mathcal{A}$-genus constitutes a lower bound for the $\mathcal{A}$-category of a G-space. Indeed, for any G-space X and family $\mathcal{A}$ we have $\mathcal{A}\text{-genus}(X) \leq \mathcal{A}\text{-cat}(X)$, see [8, Proposition 2.9]. The interested reader can check in loc. cit. explicit examples where such inequality is indeed strict. The following proposition summarizes the basic properties of $\mathcal{A}$-genus that will be of most use for our purposes. In particular, we have to highlight the characterization of $\mathcal{A}$-genus in terms of the join of spaces from the family $\mathcal{A}$.

Proposition 6.1.3 ([8] Proposition 2.9, Proposition 2.15 and Proposition 2.17)**.** *Let G be a group, X a G-space and $\mathcal{A}$ a family of G spaces. The* $\mathcal{A}$-genus *satisfies the following properties:*

(a) $\mathcal{A}\text{-genus}(X)$ *is the smallest integer $k \geq 0$ such that there exists $A_0, \cdots, A_k \in \mathcal{A}$ and a G-equivariant map of the form $X \to A_0 * \cdots * A_k$.*

(b) Monotonicity: *If there exists a G-equivariant map* $X \to Y$ *then* $\mathcal{A}\text{-genus}(X) \leq \mathcal{A}\text{-genus}(Y)$

(c) Normalization *If* $A \in \mathcal{A}$ *then* $\mathcal{A}\text{-genus}(A) = 0$.

(d) Let $H \leq G$ *closed, consider a family* $\mathcal{A}$ *of G-spaces and a family* $\mathcal{B}$ *of H-spaces. Then, for any G-space X*

$$\mathcal{B}\text{-genus}(X) \leq (\mathcal{A}\text{-genus}(X) + 1)(\max\{\mathcal{B}\text{-genus}(A) : A \in \mathcal{A}\} + 1) - 1$$

Given that we aim to identify $\mathcal{A}$-genus with certain sectional categories, we would want an analogous to the cohomological lower bound in the context of $\mathcal{A}$-genus. This role is played by a notion of length relative to the family $\mathcal{A}$. Let h^* be a multiplicative G-equivariant cohomology theory, and $I \subseteq h^*(*)$ an ideal. The $(\mathcal{A}, h^*, I)$-length of a G-space X, denoted by $l_{\mathcal{A},h^*,I}(X)$ (or simply $l(X)$ when the required imputs are clear) is defined to be the smallest integer $k \geq 0$ such that there exists $A_0, \cdots, A_k \in \mathcal{A}$ with the property that, for any class $\alpha_i \in I \cap \ker[h^*(*) \to h^*(A_i)]$ for every $0 \leq i \leq k$, the cup product

$$p_X^*(\alpha_1) \cup \cdots \cup p_X^*(\alpha_k) = 0$$

where here p_X denotes the map that collapses X to one point, $p_X : X \to *$. Using the properties of the length (see [8, Chapter 4]) we can see that, if $X \to A_0 * \cdots * A_k$ is a G-equivariant map as in (a) of Proposition 6.1.3, then we have the inequality

$$l(X) \leq l(A_0 * \cdots * A_k) \leq \sum_{i=1}^{k} l(A_i)$$

and therefore we have the expected relationship between $\mathcal{A}$-genus and the length, $l(X) \leq \mathcal{A}\text{-genus}(X)$ for every G-space X, see [8, Corollary 4.9].

6.2 Sectional category and sequential TC as $\mathcal{A}$-genus

In this section we obtain the aforementioned characterization of the sectional category of a group monomorphism and the r^{th}-sequential topological complexity of a $K(G,1)$-space for every $r \geq 2$ in terms of $\mathcal{A}$-genus. In fact, we will prove a more general statement, that the sectional category of every connected covering can be seen as an $\mathcal{A}$-genus for a suitable family $\mathcal{A}$. From there, we will easily infer the rest of characterizations. All the groups considered throughout this section will be taken as discrete.

Theorem 6.2.1. *Let X be a path connected CW-complex. If* $q : \widehat{X} \to X$ *is a connected covering, then*

$$\text{secat}(q) = \mathcal{A}\text{-genus}(\tilde{X})$$

where $\mathcal{A} = \left\{ \pi_1(X) \middle/ \pi_1(\widehat{X}) \right\}$.

Proof. First recall that, by means of the associated bundle construction, we can consider the connected covering $q\colon \widehat{X} \to X$ as a bundle

$$q_0 : \widetilde{X} \times_{\pi_1(X)} \left(\pi_1(X) \big/ \pi_1(\widehat{X}) \right) \to X$$

associated to the principal $\pi_1(X)$-bundle of the universal covering $\widetilde{X} \to X$.

Take $\{U_i\}_{0 \le i \le n}$ an open covering of X such that for each $0 \le i \le n$ there exists $s_i\colon U \to \widehat{X}$ a local section of the fibration q over U_i. Notice that the restriction of q_0 to U_i is again a covering. Moreover, if q has a local section over U, then there is a naturally induced local section of q_0. By Theorem 2.3.6, the sections of

$$q_0\colon q_0^{-1}(U_i) \times_{\pi_1(X)} \left(\pi_1(X) \big/ \pi_1(\widehat{X}) \right)$$

are in one-to-one correspondence with $\pi_1(X)$-equivariant maps

$$q_0^{-1}(U_i) \to \pi_1(X) \big/ \pi_1(\widehat{X}) \,.$$

Observe that $\{q_0^{-1}(U_i)\}_{0 \le i \le n}$ constitutes an open cover of $\widetilde{X}$ for $\mathcal{A}$-genus, with

$$\mathcal{A} = \left\{ \pi_1(X) \big/ \pi_1(\widehat{X}) \right\},$$

hence we conclude the equality

$$\operatorname{secat}(q) = \operatorname{secat}(q_0) = \mathcal{A}\text{-genus}(\widetilde{X}).$$

□

An immediate consequence of the previous theorem and our definition of $\operatorname{secat}(H \hookrightarrow G)$ is the characterization of the sectional category of any subgroup inclusion in terms of the $\mathcal{A}$-genus.

Corollary 6.2.2. *Let G be a torsion-free group, and $H \leqslant G$. Then we have*

$$\operatorname{secat}(H \hookrightarrow G) = \mathcal{A}\text{-genus}(EG)$$

where $\mathcal{A} = \{G/H\}$

Proof. As introduced in Definition 4.1.2, $\operatorname{secat}(\iota\colon H \hookrightarrow G)$ coincides with the sectional category of the fibration

$$K(\iota, 1)\colon K(H,1) \to K(G,1).$$

The covering map $EG/H \to EG/G$ associated to H provides an explicit fibration whose sectional categorry corresponds with $\operatorname{secat}(H \hookrightarrow G)$. Therefore, the claim follows immediately from Theorem 6.2.1. □

Remark 6.2.3. Notice that an alternative proof of Corollary 6.2.2 is also implicit in the proof of Theorem 4.1.4. Indeed, such proof indicates that $\operatorname{secat}(H \hookrightarrow G)$ corresponds with the smallest integer $k \ge 0$ such that there exists an equivariant map of the form

$$EG \to *^{n+1}(G/H).$$

Thus the result follows from Proposition 6.1.3 (a).

Now we turn our attention to the corresponding characterization for sequential topological complexities. In this caseFor the case of the iterated diagonal subgroup inclusion $\Delta_{\pi,r} \hookrightarrow \pi^r$, we can reason as follows

Proposition 6.2.4. *Let $r \in \mathbb{N}$ with $r \geq 2$, and let X be a path connected CW-complex with $\pi_1(X) = \pi$. Put $\mathcal{A} := \{\pi^k / \Delta_{\pi,r}\}$. The following holds:*

(1) $\mathrm{TC}_r(X) \geq \mathcal{A}\text{-genus}(\widetilde{X}^r)$.

(2) Furthermore, if X is aspherical, then $\mathrm{TC}_r(X) = \mathcal{A}\text{-genus}(\widetilde{X}^r)$.

Proof. (a) Let $q\colon \widehat{X^r} \to X^r$ be the connected covering associated to the diagonal subgroup $\Delta_{\pi,r} \leqslant \pi^r$. Recall that $q_*(\pi_1(\widehat{X^r}, \hat{x}_0)) \leqslant \pi_1(X^r, x_0)$ is either the diagonal group $\Delta_{X,r}$ or a subgroup of π^r conjugated to it, depending just on the choice of basepoint $x_0 \in X$. Indeed, we can realize $\widehat{X^r}$ as

$$\widehat{X^r} = \widetilde{X}^r \Big/ \Delta_{\pi,r}\,,$$

i.e. as the orbit space of the $\Delta_{\pi,r}$-action on $\widetilde{X}^r$ obtained by restricting the π^r-action that is given as the r-fold product of the π-action on $\widetilde{X}$ by deck transformations.

As the fibration $e_r^X\colon X^{J_r} \to X^r$ is the standard fibrational substitute of the r-iterated diagonal map $\Delta_{X,r}\colon X \to X^r$, we know that, up to a choice of basepoint,

$$(e_r^X)_*(\pi_1(X^{J_r})) = \Delta_{\pi,r}.$$

Therefore, we know that there exist basepoints $x \in X^{J_r}$ and $\hat{x} \in \widehat{X^r}$ such that

$$(e_r^X)_*(\pi_1(X^{J_r}, x)) = q_*(\pi_1(\widehat{X^r}, \hat{x}))$$

and by the lifting criterion for coverings there exists a map $h\colon X^{J_r} \to \widehat{X^r}$ lifting the fibration e_r^X. As such, we can construct a commutative diagram of the form

$$\begin{array}{ccc} X^{J_r} & \overset{h}{\dashrightarrow} & \widehat{X^r} \\ {\scriptstyle e_r^X}\big\downarrow & & \big\downarrow{\scriptstyle q} \\ X^r & \xrightarrow{=} & X^r. \end{array}$$

As a consequence of statement (d) in Theorem 3.2.8, one observes that

$$\mathrm{TC}_r(X) \overset{(3.2.3)}{=} \mathrm{secat}(e_r^X) \geq \mathrm{secat}(q) = \mathcal{A}\text{-genus}(\widetilde{X^r})$$

where the last equality is a consequence of Theorem 6.2.1 for the family $\mathcal{A} = \left\{ \pi^r \Big/ \Delta_{\pi,r} \right\}$.

(b) Assume now that $X = K(\pi, 1)$. Given the homotopy equivalence $X^{J_r} \simeq X$, we have that X^{J_r} is an aspherical space. Under the hypothesis, we also know that the connected cover space $\widehat{X^r}$ is aspherical as well and, by the isomorphism induced at the level of fundamental groups, the map h becomes an homotopy equivalence between $X \simeq X^{J_r}$ and $\widehat{X^r}$. Therefore we obtain the chain of equalities

$$\mathrm{TC}_r(X) = \mathrm{secat}(\Delta_r) = \mathrm{secat}(q \circ h) = \mathrm{secat}(q) = \mathcal{A}\text{-genus}(\widetilde{X^r})$$

which finishes the proof.

□

Remark 6.2.5. If instead of using the fibrational substitute of the diagonal map we would want to see the proof of Proposition 6.2.4 directly from the map $p_r\colon PX \to X^r$ as in Definition 3.2.15, we can argue as follows. There is a well-defined continuous map

$$\phi_X : PX \to \widehat{X^r}, \quad \phi(\gamma) = \rho(\tilde{\gamma}(0), \tilde{\gamma}(\tfrac{1}{r-1}), \ldots, \tilde{\gamma}(\tfrac{r-2}{r-1}), \tilde{\gamma}(1)),$$

where $\tilde{\gamma}$ denotes a lift of γ to $\widetilde{X}$. (This map was first studied in the case of $r = 2$ in [59, Theorem 4.1].) Then the following diagram commutes:

$$\begin{array}{ccc} PX & \xrightarrow{\quad\phi_X\quad} & \widehat{X^r} \\ & {\scriptstyle p_r}\searrow \qquad \swarrow{\scriptstyle q} & \\ & X^r & \end{array}$$

where $q\colon \widehat{X} \to X^r$ is again the connected cover associated to the diagonal subgroup $\Delta_{\pi,r}$. If X is aspherical, then ϕ_K is a fibre homotopy equivalence since both PX and $\widehat{X^r}$ are of type $K(\pi, 1)$ and both of their fundamental groups map isomorphically onto $\Delta_{\pi,r} \subset \pi^r$. Since ϕ_X commutes with the two fibrations, it follows from Dold's theorem, see [96, Section 7.5] that ϕ_X is a fibre homotopy equivalence, and so $\mathrm{secat}(p_r) = \mathrm{secat}(q)$, and we can then conclude the proof as above.

6.2.1 Consequences of the characterization

The characterization of Theorem 6.2.1 allows as well to derive new bounds for $\mathrm{secat}(H \hookrightarrow G)$ in terms of genus of classifying spaces for subgroup families and Bredon cohomological dimension.

Proposition 6.2.6. *Let G be a torsion-free group, $H \leqslant G$ and $\mathcal{A} = \{G/H\}$.*

(a) Given $\mathcal{F}$ a full family of subgroups of G we have that

$$\mathrm{secat}(H \hookrightarrow G) \leq \mathcal{A}\text{-genus}(E_{\mathcal{F}}(G)).$$

(b) For any subgroup $K \leqslant G$ subconjugate to H such that $\mathrm{cd}_{\langle K\rangle}G \geq 3$ we have

$$\mathrm{secat}(H \hookrightarrow G) \leq \mathrm{cd}_{\langle K\rangle}G.$$

(c) Under the hypothesis of (b), if $K \trianglelefteq G$ then

$$\mathrm{secat}(H \hookrightarrow G) \leq \mathrm{cd}(G/K).$$

Proof. (a) By the universal property of the classifying space with respect to a full family of subgroups, for any such family $\mathcal{F}$ there exists a G-equivariant map

$$\rho\colon EG \to E_{\mathcal{F}}(G).$$

By virtue of monotonicity of $\mathcal{A}$-genus (property (b) of Proposition 6.1.3) and Theorem 6.2.4, this yields

$$\mathrm{secat}(H \hookrightarrow G) = \mathcal{A}\text{-genus}(EG) \leq \mathcal{A}\text{-genus}(E_{\mathcal{F}}(G)).$$

(b) Let $K \leqslant G$ subconjugated to H. By the generalized Eilenberg-Ganea theorem for families, Theorem 2.3.12, there exists an n-dimensional model of the space $E_{\langle K\rangle}(G)$. As the infinite join of coset spaces is a model for the classifying space with respect to the family $\langle H\rangle$, recall (2.3.3), the equivariant Whitehead Theorem (see Theorem 2.3.3) implies that there exists a G-equivariant map

$$EG \to *^{n+1}(G/K).$$

But as the subgroup K is taken as subconjugated to H, the maps $G/K \to G/H$ induce G-equivariant maps at the level of joins, thus there exists a G-equivariant map

$$EG \to *^{n+1}(G/K) \to *^{n+1}(G/H).$$

The claim now follows from the characterization of $\mathcal{A}$-genus in terms of the join, property (a) of Proposition 6.1.3.

(c) Finally, under the hypothesis of statement (b), assume further that the subgroup is normal, $K \trianglelefteq G$. Then, Proposition 4.2.15 informs us that $E(G/K)$ is in fact a model for $E_{\langle K\rangle}(G)$. Using (b) above we conclude the inequality

$$\mathrm{secat}(H \hookrightarrow G) \leq \mathrm{cd}_{\langle K\rangle} G = \mathrm{cd}(G/K).$$

□

Remark 6.2.7. For any $r \in \mathbb{N}$ with $r \geq 2$, whenever we specialize Proposition 6.2.6 to the case of the iterated diagonal inclusion $\Delta_{\pi,r} \hookrightarrow \pi^r$, and $K \leqslant \pi^r$ subconjugated to $\Delta_{r,\pi}$, we obtain the following new bounds for $\mathrm{TC}_r(\pi)$

(a) $\mathrm{TC}_r(\pi) \leq \mathcal{A}\text{-genus}(E_{\mathcal{F}}(\pi^r))$.

(b) $\mathrm{TC}(\pi) \leq \mathrm{cd}_{\langle K\rangle} \pi^r$.

(c) $\mathrm{TC}_r(\pi) \leq \mathrm{cd}(\pi^r/K)$ if $K \trianglelefteq \pi^r$.

Observe that, if in statement (b) we take $H = \Delta_{r,\pi} \cong \pi$, we easily recover the upper bound from [58, Corollary 3.2].

Moreover, consider a central subgroup $H \leqslant \pi$. Its image through the iterated r-diagonal $\Delta_r(H) \leqslant \pi^r$ is a normal subgroup of π^r. Under these assumptions, statement (c) of Proposition 6.2.6 recovers the well-known result from M. Grant, [69, Proposition 3.7] for $r = 2$, and gives a generalization for $r > 2$.

It is interesting to note that the usual zero-divisors cup length cohomological lower bound can be recovered from the full generality of the properties of the length. Recall that the Borel equivariant cohomology of a G-space X is defined by

$$H^*_G(X; M) = H^*(EG \times_G X; M).$$

Now take h^* as the Borel equivariant cohomology theory, and $I = h^*(*) \simeq H^*(B\pi^r)$. For any subgroup $H \leq \pi^r$ we have $h^*(\pi^r/H) \simeq H^*(BH)$. Hence, for the iterated diagonal inclusion $\Delta_{\pi,r} \hookrightarrow \pi^r$ this becomes

$$I \cap \ker[h^*(*) \to h^*(\pi^r/\Delta_{\pi,r})] = \ker[H^*(B\pi^r) \to H^*(B\Delta_{\pi,r})],$$

and the induced map $p^*_{E\pi^r}$ is an isomorphism.

An immediate but interesting consequence of Theorem 6.2.1, Proposition 6.2.4 and property (d) of Proposition 6.1.3 is the following bound for sectional category and sequential topological complexities with respect to subgroups of a given group.

Corollary 6.2.8. *Let π be a free torsion group with subgroups $H, K \leqslant \pi$, and $J \leqslant H$. The following inequality holds*

$$\mathrm{secat}(J \hookrightarrow H) \leq (\mathrm{secat}(K \hookrightarrow \pi) + 1)(\mathcal{B}\text{-genus}((\pi/K)) + 1) - 1$$

for $\mathcal{B} = \{H/J\}$.

In particular, for the specific case of the iterated diagonal inclusions $\Delta_{H,r} \hookrightarrow H^r$ and $\Delta_{\pi,r} \hookrightarrow \pi^r$ the above inequality yields

$$\mathrm{TC}_r(H) \leq (\mathrm{TC}_r(\pi) + 1)(\mathcal{B}\text{-genus}(\pi^r/\Delta_{\pi,r}) + 1) - 1$$

where $\mathcal{B} = \{H^r/\Delta_{H,r}\}$ and $r \in \mathbb{N}$ with $r \geq 2$.

Proof. For the first part of the statement, consider $\mathcal{A} = \pi/K$ and $\mathcal{B} = \{H/J\}$. By Theorem 6.2.1 we have the identifications

$$\mathrm{secat}(K \hookrightarrow \pi) = \mathcal{A}\text{-genus}(E\pi), \qquad \mathrm{secat}(J \hookrightarrow H) = \mathcal{B}\text{-genus}(E\pi)$$

which, combined with Proposition 6.1.3 (d), shows the claim.

For the second part, if we define $\mathcal{A} = \{\pi^r/\Delta_{\pi,r}\}$ and $\mathcal{B} = \{H^r/\Delta_{H,r}\}$, Proposition 6.2.4 (2) gives the characterizations

$$\mathrm{TC}_r(\pi) = \mathcal{A}\text{-genus}(E\pi), \qquad \mathrm{TC}_r(H) = \mathcal{B}\text{-genus}(E\pi)$$

hence the result follows once again from Proposition 6.1.3 (d). □

Notice that the most significant aspect of this bound is that it gives us a explicit computable measure of how much the sequential topological complexities of groups fail to be monotone under subgroup inclusions. In particular, we can exploit such measure to explore new examples of lower bounds with respect to subgroups. Here we present the case of the semidirect product of groups.

Corollary 6.2.9. *Let H and K be torsion free groups. Then we have* $\mathrm{TC}_r(H \rtimes K) \geq \mathrm{TC}_r(K)$

Proof. Put $G := H \rtimes K$. As the semidirect product G is given by a group extension of the form

$$\{1\} \to H \to G = H \rtimes K \xrightarrow{p} K \to \{1\}$$

and $K \leqslant G$, we have an induced K-equivariant map

$$G^r \Big/ \Delta_{G,r} \xrightarrow{(p\times\cdots\times p)} K^r \Big/ \Delta_{K,r} .$$

Corollary 6.2.8 gives us the following inequality

$$\mathrm{TC}_r(K) \leq (\mathrm{TC}_r(G) + 1)(\mathcal{B}\text{-genus}(G^k/\Delta_{G,r}) + 1) - 1$$

for the family $\mathcal{B} = \{(K^r)/\Delta_{K,r}\}$. But statements (b) and (c) of Proposition 6.1.3 indicate that

$$\mathcal{B}\text{-genus}\left(G^r \Big/ \Delta_{G,r}\right) \leq \mathcal{B}\text{-genus}\left(K^r \Big/ \Delta_{K,r}\right) = 0$$

so the claim follows. □

Compare our result with the alternative lower bound by the cohomological dimension given by M. Grant, G. Lupton and J. Oprea, [71, Corollary 1.3] for $r = 2$.

For any subgroup $H \leqslant \pi \times \pi$ containing the diagonal subgroup Δ_π consider the family $\mathcal{B} = \{(\pi \times \pi)/H\}$. Given that the natural projection $(\pi \times \pi)/\Delta_\pi \to (\pi \times \pi)/H$ is a $(\pi \times \pi)$-equivariant map, we see that $\mathcal{B}\text{-genus}((\pi \times \pi)/\Delta_\pi) = 0$. Then, by Proposition 6.1.3(d) we get the inequality:

$$\mathcal{B}\text{-genus}(E(\pi \times \pi)) \leq \mathcal{A}\text{-genus}(E(\pi \times \pi)) = \mathrm{TC}(\pi).$$

Now, observe that there is a correspondence between subgroups of $\pi \times \pi$ containing the diagonal subgroup Δ_π and normal subgroups of π. To see this, first assume $\Delta_\pi \leqslant H \leqslant \pi \times \pi$, and define the subgroup of π determined by

$$K := \{x \mid (1, x) \in H\}.$$

This is clearly a normal subgroup, since

$$(g, g)(1, x)(g^{-1}, g^{-1}) = (1, gxg^{-1}) \qquad \forall (g, g) \in \pi \times \pi$$

and the left side of the equality is in H, since H is an overgroup of the diagonal subgroup Δ_π. Now, if $K \trianglelefteq \pi$, we can easily define its associated overgroup of Δ_π by

$$H_K := \{(g, h) \mid g^{-1}h \in K\}.$$

From here, notice that, since $\mathcal{B}\text{-genus}(E(\pi \times \pi)) = \mathrm{secat}(H \hookrightarrow \pi \times \pi)$ by Corollary 6.2.2 and repeated application of Proposition 6.1.3 (d) we get a descending sequence of bounds

$$0 \leq \cdots \leq \mathrm{secat}(H_{K_{i+1}} \hookrightarrow \pi \times \pi) \leq \mathrm{secat}(H_{K_i} \hookrightarrow \pi \times \pi) \leq \cdots \leq \mathrm{TC}(\pi \times \pi)$$

where, for each index $i \geq 0$, the group H_{K_i} is the overgroup of Δ_π associated to the normal subgroup $K_i \trianglelefteq \pi \times \pi$ for an ascending chain of normal subgroups of $\pi \times \pi$ of the form

$$\Delta_\pi = K_0 \leqslant K_1 \leqslant \cdots \leqslant K_i \leqslant K_{i+1} \leqslant \cdots \leqslant \pi \times \pi$$

6.2.2 Thoughts on proper genus and proper topological complexity of groups

We will finish this chapter with some very quick reflections on $\mathcal{A}$-genus and topological complexity for proper actions of groups, that might be of interest for future work on the matter. Let G an arbitrary discrete group, this time possibly with torsion. Define the family of finite subgroups of G, denoted $\mathcal{F}in := \{H \leqslant G \mid |H| < \infty\}$. The classifying space with respect to such family is known as the *classifying space for proper actions* of G, and it is usually denoted by $\underline{E}G$. Its orbit space with respect to the natural G-action $\underline{E}G/G$ is denoted by $\underline{B}G$. This space receives the name of *classifying space for proper G-bundles*, as it classifies proper G-bundles in an analogous way in which BG classifies principal G-bundles (as shown in [10]). Clearly, $\dim_G(\underline{E}G) = 0$ if and only if G is finite. In case G is torsion free, it is also evident that $\underline{E}G \simeq EG$. So the truly interesting case for us is whenever G is a group with torsion. As we know, for such groups the topological complexity is, indeed, infinite. However, there is a powerful realization result in the spirit of Kan-Thurston involving the classifying space for proper actions, due to I. Leary and B. Nucinkis.

Proposition 6.2.10 (Theorem 1′ of [85])**.** *For any CW-complex X there exists a group G_X such that $\underline{B}G_X$ is homotopy equivalent to X. The group G_X has a torsion-free subgroup K_X of index two. If X is finite, there is a finite model for $\underline{B}K_X$.*

Consequentially, every connected CW-complex has the homotopy type of a $\underline{B}G$. Moreover, as it is shown in the proof of Proposition 6.2.10, the group is not torsion free except in the case that X is one-dimensional. As such, the previous result provides us with a wide range of cases in which $\underline{\mathrm{TC}}(G)$ is finite even though G has torsion. This result motivates our next definition.

Definition 6.2.11. Define the *G-proper topological complexity* of G, denoted by $\underline{\mathrm{TC}}(G)$, as $\underline{\mathrm{TC}}(G) = \mathrm{TC}(\underline{B}G)$.

It is important to distinguish this notion from the previously defined concept of proper topological complexity, which deals with different kind of information, and it is defined by means of proper homotopy theory techniques. The main interest of $\underline{\mathrm{TC}}(G)$ lies in the fact that it allows us to recover the usual notion of topological complexity of the group G whenever G is torsion free, but it also potentially provides more information in good enough cases when G has torsion but has good properties with respect its proper actions (in the sense of being realized by a finitely dimensional CW-complex by means of the procedure from Proposition 6.2.10).

Remark 6.2.12. As it usually happens in Mathematics, the problem lies within finding concrete examples. One of the possible approaches is through the usage of the nullification functor. Let A and X be topological spaces, we say that X is A-null if the mapping space $\mathrm{Map}(A, X)$ is homotopically equivalent to X via $X \to \mathrm{Map}(A, X)$, the inclusion of constant maps. We define the A-nullification of X as an endofunctor $\mathbb{P}_A : \mathrm{Spaces} \to \mathrm{Spaces}$ that takes every space X to a corresponding A-null space $\mathbb{P}_A(X)$ such that there exists a universal map

$X \to \mathbb{P}_A(X)$ which induces a weak homotopy equivalence $\mathrm{Map}(\mathbb{P}_A(X), Y) \simeq \mathrm{Map}(X, Y)$ for every A-null space Y. It is possible to see that any other A-null space U satisfying that property is homotopically equivalent to $\mathbb{P}_A(X)$. It is not within our objectives to develop further the machinery required or its properties, we refer the interested reader to [45] for a thoroughful treatment of these tools. Consider now the set of all prime numbers $P = \{p_1, p_2, \cdots\}$ and define the space $W_n = \bigvee_{i=1}^{n} B\mathbb{Z}_{p_i}$ and $W_\infty = \bigvee_P B\mathbb{Z}_p$, where p ranges over the set of all prime numbers. The interesting point is that for any G such that there exists a finite dimensional model for $\underline{B}G$, there is a homotopy equivalence $\underline{B}G \simeq \mathbb{P}_{W_\infty}(BG)$, see [62, Theorem 3.2]. Colourful examples through this technique can be found for some wallpaper groups (see [109] for precise definitions), thanks to the computations carried out by R. Flores in [62].

Naturally, we can always consider the corresponding $\mathcal{A}$-genus of an arbitrary discrete group G with respect to the family of its finite subgroups.

Definition 6.2.13. Let G an arbitrary group, and $\mathcal{F}in$ the closed family of finite subgroups. Define the *proper genus* of the group G by $\underline{\mathrm{genus}}(G) := \mathcal{F}in\text{-genus}(\underline{E}G)$.

In view of the characterization from Proposition 6.2.4, it is natural to ask whether the proper versions of topological complexity and genus coincide or not. Before adressing that matter, we will show a dimensional lower bound for the proper genus of a group.

Proposition 6.2.14. *Let G be a discrete group such that there is a finite dimensional model for $\underline{B}G$ satisfying $H^n(\underline{B}G; A) \neq 0$ for some $n \in \mathbb{N}$ and some coefficient system A. Then we have $\underline{\mathrm{genus}}(G) \geq n$.*

Proof. Let $\underline{\mathrm{genus}}(G) = k$. By Proposition 6.1.3 (a) there exists a G-equivariant map

$$f \colon \underline{E}G \to *_{i=0}^{k+1}(G/F_i)$$

where for each $i \in \{0, 1, \cdots, k+1\}$ the group $F_i \in \mathcal{F}in$. The join space $*_{i=0}^{k+1}(G/F_i)$ is naturally a proper G-CW complex, where the isotropy subgroup of each point $(t_0, a_0, t_1, a_1, \cdots, t_k, a_k)$ corresponds with the intersection group $\bigcap_{a_i} G_{a_i}$. By the universal property of $\underline{E}G$, there exists a G-equivariant map

$$g \colon *_{i=0}^{k+1}(G/F_i) \to \underline{E}G$$

satisfying that $g \circ f$ is G-homotopically equivalent to the identity. Passing to the quotient, this yields a homotopically commutative diagram of CW-complexes

$$\begin{array}{ccc} \underline{B}G & \xrightarrow{\overline{f}} & [*_{i=0}^{k+1}(G/F_i)]\big/G \\ & \searrow & \big\downarrow \overline{g} \\ & & \underline{B}G. \end{array}$$

Due to the fact that each coset space G/F_i is discrete for every $F_i \in \mathcal{F}in$, we observe that $\dim\left([*_{i=0}^{k+1}(G/F_i)]\big/G\right) \leq \dim\left(*_{i=0}^{k+1}(G/F_i)\right) = k$. But the induced composite map in cohomology

$$H^n(\underline{B}G; A) \xrightarrow{\overline{g}^*} H^n\left([*_{i=0}^{k+1}(G/F_i)]\big/G\,; A\right) \xrightarrow{\overline{f}^*} H^n(\underline{B}G; A)$$

is obviously an isomorphism, hence by dimensional reasons it must hold that $k \geq n$, which shows the claim. □

We conclude this chapter with a couple of examples of this lower bound.

Example 6.2.15. (a) Suppose a group G such that its classifying space for proper G-bundles $\underline{B}G$ is a finite dimensional CW-complex with the homotopy type of a n-dimensional sphere S^n. By Proposition 6.2.14, we have that $\underline{\text{genus}}(G) \geq n$. However,

$$\underline{\text{TC}}(G) = \text{TC}(\underline{B}G) = \text{TC}(S^n) = \begin{cases} 1 & n \text{ odd} \\ 2 & n \text{ even} \end{cases}$$

Through this example, we can conclude that the difference between $\underline{\text{genus}}(G)$ and $\underline{\text{TC}}(G)$ may be arbitrarily large.

(b) More generally, we can consider the case of virtual Poincaré duality groups. Recall that given a group G we say that G virtually satisfies a property if there exists a subgroup $H \leqslant G$ of finite index with such property. R. Kim proved a specialization of Proposition 6.2.10, see [82, Theorem 1, Corollary 2], stating that for any finite connected simplicial complex X, there exists a virtually torsion-free group G with $\underline{E}G$ a cocompact manifold such that $\underline{B}G$ is homotopy equivalent to X. Furthermore, he also shown that for any finite connected simplicial complex X, there is a virtual Poincaré duality group G such that $\underline{B}G$ is homotopy equivalent to X.

Thus, for a finite simplicial complex X with $H^n(X; A) \neq 0$ for some $n > 1$ and some coefficient system A, let H be a torsion free Poincaré duality subgroup of a group G, such that $|G : H| \leq \infty$, satisfying that $\underline{E}G$ is a cocompact manifold and $\underline{B}G \simeq X$, as in [82, Theorem 1, Corollary 2]. By the cocompacity of $\underline{E}G$ we know that $\underline{B}G$ has a finite dimensional model, thus by Proposition 6.2.14, we have $\underline{\text{genus}}(G) \geq n$.

Part III

Topological complexity and symmetry

CHAPTER 7

TC in presence of symmetries: on properties of effective topological complexity

Introduction

Since the inception of the concept of topological complexity, we have witnessed the emergence of several distinct variants, depending either on different versions of the motion planning problem (recall, for example, the sequential or parametrized topological complexities seen in previous chapters), or on specific kind of information from the motion planning that we might be interested in analyzing. One of those kinds of ”specific information" concerns the impact of the symmetries that often appear in the configuration spaces, and that one might want to take into account whenever studying the instability of the motion planning problem. Formally, those symmetries are seen as actions of groups on the base topological space X and, as such, this naturally leads to the inception of equivariant versions of topological complexity. There are several non-equivalent approaches to the matter, such as the *equivariant* topological complexity from H. Colman and M. Grant [33], the *strongly equivariant* TC developed by A. Dranishnikov in [41] as a variant of the former, the *invariant* TC of W. Lubawski and W. Marzantowicz introduced in [89], or the more recent notions of *effectual* topological complexity (by N. Cadavid-Aguilar, J. González, B. Gutiérrez and C. A. Ipanaque-Zapata, [23]) and *orbital* TC (defined by E. Balzer and E. Torres-Giese, [6]). We will briefly introduced their definitions in the next section, but without entering into many details.

However, these versions of equivariant topological complexity, while mathematically relevant on their own, do not take into account the possibility of “easing" the task of the motion planning through the use of the symmetries of the space. Consider, for example, the case of a robotic arm with two identical pliers, such as the represented in Figure 7.1.

Observe that, while both states are different, any object that has to be manipulated by the arm can be grabbed equally well in both cases. Situations equivalent to this one are

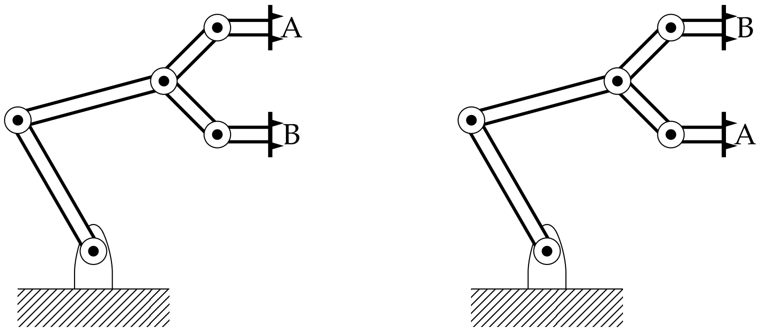


Figure 7.1: A mechanical arm in physically different, but functionally equivalent states, since grips A and B are indistinguishable.

extremely common in the world of mechanical systems. The problem is that the original approach to topological complexity does not take this sort of phenomena into account. And yet, the example above suggests that symmetries in configuration spaces can simplify the task of motion planning, given that, even though symmetric positions are physically different, they can be considered as functionally equivalent. Therefore every planning algorithm instructing a robot how to move between all possible states is a waste of effort, and it can be made easier if we take into account this functional equivalences. Given that most of the equivariant versions of TC do not exploit such potential simplifications, and in order to study this possibilites, Z. Błaszczyk and M. Kaluba introduced in [16] a new invariant, with precisely this foundational idea, which they baptised as *effective* topological complexity. In light of this notion, the effective motion planners considered in this context output paths that are tipically no longer continuous, but with discontinuities parametrized by the symmetries of the configuration space. As such, whenever a mechanical system follows such a path and runs into a point of discontinuity, it re-interprets its position accordingly within a batch of symmetric positions, and then resumes normal movement.

So far, the effective topological complexity remains a poorly understood variant of TC. The purpose of this chapter is to contribute to the understanding of said invariant, by investigating some of its properties. In particular, we will introduce and study the notion of effective Lusternik-Schnirelmann category, which will play the role of the usual LS-category in the effective setting. We will also study the relationship between effective LS-category and topological complexity with some possible properties of the orbit projection map with respect to the group action, giving plenty of examples of computations and bounds derived from our results. The chapter will conclude with some dimensional conditions for the non-vanishing of effective topological complexity at stage two of compact G-ANR spaces with finite group actions.

The contents of this chapter are featured in [15].

7.1 Equivariant notions of topological complexity

Before we proceed to jump into the intricacies of the effective topological complexity, we will dedicate a section to give a small briefing of the main notions of equivariant topological complexities present in the literature. As our main interest lies essentially in the effective variant, we will not delve deeply into the details. We refer the reader interested in going deeper into the matter to the compact survey by A. Ángel and H. Colman [2], or to the more recent one by M. Grant [68].

The first version of topological complexity to appear (and also probably the most natural from a purely mathematical point of view) was the equivariant topological complexity, developed by H. Colman and M. Grant in [33]. If X is a G-space, there is a naturally induced action on the path space PX and on $X \times X$ defined by

$$G \times PX \to PX, \qquad G \times (X \times X) \to X \times X,$$

$$g(\gamma)(t) = g(\gamma(t)), \qquad g(x, y) = (gx, gy).$$

Under such actions, the endpoints evaluation map $\pi \colon PX \to X \times X$ is a G-fibration, and one can defines the equivariant topological complexity as follows.

Definition 7.1.1. The *equivariant* topological complexity of X (denoted by $\mathrm{TC}_G(X)$) is the least integer $n \geq 0$ such that there exists an open cover of $X \times X$ by G-invariant sets $\{U_0, U_1, \cdots, U_n\}$, where for each $0 \leq i \leq n$ there is a G-equivariant map

$$s_i \colon U_i \to PX$$

making the following diagram commutative

$$\begin{array}{ccc} & & PX \\ & \overset{s_i}{\nearrow} & \downarrow \pi \\ U_i, & \longrightarrow & X \times X \end{array}$$

Althoug this definition is probably the most natural for an equivariant version of topological complexity, it has some limitations. The main one from our interests in this chapter is that, instead of simplifying the motion planning, the introduction of the information coming from the symmetries complexifies the task, as one has the inequality

$$\mathrm{TC}(X) \leq \mathrm{TC}_G(X)$$

for any G-space X. Moreover, this inequality can not only be strict, but also arbitrary large.

Example 7.1.2 ([68], Example 3.8)**.** Let $\mathbb{Z}_2$ act on $S^1 \subseteq \mathbb{C}$ by complex conjugation. Then one has the following equalities

$$\mathrm{TC}_G(S^1) = \infty \qquad \mathrm{TC}(S^1) = 1.$$

Another significant drawback related to the previous one is that, contrary to what one would expect, the equivariant topological complexity does not necessarily coincide with the classical one for free G-spaces. One of the reasons explaining this phenomena is that the definition of $\mathrm{TC}_G(X)$ involves the diagonal action of G on the product space, while the definition of $\mathrm{TC}(X/G)$ involves mostly the quotient space of $X \times X$ by the action of $G \times G$.

In order to adress this issue, W. Lubawski and W. Marzantowicz, in [89], took a different approach. Define the fibered path space over the orbit space as

$$PX \times_{X/G} PX = \{(\alpha, \beta) \in PX \times PX \mid G\alpha(1) = G\beta(0)\}.$$

One can think of this as the space of broken paths in the configuration space X that are continuous except for one point, where they are allowed to jump to another point in their same orbit by the action of G. The product group $G \times G$ acts on $PX \times_{X/G} PX$ by putting $(g, h)(\alpha, \beta) = (g\alpha, h\beta)$. The natural projection map defined by

$$p_X \colon PX \times_{X/G} PX \to X \times X \qquad p_X(\alpha, \beta) = \{\alpha(0), \beta(1)\}$$

is a $(G \times G)$-fibration, under the component-wise action. This motivate the definition of the invariant topological complexity.

Definition 7.1.3. The *invariant* topological complexity of X (written $\mathrm{TC}^G(X)$) is the least integer $n \geq 0$ such that there exists an open cover of $X \times X$ by $(G \times G)$-invariant sets $\{U_0, U_1, \cdots, U_n\}$ where for each $0 \leq i \leq n$ there is a $(G \times G)$-equivariant local section

$$s_i \colon U_i \to PX \times_{X/G} PX$$

making the following diagram commutative

$$\begin{array}{ccc} & & PX \times_{X/G} PX \\ & \overset{s_i}{\nearrow} & \downarrow p_X \\ U_i, & \longrightarrow & X \times X \end{array}$$

It is quite straightforward from the definition that, as it was intended, the invariant topological complexity is bounded from below by the topological complexity of the orbit space, i.e.

$$\mathrm{TC}(X/G) \leq \mathrm{TC}^G(X)$$

for any G-space X. The following result summarizes some of the properties of both equivariant and invariant TC with respect to fixed points sets.

Proposition 7.1.4 ([33] Corollary 5.4 and [89] Corollary 3.26)**.** *For any subgroups $H, K \leqslant G$ one has*

$$\mathrm{TC}(X^H) \leq \mathrm{TC}_G(X) \qquad \mathrm{TC}_K(X) \leq \mathrm{TC}_G(X)$$

For any G-space X we have $\mathrm{TC}(X^G) \leq \mathrm{TC}^G(X)$

There is a natural version of LS-category in the equivariant setting, originally defined by Marzantowicz in [95], the so called Lusternik-Schnirelmann G-category, which is defined as follows.

Definition 7.1.5. For a G-space X we say that a G-invariant open subset $U \subseteq X$ is *G-categorical* if the inclusion $U \hookrightarrow X$ is G-homotopic to a G-equivariant map with values in a single orbit.

The *Lusternik-Schnirelmann G-category* of X, denoted by $\mathrm{cat}_G(X)$, is defined as the smallest integer $m \geq 0$ such that there exists an open cover of X by $m+1$ G-categorical open subsets.

Indeed, for both equivariant and invariant topological complexity, a lower bound in terms of LS G-category can be found, at least in cases where the set of fixed points is non-empty.

Proposition 7.1.6 (Proposition 5.7, Corollary 5.8 of [33] and Proposition 2.7 of [17])**.** *Let X be a G-space.*

(a) If X is G-connected, then $\mathrm{TC}_G(X) \leq \mathrm{cat}_G(X \times X)$.

(b) If X is completely nrmal, G-connected and with at least some fixed point $x \in X^G$, then

$$\mathrm{cat}_G(X) \leq \mathrm{TC}_G(X) \leq 2\mathrm{cat}_G(X).$$

(c) If X has a fixed point $x_0 \in X^G$ then $\mathrm{cat}_G(X) \leq \mathrm{TC}^G(X)$.

Despite the nice properties of invariant topological complexity, there is an obvious setback in its relationship with the topological complexity of the base G-space, coming from Proposition 7.1.4: $\mathrm{TC}^G(X)$ can be arbitrary larger than $\mathrm{TC}(X)$. In fact, there is no obvious bound relating $\mathrm{TC}^G(X)$ and $\mathrm{TC}(X)$, and both inequalities may occur. Consequently, it is not a valid tool for the specific purpose of reducing the complexity of the motion planning based on the symmetries of the configuration space.

We will conclude this section mentioning briefly three other equivariant versions of topological complexity.

Definition 7.1.7. Let X be a G-space. We define

(a) The *strongly equivariant* topological complexity of X, denoted by $\mathrm{TC}^*_G(X)$, as the smallest integer $n \geq 0$ such that there exists a $(G \times G)$-invariant open cover $\{U_0, U_1, \cdots, U_n\}$ of $X \times X$ (seen as a $(G \times G)$-space via the component-wise action) such that for each $0 \leq i \leq n$ there exists a G-equivariant local section $s_i \colon U_i \to PX$ making the following diagram commutative

$$\begin{array}{ccc} & & PX \\ & \overset{s_i}{\nearrow} & \downarrow{\scriptstyle \pi} \\ U_i, & \longrightarrow & X \times X \end{array}$$

(b) The *effectual topological complexity* of X as the sectional category

$$\mathrm{TC}^G_{\mathrm{effl}}(X) = \mathrm{secat}(\epsilon)$$

where ϵ denotes the composite map

$$\epsilon\colon PX \xrightarrow{\pi} X \times X \xrightarrow{\mathrm{id}\times\rho_X} X \times (X/G)$$

(c) The *orbital topological complexity* of X as

$$\mathrm{TC}^G_{\mathrm{orb}}(X) = \mathrm{secat}(PX \xrightarrow{\rho_X\circ\pi} X/G).$$

The strongly equivariant topological complexity was introduced by A. Dranishnikov in [41], as an alternative and more restrictive version of the equivariant topological complexity of Colman and Grant. His aim was, indeed, to produce new upper bounds of ordinary topological complexity, and clearly for any G-space X we have $\mathrm{TC}(X) \leq \mathrm{TC}_G(X) \leq \mathrm{TC}^*_G(X)$.

The effectual topological complexity was introduced by N. Cadavid-Aguilar, J. González, B. Gutiérrez and C.A. Ipanaque-Zapata in [23] in close relationship with the effective topological complexity, while the orbital TC developed by E. Balzer and E. Torres-Giese in [6] was developed as a variant of the effectual one. Both are indeed upper bounds for the effective topological complexity.

7.2 Effective Topological Complexity.

We will devote this section to provide a quick review of our main notion of interest, that of effective topological complexity. As such, We recall both its construction and the most useful properties that were first proved in [16].

Given a topological group G acting on a pointed CW-complex X, and $k \geq 1$ an integer, define the *k-broken path space* by

$$\mathcal{P}_k(X) = \{(\gamma_1, \cdots, \gamma_k) \in (PX)^k \mid G\gamma_i(1) = G\gamma_{i+1}(0) \text{ for } 1 \leq i \leq k\}.$$

This is obviously a generalization of the fibered path spaces over the orbit space from Lubawski and Marzantowicz, and they can be seen as the spaces of paths broken into k continuous components, with discontinuities consisting on controlled jumps between points in the same G-orbit. In particular, for dimensions one and two we have the obvious equalities

$$\mathcal{P}_1(X) = PX \qquad \mathcal{P}_2(X) = PX \times_{X/G} PX.$$

Denote by $\rho_X\colon X \to X/G$ the projection of X onto its orbit space, and by $\delta_X\colon \daleth(X) \to X \times X$ the inclusion of the saturated diagonal into $X \times X$. Recall that the saturated diagonal corresponds with the subset

$$\daleth(X) = \{(g_1x, g_2x) \in X \times X \mid g_1, g_2 \in G \text{ and } x \in X\}.$$

In section 7 we will see another characterization of the saturated diagonal by dividing it into "slices" indexed by elements of the group, that will come in handy in order to use Mayer-Vietoris to find bounds for its cohomological dimension.

Now we need to define the generalized path-space fibrations that encapsulate the desired information about the symmetries in the configuration space. Define the map $\pi_k\colon \mathcal{P}_k(X) \to X \times X$ by

$$\pi_k(\gamma_1, \cdots, \gamma_k) = (\gamma_1(0), \gamma_k(1)).$$

Indeed, this can be seen as a fibration in the following manner: If $\pi\colon PX \to X \times X$ denotes the pathspace fibration, take the restriction of the product fibration $(\pi)^k$ to the subspace $X \times \daleth(X)^{k-1} \times X \subseteq (X \times X)^k$. The outcome of this is, again, a fibration

$$p_k\colon \mathcal{P}_k \to X \times \daleth(X)^{k-1} \times X$$

fitting in a pullback diagram of the form

$$\begin{array}{ccc} \mathcal{P}_k(X) & \xrightarrow{\qquad\qquad\qquad} & (PX)^k \\ {\scriptstyle p_k}\big\downarrow & & \big\downarrow{\scriptstyle (\pi)^k} \\ X \times \daleth^{k-1}(X) \times X & \xrightarrow{\mathrm{id}_X \times (\delta_X)^{k-1} \times \mathrm{id}_X} & (X \times X)^k. \end{array}$$

As such, composing p_k with the projection onto the first and last factors, we obtain π_k.

Definition 7.2.1. With the notation above, define

- A (G,k)-*motion planner* on an open subset $U \subset X \times X$ is defined as local homotopy section of π_k over U, that is, a map $s\colon U \to \mathcal{P}_k(X)$ such that $\pi_k \circ s \simeq \mathrm{id}_U$.
- The *k-stage effective path space fibration* as the above defined map
$$\pi_k\colon \mathcal{P}_k(X) \to X \times X \qquad \pi_k(\alpha_1, \cdots, \alpha_k) = (\alpha_1(0), \alpha_k(1)).$$
- The *k-stage effective topological complexity*, denoted by $\mathrm{TC}^{G,k}(X)$, as the smallest integer $n \geq 0$ such that there exists an open cover of $X \times X$ by $n+1$ sets admiting (G,k)-motion planners. Equivalently,
$$\mathrm{TC}^{G,k}(X) = \mathrm{secat}(\pi_k).$$

The following lemma condenses some of the most basic properties of $\mathrm{TC}^{G,k}(X)$ introduced in [16]:

Lemma 7.2.2 ([16, Lemma 3.2, Theorem 3.3])**.** $\mathrm{TC}^{G,k}(X)$ *satisfies the following properties:*

(1) The following inequalities hold for any $k \geq 1$ and any subgroup $H \leq G$:

(a) $\mathrm{TC}^{G,k}(X) \leq \mathrm{TC}^{H,k}(X)$.

(b) $\mathrm{TC}^{G,k+1}(X) \leq \mathrm{TC}^{G,k}(X)$.

(2) If there exists a G-map $f\colon X \to Y$ and a map $g\colon Y \to X$ such that $f \circ g \simeq \mathrm{id}_Y$ then

$$\mathrm{TC}^{G,k}(Y) \leq \mathrm{TC}^{G,k}(X).$$

In particular, if X and Y are G-homotopically equivalent, then

$$\mathrm{TC}^{G,k}(X) = \mathrm{TC}^{G,k}(Y).$$

Definition 7.2.3. Let $k_0 \geq 1$ be the minimal integer such that $\mathrm{TC}^{G,k}(X) = \mathrm{TC}^{G,k+1}(X)$ for $k \geq k_0$. We define the *effective topological complexity* of X as

$$\mathrm{TC}^{G,\infty}(X) = \mathrm{TC}^{G,k_0}(X).$$

Equivalently

$$\mathrm{TC}^{G,\infty}(X) = \min\{\mathrm{TC}^{G,k}(X) \mid k \geq 1\}.$$

Notice here a striking difference of effective topological complexity: unlike the previously developed notions of equivariant or invariant TC, the effective version does not require the motion planners to be equivariant. In this setting, the symmetries are just employed, in light of Lemma 7.2.2, to reduce the complexity of the motion planning task, making sure that, no matter the action, it will never be more complex than the one given by the original topological complexity.

One of the main focus of the original paper from Błaszczyk and Kaluba revolves around providing a full study of the effective topological complexity of $\mathbb{Z}_p$ spheres of any dimension. As bedrock examples that come in handy in many situations, we summarize here the classification provided in [16]:

Proposition 7.2.4 ([16, Corollary 5.10])**.** *Let p be a prime. Suppose $\mathbb{Z}_p$ acting on S^n with an r-dimensional fixed point set, for $-1 \leq r \leq n-1$ ($r = -1$ meaning free action).*

- *If $p > 2$ then* $\mathrm{TC}^{\mathbb{Z}_p,\infty}(S^n) = \mathrm{TC}(S^n) = \begin{cases} 1 & \text{if } n \text{ is odd} \\ 2 & \text{if } n \text{ is even, } n > 0. \end{cases}$
- *If $p = 2$, then $\mathrm{TC}^{\mathbb{Z},\infty}(S^n)$ depends on r as indicated in the following table:*

<table>
<tr><th></th><th>O. PRESERVING</th><th>O. REVERSING</th></tr>
<tr><td>$r = -1$</td><td colspan="2">1</td></tr>
<tr><td rowspan="2">$0 \leq r \leq n-2$</td><td colspan="2">1
for n – odd</td></tr>
<tr><td>2
for n – even</td><td>1
for n – even, if linear</td></tr>
<tr><td>$r = n-1$</td><td>—
not possible</td><td>0
if linear</td></tr>
</table>

where recall that a group G acts linearly on a n-dimensional sphere S^n if there exists a real vector space V of dimension $n+1$, with S^n seen as the unit sphere of V, and with a linear action of G such that S^n is a G-invariant subspace of V.

Remark 7.2.5. In[23] Cadavid-Aguilar, González, Gutiérrez and Ipanaque-Zapata introduced a tweaked and alternative version of effective topological complexity, different to the original one defined above. They defined a variant of the k-broken path space, $Q_k^G(X)$, as the subspace of $(PX \times G)^{k-1} \times PX$ consisting of the tuples $(\alpha_1, g_1, \cdots, \alpha_{k-1}, g_{k-1}, \alpha_k)$ such that $\alpha_i(1) \cdot g_i = \alpha_{i+1}(0)$. Defining the G-twisted evaluation map

$$\begin{aligned} \epsilon_k \colon Q_k^G &\longrightarrow X \times X \\ (\alpha_1, g_1, \cdots, g_{k-1}, \alpha_k) &\mapsto (\alpha_1(0), \alpha_k(1)) \end{aligned}$$

their alternative notion of k-effective topological complexity is defined as

$$\mathrm{TC}^{G,k}_{\mathrm{effv}}(X) = \mathrm{secat}(\epsilon_k).$$

As it becomes apparent at first glance, the space $Q_k^G(X)$ is designed to encode the precise leaping that assembles a broken path, and this constitutes the main difference with the original notion. Indeed,

$$\mathrm{TC}^{G,k}_{\mathrm{effv}}(X) = \mathrm{TC}^{G,k+1}_{\mathrm{effv}}(X)$$

for every $k \geq 2$ and moreover

$$\mathrm{TC}^{G,k}(X) \leq \mathrm{TC}^{G,k}_{\mathrm{effv}}(X). \tag{7.2.1}$$

The main advantage of this approach lies in their significant simplification with respect to the original definition. However, while conceptually interesting on its own, this alternative vision of effective TC does not adress the foundational idea of reducing to the minimum possible the complexity of the motion planning problem through the use of its symmetries, save for nicely enough cases, as inequality 7.2.1 shows. In fact, this simplification might be a tad excessive for some purposes, as the original notion of effective TC may fall non-trivially below dimension 2 (we illustrate a basic example later on, see Proposition 7.7.7). Consequently, we think that the further development of the original notion continues to be a worthwhile enterprise.

It is also interesting to remark that, based upon this alternative notion of effective topological complexity, Balzer and Torres-Giese introduced in [6] the sequential version of effective TC in the sense of [23], which coincides, at dimension two, with $\mathrm{TC}^G_{\mathrm{effv}}$.

7.3 The global effective path space

We will start our study on the properties of the effective topological complexity by taking a look in this section at the broken path spaces themselves. In particular, we define a notion of "final" or global effective path space, encompassing the information of all the broken path spaces for each stage $k \geq 0$.

Consider, for each integer $k \geq 0$, the inclusion $\mathcal{P}_k(X) \overset{\iota_k}{\hookrightarrow} \mathcal{P}_{k+1}(X)$ defined by

$$\iota_0(x) = c_x \in PX$$

and, for every $k > 0$

$$\iota_k((\gamma_1, \cdots, \gamma_k)) = (\gamma_1, \cdots, \gamma_k, c_{\gamma_k(1)}) \in \mathcal{P}_{k+1}(X).$$

As one could easily expect from the definition, these inclusions are well behaved.

Lemma 7.3.1. *For each integer $k \geq 0$, the inclusion $\mathcal{P}_k(X) \overset{\iota_k}{\hookrightarrow} \mathcal{P}_{k+1}(X)$ embeds the broken path space $\mathcal{P}_k(X)$ as a closed subspace of $\mathcal{P}_{k+1}(X)$.*

Proof. The case $k = 0$ is straightforward. For any $k \geq 1$ define the projection map that sends any broken path in $\mathcal{P}_{k+1}(X)$ to its last component as a path in PX, i.e.

$$\varphi_{k+1} \colon \mathcal{P}_{k+1}(X) \to PX \qquad \varphi_{k+1}(\gamma_1, \cdots, \gamma_{k+1}) = \gamma_{k+1}.$$

It is immediate to check that this is a well defined continuous map, and such that

$$\phi_{k+1} = p_{k+1}|_{\mathcal{P}_{k+1}(X)}$$

where $p_{k+1} \colon (PX)^{k+1} \to PX$ is just the obvious projection map into the $(k+1)$-coordinate. Now, consider the obvious inclusion $i \colon X \to PX$ given by $i(x) := c_x$. Given that X is taken to be a Hausdorff space, PX is Hausdorff as well and, as we assumed X to be compact, $i(X)$ is compact in PX and, therefore, closed. Observe that

$$\iota_k(\mathcal{P}_{k-1}(X)) = \varphi_{k+1}^{-1}(i(X))$$

and thus the claim follows from the continuity of φ_{k+1}. □

Definition 7.3.2. We define the *global effective path space*, denoted by $\mathcal{P}_\infty(X)$ as $\operatorname{colim} \mathcal{P}_k(X)$ with respect to the chain of inclusions

$$X \overset{\iota_0}{\hookrightarrow} PX \overset{\iota_1}{\hookrightarrow} \mathcal{P}_2(X) \overset{\iota_2}{\hookrightarrow} \cdots \mathcal{P}_k(X) \overset{\iota_k}{\hookrightarrow} \mathcal{P}_{k+1}(X) \overset{\iota_{k+1}}{\hookrightarrow} \cdots$$

endowed with the final (colimit) topology.

It is clear that we can visualize the building at each stage of the global effective path space through the chain of inclusions considered above as a sort of "cellular attachment", in the following sense: for each integer $n \geq 0$, the broken path space at stage n can be seen as fitting the pushout diagram

$$\begin{array}{ccc} \overline{\mathcal{P}_{n-1}(X)} & \xrightarrow{\varphi_{n-1}} & \mathcal{P}_{n-1}(X) \\ \big\downarrow & & \big\downarrow \iota_{n-1} \\ \mathcal{P}_n(X) \setminus \mathcal{P}^\circ_{n-1}(X) & \longrightarrow & \mathcal{P}_n(X) \end{array}$$

where $\mathcal{P}_{-1}(X) = \emptyset$ and obviously $\mathcal{P}_0(X) \setminus \mathcal{P}^\circ_{-1}(X) = \mathcal{P}_0(X) = X$.

Given that the global effective path space $\mathcal{P}_\infty(X)$ is endowed with the weak (colimit) topology, any subset $U \subset \mathcal{P}_\infty(X)$ is open (respectively closed) if and only if, for every $n \geq 0$, the intersection $U \cap \mathcal{P}_n(X)$ is open (respectively closed).

Proposition 7.3.3. *Let $S \colon F \to \mathcal{P}_\infty(X)$ be a continuous map. If F is both Hausdorff and compact, then S factors through $\mathcal{P}_n(X)$ for some integer n.*

Proof. Take the image $S(F) \subset \mathcal{P}_\infty(X)$ as a compact subset. Let us assume that the statement is false. As such, there exists an infinite subset of integers $\mathcal{J}$ such that the intersection $S(F) \bigcap (\mathcal{P}_k(X) \setminus \mathcal{P}_{k-1}(X))$ is non-empty for every $k \in \mathcal{J}$. Now, for each $k \in \mathcal{J}$, take exactly one distinguished element of such intersection

$$x_k \in S(F) \bigcap (\mathcal{P}_k(X) \setminus \mathcal{P}_{k-1}(X))$$

which defines a sequence of integers $J = \{x_k\}_{k \in \mathcal{J}}$ of infinite length, such that that $x_n \neq x_m$ if $n \neq m$ by its definition.

Given that $J \subset \mathcal{P}_\infty(X)$ with induced topology, any subset of J is open (alternatively closed) in $\mathcal{P}_\infty(X)$ if and only if its intersection $J \cap \mathcal{P}_k(X)$ is open/closed in $\mathcal{P}_k(X)$ for each $k \geq 0$. Now, for each $x_k \in J$ notice that

$$\{x_k\} \cap \mathcal{P}_r(X) = \begin{cases} \varnothing & \text{if } r < k \\ \{x_k\} & \text{if } r \geq k. \end{cases}$$

It is clear that $\mathcal{P}_r(X)$ and $\mathcal{P}_\infty(X)$ are Hausdorff spaces, given that PX is Hausdorff, and $\mathcal{P}_r(X)$ and $\mathcal{P}_\infty(X)$ can be seen as a subspaces of (in)finite products of copies of PX. As such, the one point set $\{x_k\}$ is closed. For any $r \in \mathcal{J}$ the intersection $J \cap \mathcal{P}_r(X) = \{x_{k_i}\}_{k_i \in \mathcal{J}}$ for $k_i \leq r$, and therefore $J \cap \mathcal{P}_r(X)$ is expressible as a finite union of closed subsets for every r, hence is closed. This implies that J is a closed subset of $\mathcal{P}_\infty(X)$. Now, it is straightforward to show that every subset of J is closed, and consequently J is equipped with the discrete topology which, by the hypothesis of compacity, contradicts the assumption. □

Consider now each broken path space $\mathcal{P}_k(X)$ as a G^k-space via the component-wise action. The space $X \times X$ has a natural structure as a $(G \times G)$-space, but it can be seen also as a G^k-space via precomposition of the $(G \times G)$-action with the projection $G^k \to G \times G$ onto the first and last coordinates. In this manner $\pi_k \colon \mathcal{P}_k(X) \to X \times X$ becomes a G^k-equivariant map, and one obtains the following commutative diagram.

$$\begin{array}{ccc} \mathcal{P}_k(X) & \xrightarrow{\quad \pi_k \quad} & X \times X \\ \downarrow & & \downarrow \\ \mathcal{P}_k(X)/G^k & \longrightarrow & X/G \times X/G \\ & \searrow \qquad \nearrow & \\ & P(X/G) & \end{array}$$

Here the vertical maps are orbit projections, the lower horizontal map is induced by π_k, the oblique map on the left is the concatenation of a sequence of k paths in X/G, and the oblique map on the right is the path space fibration for X/G. Composing the orbit projection for $\mathcal{P}_k(X)$ and the concatenation of paths in $\mathcal{P}_k(X)/G^k$, we get the obvious commutative diagram

$$\begin{array}{ccc} \mathcal{P}_k(X) & \longrightarrow & X \times X \\ {\scriptstyle \theta_k}\big\downarrow & & \big\downarrow{\scriptstyle \rho_{X\times X}} \\ P(X/G) & \longrightarrow & X/G \times X/G. \end{array}$$

Proposition 7.3.4. *Let X be a G space and suppose that, for some $k > 0$, there exists a continuous map*

$$\overline{s_k}\colon P(X/G) \to \mathcal{P}_k(X)$$

such that $\theta_k \circ \overline{s_k} = \mathrm{id}_{P(X/G)}$. Then $\mathrm{TC}^{G,k+2}(X) \leq \mathrm{TC}(X/G)$ and therefore

$$\mathrm{TC}^{G,\infty}(X) \leq \mathrm{TC}(X/G).$$

Proof. Let $n := \mathrm{TC}(X/G)$. Consider $\{V_i\}_{0\leq i\leq n}$ with $V_i \subset X/G \times X/G$ and $s_i\colon V_i \to P(X/G)$ a local section for the path space fibration $\pi\colon P(X/G) \to X/G \times X/G$ for every $0 \leq i \leq n$.

Define for each $0 \leq i \leq n$ the open set $U_i = (\rho_X \times \rho_X)^{-1}(V_i)$. The map $\overline{s_k}$ restricted to U_i is of the form $\overline{s_k}(\overline{\gamma}) = (\gamma_1, \cdots, \gamma_k)$ with the obvious condition $G\gamma_j(1) = G\gamma_{j+1}(0)$ for every $0 \leq j \leq k$. Now define, for each U_i, a map

$$\xi_i\colon U_i \to \mathcal{P}_{k+2}(X) \qquad \xi_i(x,y) = (c_x, \bar{s}_k(\overline{\gamma}), c_y)$$

for $\overline{\gamma}(0) = [x]$ and $\overline{\gamma}(1) = [y]$. It is immediate from its definition that $\pi_k \circ \xi_i = \mathrm{id}_{U_i}$ and thus $\{U_i\}_{0\leq i\leq n}$ constitutes a categorical cover for $\mathrm{TC}^{G,k+2}(X)$. Consequently

$$\mathrm{TC}^{G,k+2}(X) \leq n = \mathrm{TC}(X/G).$$

□

Remark 7.3.5. It is important to note that the section $\overline{s_k}$ assumed before is not necessarily induced by a section $s\colon X/G \to X$ of the orbit map ρ_X. We will see that, if such a section s exists, then Proposition 7.3.4 is just an immediate consequence of Theorem 7.6.1, that we will state and prove in a later section.

7.4 Effective LS-category

In most cases topological complexity is a significantly difficult invariant to compute, one for which no general systematic way of calculation exists. One of the possible approaches to give estimates for TC relies in its well known bounds by Lusternik-Schnirelmann category which is, in most cases, an easier invariant to compute, and it is generally better understood than its counterpart.

As we discussed in Section 7.1, there is a natural version of LS-category in the equivariant setting, the Lusternik-Schnirelmann G-category. Indeed, as we saw for both equivariant and invariant topological complexity, a lower bound in terms of LS G-category can be found, at least in cases where the set of fixed points is non-empty, see Proposition 7.1.6. However, as pointed out by Z. Błaszczyk and M. Kaluba in [16, Section 7], such lower bound is not possible for effective topological complexity. Moreover, they noted that this impossibility do not stem from the particular definition of the invariant, but rather from the philosophy behind it, i.e. such bound would be impossible to accomplish for any other homotopy invariant $\mathcal{TC}$ with the property $\mathcal{TC}(X) \leq \mathrm{TC}^{G,\infty}(X)$. In fact, such an anomalous behaviour in the effective setting is hardly surprising. After all, unlike the cases of (strongly) equivariant and invariant TC, the effective motion planners are not required to be equivariant.

Given the additional layer of difficulty that the effective topological complexity carries, it is only natural to ponder whether a category lower bound can be laid down in the effective setting. The unfeasibility of the LS G-category points out to the necessity of considering a new candidate, an analogue of usual LS-category for the effective setting. In this section we will fill such void, and we will develop a notion of effective Lusternik-Schnirelmann category, which we will show that behaves analogously in the effective setting as the classic LS-category does in the classic one.

Recall that, given a fibration $f\colon X \to Y$, property (1) in Theorem 3.2.8 gives the upper bound $\mathrm{cat}(Y) \geq \mathrm{secat}(f)$. Let $x_0 \in X$ such that P_*X is the space of paths starting at x_0, and consider the inclusion $X \hookrightarrow X \times X$ by $x \mapsto (x_0, x)$. There is an obvious pullback diagram of the form

$$\begin{array}{ccc} P_*X & \longrightarrow & PX \\ {\scriptstyle \mathrm{ev}_1}\downarrow & & \downarrow{\scriptstyle \pi} \\ X & \hookrightarrow & X \times X \end{array}$$

and taking into account that $\mathrm{cat}(X) = \mathrm{secat}(\mathrm{ev}_1)$, we have the classic chain of inequalities relating category and TC

$$\mathrm{cat}(X) \leq \mathrm{TC}(X) \leq \mathrm{cat}(X \times X) \leq 2\mathrm{cat}(X). \tag{7.4.1}$$

We can further generalize the previous pullback diagram considering an analogous pullback diagram associated, for each $k > 0$, to the k-effective fibration π_k :

$$\begin{array}{ccc} P_*^k(X) & \longrightarrow & \mathcal{P}_k(X) \\ {\scriptstyle q_k}\downarrow & & \downarrow{\scriptstyle \pi_k} \\ X & \hookrightarrow & X \times X. \end{array}$$

In this way we obtain a fibration $q_k\colon P_*^k(X) \to X$ as a pullback of π_k by the inclusion of $X \hookrightarrow X \times X$. Those are, precisely, the fibrations that encode in the effective setting the relationship analogous to the one the usual LS-category had with the standard TC. Thus, the definition comes naturally.

Definition 7.4.1. For an integer $k \geq 1$ we define the k-effective Lusternik-Schnirelmann category of a G-space X as $\mathrm{cat}^{G,k}(X) = \mathrm{secat}(q_k)$. The *effective LS-category* of X, thus, is defined as

$$\mathrm{cat}^{G,\infty}(X) = \min\{\mathrm{cat}^{G,k}(X) \mid k \geq 1\}.$$

It is straightforward from the definition that $q_1 = \mathrm{ev}_1$, so $\mathrm{cat}^{G,1}(X) = \mathrm{cat}(X)$. Indeed, the classic chain of inequalities relating LS-category and topological complexity 7.4.1 can be generalised to the effective setting in a natural way:

Theorem 7.4.2. *For X a G-space, the following chain of inequalities holds:*

$$\mathrm{cat}^{G,\infty}(X) \leq \mathrm{TC}^{G,\infty}(X) \leq \mathrm{cat}^{G\times G,\infty}(X\times X) \leq 2\mathrm{cat}^{G,\infty}(X). \tag{7.4.2}$$

Proof. For the first inequality, observe that q_k is defined as a pullback fibration of the k-effective fibration π_k, so the inequality holds by (3) in Theorem 3.2.8.

To show the second inequality, first notice that we can immediately identify $P^k_*(X\times X) = P^k_*(X)\times P^k_*(X)$. Consider a categorical cover $\{U_j\}_{0\leq j\leq n}$ of $X\times X$ for q_k, take any of its open subsets $U_i \subset X\times X$ and a local section s_{U_j} of q_k over U, defined as

$$s_{U_j} := ((s_1,\cdots,s_k),(s'_1,\cdots s'_k))$$

where, for each $1\leq i\leq k$, the entries s_i and s'_i correspond with components of the local section to the i-th coordinate of $P^k_*(X)$ for each of the two copies of X in the cartesian product. Define now a map from U_i to the $(2k-1)$-broken path space

$$\sigma_{U_j}\colon U_j \to \mathcal{P}_{2k-1}(X) \qquad \sigma_{U_j}(x,y) = (s_k(x,y)^{-1},\cdots,s_1(x,y)^{-1} * s'_1(x,y),\cdots,s'_k(x,y))$$

where, for each index $1\leq i\leq k$, we denote by $s_i(x,y)^{-1}$ the path walked in reverse orientation, and $s_1(x,y) * s'_1(x,y)$ is just the corresponding concatenation of paths. One checks that this map determines a local section for the fibration π_{2k-1} over U_j for each of the possible choices of U_j in the categorical cover, hence

$$\mathrm{TC}^{G,\infty}(X) \leq \mathrm{cat}^{G,\infty}(X\times X).$$

Finally, the last inequality is just a consequence of property Theorem 3.2.8(4). □

From the definition and Theorem 3.2.8 it is obvious that, in analogy with the effective topological case

$$\mathrm{cat}^{G,\infty}(X) \leq \mathrm{cat}^{G,k}(X) \leq \mathrm{cat}(X). \tag{7.4.3}$$

As such, combining 7.4.3, Proposition 7.4.2 (and a consequence derived from it that we will make explicit in the next section, Corollary 7.5.4) and Theorem 7.2.4, we immediately derive the effective LS category of $\mathbb{Z}_p$-spheres.

Corollary 7.4.3. *For any prime p, suppose $\mathbb{Z}_p$ acting on S^n. Then $\mathrm{cat}^{\mathbb{Z}_p,\infty}(S^n) = 1$.*

Recall that by the LS-category of a map $f\colon X\to Y$ we understand the minimal number of open sets in a covering of X such that f is nullhomotopic over each one of them. It is not surprising that the category of the orbit map of X with respects to the G-action turns out to be a lower bound for the effective LS category of X:

Proposition 7.4.4. *Let X be a G-space, and $\rho_X\colon X\to X/G$ the orbit map with respect to the action of G. Then $\mathrm{cat}(\rho_X) \leq \mathrm{cat}^{G,\infty}(X)$.*

Proof. Define a map $\lambda\colon P^k_*(X) \to P_*(X/G)$ by composing each path component in $P^k_*(X)$ with the orbit map $\rho_X\colon X \to X/G$, and then concatenating the resulting paths in X/G in the order prescribed by their appearance in the k-tuple. Explicitly put

$$\begin{aligned} \lambda\colon P^k_*(X) &\longrightarrow P_*(X/G) \\ (\alpha_1, \cdots, \alpha_k) &\mapsto (\rho_X \circ \alpha_1) * \cdots * (\rho_X \circ \alpha_k). \end{aligned}$$

The map λ thus defined fits inside a commutative diagram of the form

$$\begin{array}{ccc} P^k_*(X) & \xrightarrow{\lambda} & P_*(X/G) \\ {\scriptstyle q_k}\downarrow & & \downarrow{\scriptstyle q_1} \\ X & \xrightarrow[\rho_X]{} & X/G. \end{array}$$

Consider now $\{U_i\}_{0\leq i\leq n}$ an effective categorical open cover of X for $\mathrm{cat}^{G,\infty}(X)$, and take for each index $0 \leq i \leq n$ a local section $s_i\colon U_i \to P^k_*(X)$ of q_k. Define for each $0 \leq i \leq n$ a map

$$H_i\colon U_i \times I \to X/G \qquad H(x,0) = (\lambda \circ s_i)(x)(0) \qquad H(x,1) = (\lambda \circ s_i)(x)(1) = \rho_{|U_i}(x).$$

Notice that H_i defines a a homotopy between $\rho_{X_{|U_i}}$, the restriction of the orbit map on U_i, and a constant map, making $\rho_{X_{|U_i}}$ nullhomotopic and therefore showing that $\{U_i\}_{0\leq i\leq n}$ is a categorical cover for $\mathrm{cat}(\rho_X)$, so it follows $\mathrm{cat}(\rho_X) \leq \mathrm{cat}^{G,\infty}(X)$. □

Let us discuss an example on how to make use of the notion of effective LS-category to bound effective topological complexity from below.

Example 7.4.5. Consider $\mathbb{C}P^n \times \mathbb{C}P^n$ with $\mathbb{Z}_2$ acting on the product by switch of coordinates. It is clear that the action is not free, but it can be turned into a free action in a standard way by applying the Borel construction. That way, by considering the orbit projection with respect to this induced free action we end up with a 2-fold covering projection map

$$\rho\colon \mathbb{C}P^n \times \mathbb{C}P^n \times S^\infty \to (\mathbb{C}P^n \times \mathbb{C}P^n \times S^\infty)/\mathbb{Z}_2.$$

The category of a map is bounded below by the cup length of its image in cohomology, (recall Proposition 3.2.12) so we have $\mathrm{cat}(\rho) \geq \mathrm{cl}_\mathbb{R}(\mathrm{Im}\,\rho^*)$. Recall that the real cohomology ring structure of $\mathbb{C}P^n$ corresponds with

$$H^*(\mathbb{C}P^n;\mathbb{R}) = \mathbb{R}[\alpha]/(\alpha^{n+1}) \qquad |\alpha| = 2.$$

If we denote by x and y the generators of the second cohomology group of $\mathbb{C}P^n \times \mathbb{C}P^n$ corresponding to the factors of the product then by [74, Proposition 3G.1] we have that $x + y \in \mathrm{Im}\rho^*$. Given that

$$(x+y)^{2n} = \binom{2n}{n} x^n y^n \neq 0$$

we obtain that $\mathrm{cl}_\mathbb{R}(\mathrm{Im}\rho^*) \geq 2n$, and so it follows that

$$\mathrm{TC}^{\mathbb{Z}_2,\infty}(\mathbb{C}P^n \times \mathbb{C}P^n) \geq \mathrm{cat}^{\mathbb{Z}_2,\infty}(\mathbb{C}P^n \times \mathbb{C}P^n) \geq \mathrm{cat}(\rho) \geq 2n.$$

7.5 The problem of $\mathrm{TC}^{G,\infty}(X) = 0$

It is well known since the inception of the whole theory, [53, Theorem 1] that the only spaces with topological complexity equal to zero are those which are contractible. Perhaps, it is not so surprising that, given the additional layer of complexity that is involved in the definition of the effective variant, such a basic case it is still unknown. In this section we will briefly discuss the situation, and also present counter-examples to certain proposed characterizations of $\mathrm{TC}^{G,\infty}(X) = 0$.

It is immediate to check from the definition that, by design, if X is a contractible or G-contractible space, then $\mathrm{TC}^{G,\infty}(X) = 0$. The converse, however, is not true, and an easy counterexample can be constructed by considering the computation of the $\mathbb{Z}_p$-spheres of Theorem 7.2.4:

Example 7.5.1. Consider the unit n-sphere S^n, for $n \geq 1$, equipped with a $\mathbb{Z}_2$ action by involution, which interchanges the two hemispheres and leaves the equator fixed. By Theorem 7.2.4 we have that $\mathrm{TC}^{\mathbb{Z}_2,\infty}(S^n) = 0$.

Despite the failure of this reciprocity, the condition that the effective topological complexity of a G-space is zero imposes a strong condition over orbit map with respect to the action, as the following proposition shows.

Proposition 7.5.2. *Let X be a G-space such that $\mathrm{TC}^{G,\infty}(X) = 0$. Then the orbit projection map $\rho_X\colon X \to X/G$ is nullhomotopic.*

Proof. Assume that $\mathrm{TC}^{G,\infty}(X) = 0$. Then there is an integer $k \geq 0$ such that there exists a global section of the k-effective fibration π_k, i.e. a map

$$s\colon X \times X \to \mathcal{P}_k(X) \qquad \pi_k \circ s \simeq \mathrm{id}_{X\times X}.$$

Defining the map

$$\zeta_k\colon X \times X \to P(X/G), \qquad \zeta_k := \theta_k \circ s,$$

we obtain the following commutative diagram

$$\begin{array}{ccc} & & P(X/G) \\ & \overset{\zeta_k}{\nearrow} & \downarrow \\ X \times X & \xrightarrow[\rho_X \times \rho_X]{} & X/G \times X/G. \end{array}$$

Therefore, for a choice of a distinguished point $x_0 \in X$, we can define a map

$$\begin{aligned} H\colon X \times I &\longrightarrow X/G \\ (x,t) &\mapsto \zeta_k(x_0,x)(t) \end{aligned}$$

which, evaluated at $t = 0$ and $t = 1$, gives the following values

$$H(x,0) = \zeta_k(x_0,x)(0) = \rho_X(x_0), \qquad H(x,1) = \zeta_k(x_0,x)(1) = \rho_X(x).$$

Hence, H defines a homotopy between the orbit map ρ_X and the constant path $c_{\rho_X(x_0)}$, and as a result ρ_X is seen to be nullhomotopic. $\square$

Unfortunately, the converse of the previous implication, again, does not hold in general. This time the counterexample is a little bit more elaborated, though:

Example 7.5.3. Consider the six-dimensional sphere, S^6, with $\mathbb{Z}_2$ acting on it via the antipodal action. Now, take the orbit projection map $\rho\colon S^6 \to \mathbb{R}P^6$. The eighth suspension of the orbit projection

$$\Sigma^8\colon \Sigma^8 S^6 \to \Sigma^8 \mathbb{R}P^6$$

coming from the antipodal action on S^6 can be seen to be nullhomotopic (by the work of E. Rees in his PhD thesis, see [106, Corollary 2]). However, $\Sigma^8 S^6$, equipped with the corresponding involution has a 7-dimensional fixed point set. By Theorem 7.2.4, we have that

$$\mathrm{TC}^{\mathbb{Z}_2,\infty}(\Sigma^8 S^6) = 1.$$

In the previous section, we made use of our definition of effective LS-category to generalize the classic bound of topological complexity in terms of Lusternik-Schnirelmann category, see Theorem 7.4.2. It is important to notice that one of the immediate consequences of such upper and lower bound indicates an alternative approach to the problem of determining the kind of G-spaces with effective topological complexity equal to zero.

Corollary 7.5.4. *If X is a G-space,* $\mathrm{cat}^{G,\infty}(X) = 0$ *if and only if* $\mathrm{TC}^{G,\infty}(X) = 0$.

7.6 Effective topological complexity and the orbit projection

It is only natural to ponder about the relationship between the effective topological complexity of a G-space and distinguished properties of the orbit projection map associated to the G-action. In this section, we will investigate the influence of two of such properties. First, we analyze the scenario where the orbit projection map is endowed with a strict section. After that, we consider the instance where the orbit map is a fibration. In both cases, plenty of examples of computations and bounds are given.

7.6.1 Orbit map has a strict section

In the circumstance that the orbit projection by the group action is equipped with a strict section $s\colon X/G \to X$, the effective framework gets significantly simplified. By using this section, one can lift all paths in X/G to paths in the base space X, and the effective LS category and topological complexity coincide with the corresponding non-effective ones of the orbit space.

Theorem 7.6.1. *Let X be a G-space. If the orbit map $\rho_X : X \to X/G$ has a strict section $s\colon X/G \to X$, the following holds:*

(1) $\mathrm{cat}^{G,\infty}(X) = \mathrm{cat}(X/G)$.

(2) $\mathrm{TC}^{G,\infty}(X) = \mathrm{TC}(X/G)$.

Proof. Let us prove the second claim. Start by considering an open cover $\{U_i\}_{0\leq i\leq n}$ with $U_i \subset X/G \times X/G$ and a local section $\sigma_i \colon U_i \to P(X/G)$ of π_1 for each $0 \leq i \leq n$. Now put $V_i := (\rho_X \times \rho_X)^{-1}(U_i)$ an open set in $X \times X$, and consider the map induced at the level of path spaces by the section s, i.e.

$$\begin{aligned} \overline{s} \colon P(X/G) &\longrightarrow PX \\ \overline{\gamma} &\longmapsto \overline{s}(\overline{\gamma})(t) = s(\overline{\gamma}(t)). \end{aligned}$$

Now we can define a local section of the effective fibration $\pi_3 \colon \mathcal{P}_3(X) \to X \times X$, denoted $\varsigma_i \colon V_i \to \mathcal{P}_3(X)$, by the expression

$$\varsigma_i(x, y) := (c_x, \overline{s}[\sigma_i([x], [y])], c_y).$$

This shows that $\mathrm{TC}^{G,\infty}(X) \leq \mathrm{TC}(X/G)$.

For the reverse inequality, let $n := \mathrm{TC}^{G,\infty}(X)$, and consider an open cover $\{V_i\}_{0\leq i\leq n}$ of $X \times X$, and $\varsigma_i \colon V_i \to \mathcal{P}_k(X)$ as a continuous local section for the effective fibration $\pi_k \colon \mathcal{P}_k(X) \to X \times X$ for some $k > 0$ realizing $\mathrm{TC}^{G,\infty}(X)$. Define

$$\overline{\rho_x} \colon \mathcal{P}_k(X) \to P(X/G)$$

as a map induced in $\mathcal{P}_k(X)$ by the orbit map, by projecting any k-broken path $\gamma = (\gamma_1, \cdots, \gamma_k) \in \mathcal{P}_k(X)$ through the orbit map, and concatenating, for each $1 < j < k$, the end point of $\rho_X(\gamma_j)$ with the initial point of $\rho_X(\gamma_{j+1})$, i.e.

$$\overline{\rho_X}(\gamma_1, \cdots, \gamma_k) = (\rho_X \circ \gamma_1) * \cdots * (\rho_X \circ \gamma_k).$$

Finally, observe that for each $0 \leq i \leq n$ the composite map

$$\xi_i := \overline{\rho_x} \circ \varsigma_i \circ (s \times s)$$

defines a local section of $\pi_1 \colon P(X/G) \to X/G \times X/G$ over $U_i := (s \times s)^{-1}(V_i)$, and so $\mathrm{TC}(X/G) \leq \mathrm{TC}^{G,k}(X)$.

With this approach in mind, the proof of 1. is, essentially, analogous. Start by considering $\{U_i\}_{0\leq i\leq m}$, a categorical open cover for $\mathrm{cat}(X/G)$. If we regard cat as a sectional category we have, for each $0 \leq i \leq m$, a local section $\sigma_i \colon U_i \to P(X/G)$. Define now, as above, $V_i = (\rho_X \times \rho_X)^{-1}(U_i)$ and a local section for q_2 by

$$\varsigma_i(x) := (\overline{s}[\sigma_i([x])], c_x).$$

This shows that $\mathrm{cat}^{G,2}(X) \leq \mathrm{cat}(X/G)$. For the reverse inequality, if we have an open cover $\{V_i\}_{0\leq i\leq m}$ and local sections $\varsigma_i \colon V_i \to P^k_*(X)$ of the fibration q_k for some k realizing $\mathrm{cat}^{G,\infty}(X)$ then, putting $U_i = (s \times s)^{-1}$, we can define a local section of $\mathrm{ev}_1 \colon P(X/G) \to X/G$ over U_i by

$$\xi_i := \lambda_k \circ \varsigma_i \circ s$$

where λ_k is as defined in the proof of Proposition 7.4.4. □

Let explore some examples of the theorem above:

Example 7.6.2. (1) As an immediate consequence we obtain that $\mathrm{TC}^{\mathbb{Z}_2,\infty}(S^n) = 0$ when the action is the flip (i.e. reflection interchanging the hemispheres and fixing the equator). Though this was computed in [16, Proposition 5.7], Theorem 7.6.1 provides a more general and conceptual explanation to it.

(2) Recall that the unitary group $\mathrm{U}(n)$ fits inside a split short exact sequence of groups of the form

$$\mathrm{SU}(n) \hookrightarrow \mathrm{U}(n) \to \mathrm{U}(1) \cong S^1.$$

Hence, by Theorem 7.6.1

$$\mathrm{TC}^{\mathrm{SU}(n),\infty}(\mathrm{U}(n)) = \mathrm{TC}(S^1) = 1.$$

(3) Recall that the special orthogonal group, denoted $SO(n)$, is the group of orthogonal matrices in the n-dimensional euclidean space with determinant equal to 1. The principal bundle

$$\mathrm{SO}(3) \to \mathrm{SO}(4) \to S^3$$

has a section, and consequently

$$\mathrm{TC}^{\mathrm{SO}(3),\infty}(\mathrm{SO}(4)) = \mathrm{TC}(S^3) = 1.$$

Later on we will see more applications of our results to more general cases of $\mathrm{SO}(n)$.

(4) As illustrated in the previous two cases, split Lie group extensions are a rich source of examples for G-spaces equipped with strict sections for their orbit map. Other instances of split exact sequences of groups in the spirit of the previous example can be obtained in the following manner: let $p > 2$ be a prime integer, $r \geq 1$ and define the central product

$$S(p^r, p^r) = \mathrm{SU}(p^r) \times_{\Gamma_{p^r}} \mathrm{SU}(p^r)$$

where Γ_{p^r} corresponds with the diagonal cyclic subgroup of the center of order p^r. Now one can make $\mathrm{SU}(p^r)$ act on $S(p^r, p^r)$ by left action on just the first coordinate of the central product. Under this action we obtain a principal bundle

$$\mathrm{SU}(p^r) \to S(p^r, p^r) \to \mathrm{PU}(p^r)$$

and such bundle has, indeed, a global section. Hence we get

$$\mathrm{TC}^{\mathrm{SU}(p^r),\infty}(S(p^r, p^r)) = \mathrm{cat}(\mathrm{PU}(p^r)) = 3(p^r - 1)$$

where the last equality was computed in [79].

(5) Let X be a based space, and G any group. Construct the space $Z = \vee_{g \in G} X_g$ defined by $X_g = X$, and equipped with a G-action given by $hx_g = x_{hg}$ for $x_g = x \in X_g$. Then $\mathrm{TC}^{G,\infty}(Z) = \mathrm{TC}(X)$.

The last case of the previous example allows us to give an easy realization result for effective topological complexity:

Corollary 7.6.3. *Let G any finite group and $n \geq 0$ a non-negative integer. Then there exists a G-space X such that $\mathrm{TC}^{G,\infty}(X) = n$.*

Proof. Consider a space Y with $\mathrm{TC}(Y) = n$ (an easy example is $Y = T^n$). Now, construct the space $X = \vee_{g \in G} Y_g$ defined in the same manner as in Example 7.6.2 (3) above. As a consequence of Theorem 7.6.1 we have that $\mathrm{TC}^{G,\infty}(X) = \mathrm{TC}(Y) = n$. □

7.6.2 Orbit map is a fibration

In this case, the situation has richer derivations, but requires a bit more subtlety. The equality obtained in the presence of a strict section is not always possible. However, we can collapse both effective LS category and topological complexity at stage 2, and bound them both by their corresponding non-effective counterparts of the orbit space by the group action, as the following theorem shows.

Theorem 7.6.4. *Let X be a G-space such that the orbit map $\rho_X\colon X \to X/G$ is a fibration. Then:*

(1) $\mathrm{cat}^{G,\infty}(X) = \mathrm{cat}^{G,2}(X) = \mathrm{cat}(\rho_X) \leq \mathrm{cat}(X/G)$.

(2) $\mathrm{TC}^{G,\infty}(X) = \mathrm{TC}^{G,2}(X) \leq \mathrm{TC}(X/G)$.

Proof. To prove (i), let $\{U_i\}_{0 \leq i \leq n}$ be a categorical open cover of X for $\mathrm{cat}(\rho_X)$. By the hypothesis of nullhomotopy of ρ_X over every U_i, it is possible to construct a family of homotopies of the form

$$H_i\colon U_i \times I \to X/G, \qquad H_i(x,0) = \rho_X(x_0), \qquad H_i(x,1) = \rho_X(x).$$

Since ρ_X is a fibration, by the homotopy lifting property H_i can be lifted through ρ_X to a homotopy $K_i\colon U_i \times I \to X$ satisfying

$$K_i(x,0) = x_0, \qquad \rho_X \circ K_i = H_i.$$

Define now, for every $0 \leq i \leq n$, a map

$$s_i\colon U_i \to P^2_*(X) \qquad \text{by } s_i(x) = (K_i(x,\cdot), c_x).$$

It is clear that s_i constitutes a local section for the fibration $q_2\colon P^2_*(X) \to X$ over U_i, and therefore $\mathrm{cat}^{G,2}(X) \leq \mathrm{cat}(\rho_X)$. By Proposition 7.4.4, this means that

$$\mathrm{cat}^{G,\infty}(X) = \mathrm{cat}^{G,2}(X) = \mathrm{cat}(\rho_X)$$

and the last inequality of the claim follows from usual properties of the category of a map.

To prove (ii), let $\{U_i\}_{0 \leq i \leq n}$ be an open cover of $X \times X$ such that there exists, for every $0 \leq i \leq n$, a local section $s_i\colon U_i \to \mathcal{P}_k(X)$ of the k-effective fibration π_k over U_i, for some k

such that $\mathrm{TC}^{G,k}(X) = \mathrm{TC}^{G,\infty}(X)$. Recall from the proof of Theorem 7.6.1 that one can define a map $\overline{\rho_X}\colon \mathcal{P}_k(X) \to X/G \times X/G$ induced by the orbit projection map ρ_X by

$$\overline{\rho_X}(\gamma_1, \cdots, \gamma_k) = (\rho_X \circ \gamma_1) * \cdots * (\rho_X \circ \gamma_k).$$

Through the map $\overline{\rho_X}$, every local section s_i defines a homotopy

$$H_i\colon U_i \times I \to X/G$$

by putting

$$H_i((x,y),0) = \overline{\rho_X}(s_i(x,y))(0) = \rho_X(x), \qquad H_i((x,y),1) = \overline{\rho_X}(s_i(x,y))(1) = \rho_X(y).$$

Since ρ_X is a fibration by hypothesis, we have a lifting for H_i, the homotopy

$$K_i\colon U_i \times I \to X$$

satisfying

$$K_i((x,y),0) = x, \qquad \rho_X \circ K_i = H_i.$$

Through this homotopy it is possible to define a local section $\sigma_i\colon U_i \to \mathcal{P}_2(X)$ of the effective fibration π_2 over U_i, by putting

$$\sigma_i(x,y) := (K_i((x,y),\cdot), c_y)$$

which shows that $\mathrm{TC}^{G,\infty}(X) = \mathrm{TC}^{G,2}(X)$.

For the last inequality, consider $\{V_i\}_{0\le i\le m}$ an open cover of $X/G \times X/G$ such that there exists, for each $0 \le i \le m$, a local section over V_i of the path space fibration $\pi\colon P(X/G) \to X/G \times X/G$. Define for each i a homotopy

$$P_i\colon V_i \times I \to X/G$$

satisfying $P_i(([x],[y]),0) = [x]$ and $P_i(([x],[y]),1) = [y]$ for each $([x],[y]) \in V_i$, and put $W_i := (\rho_X \times \rho_X)^{-1}(V_i)$. There are induced homotopies

$$W_i \times I \xrightarrow{(\rho_X\times\rho_X)\times \mathrm{id}_I} V_i \times I \xrightarrow{P_i} X/G$$

and, since ρ_X is a fibration, we can lift them to obtain new homotopies

$$Q_i\colon W_i \times I \to X$$

such that

$$\rho_X \circ Q_i = P_i \circ ((\rho_X \times \rho_X)_{|W_i} \times \mathrm{id}_I)$$

and consequently

$$Q_i((x,y),0) = x \qquad Q_i((x,y),1) = z \in [y].$$

Through this last family of homotopies, a local section $\lambda_i\colon W_i \to \mathcal{P}_2(X)$ for the effective fibration π_2 can then be defined by putting

$$\lambda_i(x,y) := (Q_i((x,y),\cdot), c_y)$$

thus $\mathrm{TC}^{G,2}(X) \le \mathrm{TC}(X/G)$. □

Whenever G is a discrete group acting properly discontinuously on X, Theorem 7.6.4 recovers the bound of effective topological complexity by $\mathrm{TC}(X/G)$ of [23, Theorem 1.1]. However, the situation is much more interesting if we are considering actions of compact Lie groups.

Example 7.6.5. (a) Under the identification of S^1 as the topological unitary group $U(1)$, we have a very well known fibre bundle

$$S^1 \hookrightarrow S^{2n+1} \to \mathbb{C}P^n.$$

As a consequence of (2) of Theorem 7.6.4, we see that

$$\mathrm{TC}^{S^1,\infty}(S^{2n+1}) \leq \mathrm{TC}(\mathbb{C}P^n) = 2n$$

(where the value of $\mathrm{TC}(\mathbb{C}P^n)$ was computed in [59], see Theorem 3.2.14 on Chapter 3). In this case, however, the bound provided by theorem is far from a sharp one. Notice that we can consider the subgroup inclusion $\mathbb{Z}_p \leqslant S^1$ and hence, by virtue of Lemma 7.2.2 and Theorem 7.2.4 one gets

$$\mathrm{TC}^{S^1,\infty}(S^{2n+1}) \leq \mathrm{TC}^{\mathbb{Z}_p,\infty}(S^{2n+1}) = 1$$

and, as a consequence of Proposition 7.5.2, $\mathrm{TC}^{S^1,\infty}(S^{2n+1}) = 1$.

Furthermore, we can take the principal bundle associated to the classifying space of $U(1)$,

$$S^1 \hookrightarrow E\mathrm{U}(1) \to B\mathrm{U}(1) \qquad \equiv \qquad S^1 \hookrightarrow S^\infty \to \mathbb{C}P^\infty$$

and in, this case, the contractibility of S^∞ implies that $\mathrm{TC}^{S^1,\infty}(S^\infty) = 0$.

(b) It is well known that the identification map (sometimes called "realification")

$$\phi\colon \mathbb{C}^{n\times n} \to \mathbb{R}^{2n\times 2n}$$

given by putting

$$C := A + iB \mapsto \begin{bmatrix} A & -B \\ B & A \end{bmatrix}$$

allows to identify the linear group $\mathrm{U}(n)$ as a subgroup of $\mathrm{SO}(2n)$. To be more specific, it can be shown that

$$\phi(\mathrm{U}(n)) = \mathrm{SO}(2n) \cap \phi(\mathrm{GL}(n, \mathbb{C})).$$

There is then a principal $\mathrm{U}(3)$-bundle

$$\mathrm{U}(3) \hookrightarrow \mathrm{SO}(6) \to \mathbb{C}P^3$$

which, in conjunction with Theorem 7.6.4 informs us that

$$\mathrm{TC}^{\mathrm{U}(3),\infty}(\mathrm{SO}(6)) \leq \mathrm{TC}(\mathbb{C}P^3) = 6.$$

(c) Think of the S^{2n+1} sphere immersed in the $(n+1)$-dimensional complex space $\mathbb{C}^{n+1}$. Recall that the map $T\colon \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ defined as the scalar multiplication by the p-th root of unity, (i.e. $T(z) = \exp(2\pi i/p)z$ for $z \in S^{2n+1}$) generates the standard complex representation of the cyclic group $\mathbb{Z}_p$. This induces a free action on S^{2n+1} under a complex unitary map of period p. The orbit space of such action is the well known lens space L_p^{2n+1}.

We can consider, in S^{2n+1}, the scalar multiplication of $z \in S^{2n+1}$ by $\exp(2\pi ix/p)$, where $x \in \mathbb{R}$. This operation defines a group homomorphism $g\colon \mathbb{R} \to \mathrm{Aut}(S^{2n+1})$ which commutes with the induced periodic map $T\colon S^{2n+1} \to S^{2n+1}$. Consequently, g induces an action of $\mathbb{R}$ on the lens space L_p^{2n+1} through an induced homomorphism $\overline{g}\colon \mathbb{R} \to \mathrm{Aut}(L_p^{2n+1})$. If we take an integer k, it is easy to see that $\exp(2\pi ik/p) = (\exp(2\pi i/p)^k)$ which informs us that the integers act trivially on L_p^{2n+1}. Therefore, the map $\overline{g}$ factors through the exponential map and it subsequently induce an action of S^1, regarded as the circular group, on the lens space L_p^{2n+1}, defined explicitely as

$$s \cdot [z] = [\exp(2\pi ix/p)z]$$

for $z \in S^{2n+1}$, $[z] \in L_p^{2n+1}$ and $s \in S^1$, $x \in \mathbb{R}$ such that $s = \exp(2\pi ix)$. Jaworowski, in [81], demonstrated that such an action is free and, furthermore, that the orbit space under it corresponds with the complex projective space $\mathbb{C}P^n$. Therefore, by Theorem 7.6.4 we see that

$$\mathrm{TC}^{S^1,\infty}(L_p^{2n+1}) \leq \mathrm{TC}(\mathbb{C}P^n) = 2n.$$

(d) Although the situation is significantly more complicated in the case of real projective spaces, we can still make use of the known topological complexity of $\mathbb{R}P^n$ for certain values of n to derive even more examples from Theorem 7.6.4. As it is discussed in [79, Section 4], we have the following principal bundles of compact Lie groups over real projective spaces:

$$\mathrm{Sp}(1) \to \mathrm{SO}(5) \to \mathbb{R}P^7, \qquad \mathrm{SU}(3) \to \mathrm{SO}(6) \to \mathbb{R}P^7,$$

$$G_2 \to \mathrm{SO}(7) \to \mathbb{R}P^{15}, \qquad \mathrm{Spin}(7) \to \mathrm{SO}(9) \to \mathbb{R}P^{15},$$

$$G_2 \to \mathrm{PO}(8) \to \mathbb{R}P^7 \times \mathbb{R}P^7.$$

Therefore, by Theorem 7.6.4 and the computation of the topological complexity of real projective spaces in dimension 7 and 15 carried out in [59], we obtain the inequalities:

$$\mathrm{TC}^{\mathrm{Sp}(1),\infty}(\mathrm{SO}(5)) \leq 7, \quad \mathrm{TC}^{\mathrm{SU}(3),\infty}(\mathrm{SO}(6)) \leq 7,$$

$$\mathrm{TC}^{G_2,\infty}(\mathrm{SO}(7)) \leq 23, \qquad \mathrm{TC}^{\mathrm{Spin}(7),\infty}(\mathrm{SO}(9)) \leq 2,$$

$$\mathrm{TC}^{G_2,\infty}(\mathrm{PO}(8)) \leq 14.$$

Let G be a matrix Lie group, and $H \leq G$ a closed subgroup. It is a well known fact that G has the structure of a fibre bundle

$$H \hookrightarrow G \xrightarrow{\rho} G/H$$

(see, for example [73, Proposition 13.8]). In particular, Theorem 7.6.4 produces very easy upper bounds for actions of closed matrix subgroups in their immediate matrix overgroup:

Corollary 7.6.6. *Let $n \in \mathbb{N}$. Then the following holds:*

(a) $\mathrm{TC}^{\mathrm{SO}(n-1),\infty}(\mathrm{SO}(n)) = 1$ *for n even and* $\mathrm{TC}^{\mathrm{SO}(n-1),\infty}(\mathrm{SO}(n)) \leq 2$ *for n odd.*

(b) For $n \geq 2$ we have $\mathrm{TC}^{\mathrm{U}(n-1),\infty}(\mathrm{U}(n)) = 1$.

(c) For $n \geq 3$, we have $\mathrm{TC}^{\mathrm{SU}(n-1),\infty}(\mathrm{SU}(n)) = 1$.

(d) For all $n \geq 1$ we have $\mathrm{TC}^{\mathrm{Sp}(n-1),\infty}(\mathrm{Sp}(n)) = 1$.

Proof. The statements have almost analogous proofs. All of them depend on the identification of the orbit maps with fibrations with base spheres of appropiate dimension (to see a proof of these facts see, for example, [73, Section 13.2]) and on the computation of the standard topological complexity of spheres (see [52] or the computations we carried in Chapter 2).

(a) $\mathrm{SO}(n-1)$ acting over $\mathrm{SO}(n)$ fits into a fibration

$$\mathrm{SO}(n-1) \hookrightarrow \mathrm{SO}(n) \xrightarrow{\rho} \mathrm{SO}(n)\big/\mathrm{SO}(n-1) \cong S^{n-1}.$$

By (2) of Theorem 7.6.4 we know that

$$\mathrm{TC}^{\mathrm{SO}(n-1),\infty}(\mathrm{SO}(n)) \leq \mathrm{TC}(S^{n-1}) = \begin{cases} 1 & \text{for n even} \\ 2 & \text{for n odd} \end{cases}$$

(b) The action of $\mathrm{U}(n-1)$ on its overgroup $\mathrm{U}(n)$ fits into a principal bundle

$$\mathrm{U}(n-1) \hookrightarrow \mathrm{U}(n) \to \mathrm{U}(n)\big/\mathrm{U}(n-1) \cong S^{2n-1}$$

which informs us, by virtue of (2) of Theorem 7.6.4, that

$$\mathrm{TC}^{\mathrm{U}(n-1),\infty}(\mathrm{U}(n)) \leq \mathrm{TC}(S^{2n-1}) = 1.$$

(c) The subgroup $\mathrm{SU}(n-1)$ acting over $\mathrm{SU}(n)$ fits into a fibration of the form

$$\mathrm{SU}(n-1) \hookrightarrow \mathrm{SU}(n) \xrightarrow{\rho} \mathrm{SU}(n)\big/\mathrm{SU}(n-1) \cong S^{2n-1}.$$

Once again, (2) of Theorem 7.6.4 implies that

$$\mathrm{TC}^{\mathrm{SU}(n-1),\infty}(\mathrm{SU}(n)) \leq \mathrm{TC}(S^{2n-1}) = 1.$$

(d) Finally, $\mathrm{Sp}(n-1)$ acting over $\mathrm{Sp}(n)$ makes the orbit map projection into a fibration

$$\mathrm{Sp}(n-1) \hookrightarrow \mathrm{Sp}(n) \xrightarrow{\rho} \mathrm{Sp}(n)\big/\mathrm{Sp}(n-1) \cong S^{4n-1}.$$

As previously, by (2) of Theorem 7.6.4

$$\mathrm{TC}^{\mathrm{Sp}(n-1),\infty}(\mathrm{Sp}(n)) \leq \mathrm{TC}(S^{4n-1}) = 1.$$

Finally note that, as a consequence of Proposition 7.5.2, the previously determined upper bounds by 1 are, indeed, sharp equalities. □

For any pair of numbers $n, k \in \mathbb{N}$, with $k < n$, we say that a (compact) *Stiefel* manifold over a field $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$, denoted by $V_k(\mathbb{F}^n)$, is the set of k-orthonormal tuples of vectors in $\mathbb{F}^n$, with the subspace topology in $\mathbb{F}^{n+k}$. Conversely, a k-Grassmannian over $\mathbb{F}^n$ is the set of all possible k-dimensional vector subspaces of $\mathbb{F}^n$. The group $O(k, \mathbb{F})$ acts freely on $V_k(\mathbb{F}^n)$, by rotating a k-frame in the space it spans. The orbits of this action are precisely the orthonormal k-frames spanning a given k-dimensional subspace, that is, the orbit map corresponds with a fibration (indeed a principal $O(k, \mathbb{F})$-bundle) of the form

$$O(k, \mathbb{F}) \hookrightarrow V_k(\mathbb{F}^n) \xrightarrow{\rho} G_k(\mathbb{F}^n).$$

If we specialize the concrete choice of the field, we obtain fibrations

$$O(k, \mathbb{R}) \hookrightarrow V_k(\mathbb{R}^n) \xrightarrow{\rho} G_k(\mathbb{R}^n) \qquad U(k, \mathbb{C}) \hookrightarrow V_k(\mathbb{C}^n) \xrightarrow{\rho} G_k(\mathbb{C}^n) \tag{7.6.1}$$

which allow us to give upper bounds for the effective topological complexity of orthogonal actions on Stiefel manifolds.

Corollary 7.6.7. *Let $k, n \in \mathbb{N}$ with $k < n$. Then the following bounds are satisfied:*

(a) $\mathrm{TC}^{O(k),\infty}(V_k(\mathbb{R}^n)) \leq 2k(n-k) - 1$

(b) $\mathrm{TC}^{U(k),\infty}(V_k(\mathbb{C}^n)) \leq 2k(n-k)$

Proof. As a consequence of the fibrations in 7.6.1 and (2) of Theorem 7.6.4, we know that

$$\mathrm{TC}^{O(k),\infty}(V_k(\mathbb{R}^n)) \leq \mathrm{TC}(G_k(\mathbb{R}^n)) \qquad \text{and} \qquad \mathrm{TC}^{U(k),\infty}(V_k(\mathbb{C}^n)) \leq \mathrm{TC}(G_k(\mathbb{C}^n)).$$

Then both claims follow from the upper bounds for the topological complexity of Grassmann manifolds computed by P. Pavešić in [105, Proposition 4.1, Theorem 4.2]. □

We will briefly recall the definition of more general orthonormal frame bundles defined over smooth manifolds, and we will apply our results to that setting. Let M be an n-dimensional (oriented) Riemann manifold, define, for every $x \in M$ the space

$$F_x(M) := \{(v_1, \cdots, v_n) \in (T_xM)^n \mid (v_1, \cdots, v_n) \text{ a positive orthonormal basis of } T_xM\}$$

and from it the *space of positive orthonormal frames* of M by putting

$$F(M) := \{(x, b) \mid x \in M, b \in F_x(M)\}.$$

Build the continuous map $p_{F(M)}\colon F(M) \to M$ by $p_{F(M)}(x,b) = x$. Then $p_{F(M)}$ has the structure of a smooth principal $\mathrm{SO}(n)$-bundle, called the bundle of positive orthonormal frames of M.

Corollary 7.6.8. *Let M a path-connected n-dimensional smooth manifold, and $p_{F(M)}\colon F(M) \to M$ defined as above. Then*

(a) $\mathrm{TC}^{\mathrm{SO}(n),\infty}(F(M)) \leq 2\dim(M)$.

(b) Furthermore, if M is parallelizable, then

$$\mathrm{TC}^{\mathrm{SO}(n),\infty}(F(M)) = \mathrm{TC}(M).$$

Proof. (a) Given that $p_{F(M)}$ is a principal bundle, we are under the assumptions of Theorem 7.6.4, hence

$$\mathrm{TC}^{\mathrm{SO}(n),\infty}(F(M)) \leq \mathrm{TC}(F(M)/\mathrm{SO}(n)) = \mathrm{TC}(M) \leq 2\dim(M)$$

where the last inequality just comes from the well-known dimensional bound of topological complexity, see Corollary 3.2.10.

(b) Under the hypothesis of M being parallelizable, $p_{F(M)}\colon F(M) \to M$ becomes a trivial $\mathrm{SO}(n)$-bundle, thus the claim follows from Theorem 7.6.1.

□

To check computations of usual topological complexity of orthonormal frame bundles we refer the readers to the analysis of S. Mescher on the matter, see [99].

Under nice enough group actions, the quotient space of a smooth manifold is itself a manifold with a smooth structure making the orbit map a fibration.

Corollary 7.6.9. *Let G be a Lie group acting smoothly, freely and properly on a connected smooth manifold M. Then*

$$\mathrm{TC}^{G,\infty}(M) \leq 2(\dim(M) - \dim(G)).$$

Proof. By the quotient manifold theorem (see [86, Theorem 21.10]) the orbit space M/G has the structure of a topological manifold with

$$\dim(M/G) = \dim(M) - \dim(G)$$

and with an unique smooth structure satisfying that the orbit map $\rho_M\colon M \to M/G$ is a smooth submersion. By the Ehresmann's fibration theorem (see [46, Theorem 8.5.10]) ρ_M is a (locally trivial) fibration, hence Theorem 7.6.4 gives us

$$\mathrm{TC}^{G,\infty}(M) = \mathrm{TC}^{G,2}(M) \leq \mathrm{TC}(M/G) \leq 2(\dim(M) - \dim(G))$$

where the last inequality just follows from the dimensional upper bound of TC. □

We can easily obtain the same inequality for locally smooth free actions though, unlike in the case above, we have to impose compacity as an additional restriction. Let us recall some terminology first. Let G be this time a compact Lie group, acting over a closed connected smooth manifold M. Since G is compact, we observe that, for any $x \in M$, the map $q_x \colon G/G_x \to G(x)$ given by $q_x(gG_X) = gx$ is a homeomorphism, and the orbit $G(x)$ is said to be of type G/G_x. We say that the G action is locally smooth if there is a linear tube $\varphi \colon G \times_H V \to M$ about every orbit of type G/H, where V is an orthogonal representation of the subgroup H. Locally smooth actions come equipped with principal orbits, i.e. orbits of type G/H such that H is subconjugated to any isotropy subgroup $G_x \leqslant G$. By virtue of [21, Theorem IV.3.8] we have that

$$\dim(M/G) = \dim(M) - \dim(P).$$

If, furthermore, the action of G is taken as free, the orbit projection map $\rho_M \colon M \to M/G$ becomes a fibration, hence Theorem 7.6.4 applies, and we have

$$\mathrm{TC}^{G,\infty}(M) \leq \mathrm{TC}(M/G) \leq 2\dim(M/G)$$

and, since $\dim(P) = \dim(G)$, we immediately get

$$\mathrm{TC}^{G,\infty}(M) \leq 2(\dim(M) - \dim(G)). \tag{7.6.2}$$

Compare both Corollary 7.6.9 and inequality 7.6.2 above with the upper bounds for usual topological complexity of smooth manifolds with locally smooth free actions obtained by M. Grant in [69].

It is possible to find conditions under which the effective LS category and the regular LS category coincide in this setting:

Corollary 7.6.10. *Let X be a G-space with basepoint $x_0 \in X$. If the orbit projection map $\rho_X \colon X \to X/G$ is a fibration and the orbit of the base point Gx_0 is contractible in X then*

$$\mathrm{cat}^{G,\infty}(X) = \mathrm{cat}(X).$$

Proof. Let $\{U_i\}_{0 \leq i \leq n}$ be a categorical cover for $\mathrm{cat}(\rho_X)$ and define, for each $U_i \subset X$ a homotopy $H_i \colon U_i \times I \to X/G$ such that

$$H_i(x,0) = \rho_X(x_0) \qquad \text{and} \qquad H_i(x,1) = \rho_X(x).$$

Given that ρ_X is a fibration, we can lift H_i to a homotopy $K_i \colon U_i \times I \to X$ satisfying

$$K_i(x,1) = x \qquad \text{and} \qquad \rho_X \circ K_i = H_i.$$

By the hypothesis of contractibility of the orbit of the basepoint, there is a continuous map $\theta \colon Gx_0 \to P_*X$ such that $\theta(x)(1) = x$ for all $x \in Gx_0$. Consequently, it is possible to define a section for the LS-cat fibration ev_1 over U_i as the concatenation

$$s_i(x) := \theta(K_i(x,0)) * H_i(x,\cdot)$$

which shows that $\mathrm{cat}(\rho_X) = \mathrm{cat}(X)$. The claim thus follows from the equality $\mathrm{cat}(\rho_X) = \mathrm{cat}^{G,\infty}(X)$ provided by (1) of Theorem 7.6.4. □

Naturally, the previous corollary is of particular interest in situations where we can assume the coincidence between LS-category and topological complexity, so the computation of effective topological complexity would be derived from the (generally easier) task of knowing the classic LS-category.

Corollary 7.6.11. *Let G be a finite group and X a free G-space such that* $\mathrm{cat}(X) = \mathrm{TC}(X)$. *Then* $\mathrm{TC}^{G,\infty}(X) = \mathrm{cat}(X)$.

Proof. The chain of inequalities in Theorem 7.4.2 shows that $\mathrm{cat}^{G,\infty}(X) \leq \mathrm{TC}^{G,\infty}(X) \leq \mathrm{TC}(X)$. By Corollary 7.6.10 above we know that $\mathrm{cat}^{G,\infty}(X) = \mathrm{cat}(X)$, and the claim follows from the hypothesis. □

We will close this subsection by mentioning some interesting examples of the previous corollaries.

Example 7.6.12. (1) Let G be a connected Lie group. In [51, Lemma 8.2], Farber proved the equality $\mathrm{TC}(G) = \mathrm{cat}(G)$. Therefore, if there exists a finite non-trivial discrete subgroup $H \leq G$, Corollary 7.6.11 implies that

$$\mathrm{TC}^{H,\infty}(G) = \mathrm{cat}(G).$$

Recall that, for example, if G is a non-nilpotent and simply connected group (such as groups of upper triangular matrices with diagonal terms equal to one) there always exists a non-trivial finite subgroup H.

(2) Generalizing Farber's result, Lupton and Scherer demonstrated in [90, Theorem 1] that, if X is a connected CW H-space, then $\mathrm{TC}(X) = \mathrm{cat}(X)$. Consequently, if G is a finite group acting freely on X, Corollary 7.6.11 applies and $\mathrm{TC}^{G,\infty}(X) = \mathrm{cat}(X)$.

(3) A particularly simple example comes from free products on spheres. Let G be a finite group acting freely on the k-dimensional torus

$$T^k = \underbrace{S^1 \times \cdots \times S^1}_{k}.$$

Then we have $\mathrm{TC}^{G,\infty}(T^k) = k$.

More generally, we can consider products of odd dimensional spheres, so if G acts freely on the product $\underbrace{S^{2n+1} \times \cdots \times S^{2n+1}}_{k}$ we obtain

$$\mathrm{TC}^{G,\infty}(\underbrace{S^{2n+1} \times \cdots \times S^{2n+1}}_{k}) = k$$

Remark 7.6.13. Notice that for a result in the spirit of Corollary 7.6.10, the hypothesis about the fibrational nature of the orbit projection map is crucial. Indeed, observe for example the case of a nilmanifold M with the characteristic transitive action of a nilpotent (contractible) Lie group G. The orbit map ρ_M has an obvious section (since the orbit space consists of a single element)

7.7 Cohomological conditions for non-vanishing of $\mathrm{TC}^{G,2}(X)$

One of the significant open problems suggested in the original article of Błaszczyk and Kaluba [16] concerned the determination of the kind of sequences that could arise as sequences of effective topological complexities. The problem is too broad and general, and it will most certainly require a specific in-depth inquiry on the matter, which goes beyond the scope of the present article. However, we will make a first contribution to the problem.

In this section we will study some cohomological conditions to determine whether or not the effective topological complexity at stage two vanishes. The set stage is not arbitrary by any means: such cohomological conditions are examined over the saturated diagonal, and we will make use of an homotopy equivalence between $\daleth(X)$ and the stage 2 broken path space $\mathcal{P}_2(X)$ to infer the aforementioned non-vanishing condition.

Let us start by noticing that the saturated diagonal $\daleth(X)$ can be easily represented as

$$\daleth(X) = \{(gx, x) \mid g \in G, x \in X\}.$$

The inclusion $\{(gx, x) \mid g \in G, x \in X\} \subset \daleth(X)$ is obvious, while for any pair $(g_1x, g_2x) \in \daleth(X)$ it is possible to define

$$(\overline{g}y, y) \in \{(gx, x) \mid g \in G, x \in X\}, \qquad \text{for } \overline{g} = g_1g_2^{-1} \text{ and } y = g_2x.$$

This, in turn, informs us that we can decompose $\daleth(X)$ as the union of "slices" of the saturated diagonal, i.e.

$$\daleth(X) = \bigcup_{g \in G} \daleth_g(X),$$

where, for each $g \in G$, we set

$$\daleth_g(X) := \{(gx, x) \mid x \in X\}.$$

This decomposition will be quite useful for the rest of our arguments throughout this section. However, before proceeding further, let us describe the homotopy equivalence between $\daleth(X)$ and the broken path space $\mathcal{P}_2(X)$.

Lemma 7.7.1. *Let X be a G space. There is a homotopy equivalence between $\daleth(X)$ and $\mathcal{P}_2(X)$.*

Proof. Start by noticing that there is an obvious inclusion

$$\iota\colon \daleth(X) \hookrightarrow \mathcal{P}_2(X) \qquad \iota(gx, x) = (c_{gx}, c_x).$$

Now, consider a map $f\colon \mathcal{P}_2(X) \to \daleth(X)$ defined as

$$f(\gamma_1, \gamma_2) = (\gamma_1(1), \gamma_2(0)).$$

It is immediate to see that the composition $f \circ \iota$ corresponds with $\mathrm{id}_{\daleth(X)}$. For the other composition, we obtain

$$(\gamma_1, \gamma_2) = \iota(\gamma_1(1), \gamma_2(0)) = (c_{\gamma_1(1)}, c_{\gamma_2(0)}).$$

Define the map $H\colon \mathcal{P}_2(X) \times I \to \mathcal{P}_2(X)$ by

$$H((\gamma_1, \gamma_2), 0) = (\gamma_1, \gamma_2), \qquad H((\gamma_1, \gamma_2), 1) = (c_{\gamma_1(1)}, c_{\gamma_2(0)}),$$
$$H((\gamma_1, \gamma_2), t) = (\gamma_1^t, \gamma_2^t) \qquad \forall 0 < t < 1$$

where for all $0 \le s \le 1$ we put

$$\gamma_1^t(s) = \gamma_1(t(1-s)+s), \qquad \gamma_2^t(s) = \gamma_2(s(1-t)).$$

One observes that said map defines a homotopy between $\iota \circ f$ and $\mathrm{id}_{\mathcal{P}_2(X)}$ and, as such, we have the homotopy equivalence $\urcorner(X) \simeq \mathcal{P}_2(X)$. □

Notice that the above lemma generalizes the homotopy equivalence between $\mathcal{P}_2(X)$ and $\urcorner(X)$ noted by Cadavid-Aguilar and González in [24] for finite free actions into arbitrary group actions.

Throughout the rest of this section, assume that G is a finite group, and X is a compact G-ANR. By [89, Theorem 3.15] this implies, in turn, that the saturated diagonal $\urcorner(X)$ becomes a $(G \times G)$-ANR, and we can apply the cohomological Mayer-Vietoris sequence for general subsets that are retractions of open subsets (check, for example, [74, Pag. 150]). Also recall that by the *cohomological dimension* of a space X we mean the largest integer $n \ge 0$ such that there exists a local coefficient system M satisfying $H^n(X; M) \neq 0$.

Lemma 7.7.2. *Let X be a G-CW complex such that $\mathrm{cd}(X^H) \le \mathrm{cd}(X)$ for all non-trivial subgroups $H \leqslant G$. Then, given any list L of non-trivial subgroups of G, we have*

$$\mathrm{cd}\left(\bigcup_{H \in L} X^H\right) < \mathrm{cd}(X) + |L| - 1.$$

Proof. We will proceed by induction. Consider the base case $|L| = 1$, then L consists of only one non-trivial subgroup H of G and hence $\mathrm{cd}(X^H) \le \mathrm{cd}(X)$ by the initial hypothesis.

Now assume that the claim is satisfied for any list of subgroups of cardinality $n-1$, and define $L := \{K_1, \cdots, K_{n-1}\} \cup \{H\}$ with $H, K_i \leqslant G$ for all $1 \le i \le n-1$. Define the sets

$$A := \bigcup_{K_i \in L} X^{K_i} \qquad \text{and} \qquad B := X^H.$$

Notice that the intersection corresponds to the following union of fixed point sets

$$A \cap B = \left(\bigcup_{K_i \in L} X^{K_i}\right) \cap X^H = \bigcup_{K_i \in L} (X^{K_i} \cap X^H) = \bigcup_{K_i \in L} X.^{\langle K_i, H\rangle}$$

By the induction hypothesis, we have the inequalities

$$\mathrm{cd}(A) < \mathrm{cd}(X) + n - 2 \qquad \mathrm{cd}(A \cap B) < \mathrm{cd}(X) + n - 2,$$

while we also have the inequality $\mathrm{cd}(B) < \mathrm{cd}(X)$ as a consequence of the initial hypothesis. Applying the Mayer-Vietoris sequence to the spaces just defined, and putting $d := \mathrm{cd}(X) + n - 2$, we obtain a sequence

$$\cdots \longrightarrow H^d(A\cup B;M) \longrightarrow H^d(A;M)\oplus H^d(B;M) \longrightarrow H^d(A\cap B;M) \longrightarrow$$
$$\longrightarrow H^{d+1}(A\cup B;M) \longrightarrow 0 \longrightarrow 0$$

where M is an arbitrary (possibly twisted) coefficient system and, by the cohomological dimensional bounds stated above, we have that $H^{d+1}(A\cup B;M)=0$, and thus we obtain that

$$\mathrm{cd}\left(\bigcup_{K_i\in L} X^{K_i}\right)\cup X^H) < \mathrm{cd}(X)+|L|-1.$$

□

The lemma above is instrumental, both in the result itself and in the argument of the proof, of the following bound of the cohomological dimension of the saturated diagonal.

In the same spirit as before, for any list of elements $L\subseteq G$, define the relative saturated diagonal with respect to L as

$$\daleth_L(X)=\bigcup_{g_i\in L}\daleth_{g_i}(X).$$

Theorem 7.7.3. *Let X be a G-CW complex such that $\mathrm{cd}(X^H)\le \mathrm{cd}(X)$ for all non-trivial subgroup $H\leqslant G$. Then, for any L list of elements of G,*

$$\mathrm{cd}(\daleth_L(X))\le \mathrm{cd}(X)+|L|-1.$$

In particular we have that

$$\mathrm{cd}(\daleth(X))\le \mathrm{cd}(X)+|G|-1.$$

Proof. The idea of this proof builds upon the argument used in the previous lemma, and we will proceed, once again, by induction. First assume we consider lists consisting on only one element. Then $\daleth_L(X)$ is just homeomorphic to X and, as such, $\mathrm{cd}(\daleth_L(X))=\mathrm{cd}(X)$.

Now, let us assume that the induction hypothesis is satisfied for any list of elements of G of length $n-1$. Define an arbitrary list of such length $L'=\{g_1,\cdots,g_{n-1}\}$, and let $L=L'\cup\{r\}$ for $G'=\{g_1,\cdots,g_{n-1}\}$ a subgroup of order $n-1$ and r an element of G not included in L'. Consider the decomposition

$$\daleth_L(X)=\bigcup_{k_i\in L}\daleth_{k_i}(X)=\left(\bigcup_{g_i\in L'}\daleth_{g_i}(X)\right)\cup\daleth_r(X).$$

Define now the sets

$$A:=\bigcup_{g_i\in L'}\daleth_{g_i}(X)\qquad B:=\daleth_r(X).$$

The intersection of these two subsets corresponds with the following set

$$A\cap B=\left(\bigcup_{g_i\in L'}\daleth_{g_i}(X)\right)\cap\daleth_r(X)=\bigcup_{g_i\in L'}\left(\daleth_{g_i}(X)\cap\daleth_r(X)\right).$$

For each $g_i \in L'$, the intersection $\urcorner_{g_i}(X) \cap \urcorner_r(X)$ is equivalent to the set

$$\{x \in X \mid (g_i x, x) = (rx, x)\},$$

which implies $r^{-1}g_i x = x$. Thus we can identify the intersection

$$\urcorner_{g_i}(X) \cap \urcorner_r(X) \cong X^{\langle r^{-1}g_i\rangle}$$

and, consequentially, the intersection above can be reformulated as an union of invariant sets of the form

$$A \cap B = \bigcup_{g_i \in L'} X^{\langle r^{-1}g_i\rangle}.$$

Define M as the collection of non-trivial subgroups of G

$$M := \{\langle r^{-1}g_1\rangle, \cdots, \langle r^{-1}g_{n-1}\rangle\}.$$

By Lemma 7.7.2 we know that $\mathrm{cd}(A \cap B) < \mathrm{cd}(X) + n - 2$. By the induction hypothesis one observes $\mathrm{cd}(A) < \mathrm{cd}(X) + n - 2$ and clearly $\mathrm{cd}(B) = \mathrm{cd}(X)$. Applying now the Mayer-Vietoris sequence as in Lemma 7.7.2 yields the exact sequence

$$\cdots \longrightarrow H^d(A \cup B; M) \longrightarrow H^d(A;M) \oplus H^d(B;M) \longrightarrow H^d(A \cap B; M) \longrightarrow H^{d+1}(A \cup B; M) \longrightarrow 0 \longrightarrow 0$$

and given the cohomological dimensional bounds stated above, we obtain

$$\mathrm{cd}(A \cup B) = \mathrm{cd}(\urcorner_L(X)) \leq \mathrm{cd}(X) + |L| - 1.$$

which gives us the desired result. □

As a consequence of the previous result, we can deduce a cohomological condition on the base space X for non-vanishing second stage effective topological complexity, reflected in the following corollary.

Corollary 7.7.4. *Under the assumptions of Theorem 7.7.3, we have that, if* $|G| \leq \mathrm{cd}(X)$, *then* $\mathrm{TC}^{G,2}(X) > 0$.

Proof. Consider the second effective fibration $\pi_2 \colon \mathcal{P}_2(X) \to X \times X$. This map induces an homomorphism in cohomology

$$H^*(X \times X; M) \xrightarrow{\pi_2^*} H^*(\mathcal{P}_2(X); M).$$

By the homotopy equivalence given in Lemma 7.7.1, this homomorphism can be seen as

$$H^*(X \times X; M) \to H^*(\urcorner(X); M)$$

and, by Proposition 7.7.3, $H^k(\urcorner(X); M) = 0$ for any $k > \mathrm{cd}(X) + |G| - 1$. However, $H^{2n}(X \times X) \neq 0$, which implies the existence of at least one non-trivial element in

$$\ker(H^*(X \times X; M) \to H^*(\urcorner(X)); M).$$

Thus, by (2) in Theorem 3.2.8, we have $\mathrm{secat}(\pi_2) = \mathrm{TC}^{G,2}(X) > 0$. □

Remark 7.7.5. In [24, Definition 7.3] the authors introduced the notion of *effective zero-divisors*. Namely, considering the inclusion of the saturated diagonal $\delta_X \colon \urcorner(X) \to X \times X$, we say that an effective zero-divisor is an element in the kernel of the induced map in cohomology

$$\delta_X^* \colon H^*(X \times X; R) \to H^*(\urcorner(X); R)$$

where cohomology is considered with arbitrary coefficients. As noted by Grant in [68], it is implicit in [24] that, if X is a free G-space with G a finite group, we have the following lower bound

$$\mathrm{TC}^{G,2}(X) = \mathrm{TC}^{G,\infty}(X) \geq \operatorname{nil}\ker(\delta_X^* \colon H^*(X \times X; R) \to H^*(\urcorner(X); R)).$$

Our Corollary 7.7.4 generalizes such lower bound (at stage two) to non necessarily free actions with prescribed cohomological dimensional bounds.

Example 7.7.6. Let us consider now the case of our space being a n-sphere (for $n > 1$) with a $\mathbb{Z}_2$-action by involution and codimension one fixed point set. $\mathbb{Z}_2$ acts on S^n by a reflection interchanging the hemispheres, and the action, as such, is linear. Adopt the notation $\mathbb{Z}_2 = \{1, g\}$, where 1 acts as the identity element. As above, take the split saturated diagonal $\urcorner(S^n)$ as the union of slices

$$\urcorner(S^n) = \urcorner_1(S^n) \bigcup \urcorner_g(S^n).$$

Similarly as before, the intersection $\urcorner_1(S^n) \bigcap \urcorner_g(S^n)$ corresponds with the set of elements of S^n such that $x = gx$, which is precisely $(S^n)^{\mathbb{Z}_2}$, the set of invariants by the action of $\mathbb{Z}_2$.

By Lemma 7.7.1, we know $\mathcal{P}_2(S^n)$ is homotopically equivalent to the saturated diagonal $\urcorner(S^n)$. Given that the dimension of the fixed point set $(S^n)^{\mathbb{Z}_2}$ is $n-1$ by the choice of the group action, we are under the hypothesis of Theorem 7.7.3 and thus, by Corollary 7.7.4, we have that $\mathrm{TC}^{\mathbb{Z}_2,2}(S^n) > 0$.

The idea of how to find the $(\mathbb{Z}_2, 2)$-motion planners is essentially analogous to [16, Proposition 5.6]. Let us recall it briefly. Define a homeomorphism $\tau \colon S^n \to S^n$ by

$$\tau(x_0, \cdots, x_n) := (-x_0, -x_2, x_1, \cdots, -x_n, x_{n-1}).$$

The two-fold motion planners are given over the open covering

$$U_1 = \{(x, y) \in S^n \times S^n \mid y \neq -x\},$$

$$U_2 = \{(x, y) \in S^n \times S^n \mid y \neq -\tau(x)\}$$

(where it is obvious that all points of the form $(x, -x)$ are indeed contained in U_2). The motion planner over U_1 is just $s_1(x, y) := s'(x, y)$, where $s'(x, y)$ denotes the shortest arc connecting two non-antipodal points x and y. Meanwhile, the motion planner $s_2 \colon U_2 \to \mathcal{P}_2(S^n)$ is defined by putting

$$s_2(x, y) := (c_x, s'(x, \tau(x)) * s'(\tau(x), y)) \qquad \forall (x, y) \in U_2.$$

But under this choice of action, by Theorem 7.2.4, we actually know that

$$\mathrm{TC}^{\mathbb{Z}_2,\infty}(S^n) = \mathrm{TC}^{\mathbb{Z}_2,3}(S^n) = 0.$$

Indeed, recall that a $(\mathbb{Z}_2, 3)$-motion planner over S^n can be defined, as shown in [16, Proposition 5.7], by

$$s(x,y) := (c_x, s'(\Gamma(x)x, N) * s'(N, \Gamma(y)y, c_y))$$

for any $x, y \in S^n$, where $N \in S^n$ denotes the north pole, and $\Gamma(x)$ is the trivial element of $\mathbb{Z}_2$ if $x \in S^n_+$, and its generator otherwise.

Notice that in the previous example we have just realized the following basic sequence for involutions on arbitrary spheres S^n, $n > 1$, with codimension one fixed point set.

Proposition 7.7.7. *Let $\mathbb{Z}_2$ act on S^n by involution, with fixed point set of codimension one. Then, the effective topological complexity sequence associated to S^n is*

$$\mathrm{TC}^{\mathbb{Z}_2,k}(S^n) = \begin{cases} 2, & k = 1, \\ 1, & k = 2, \\ 0, & k \geq 3. \end{cases}$$

Naturally, one of the clear objectives on the field would be fully determining all the sequences that appear as $\{\mathrm{TC}^{G,k}\}$-sequences. The investigation of the realization of more complex sequences, or on the generalization of the previously developed methods to higher k-stage effective topological complexities will be the subject of future work.

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List of Symbols

$*^k X$	k-join of X
$\mathcal{A}$-cat(X)	$\mathcal{A}$-category of X with respect to family $\mathcal{A}$
$\mathcal{A}$-genus(X)	$\mathcal{A}$-genus of X with respect to family $\mathcal{A}$
A^G	Invariants of A with respect of the action of G
BG	Classifying space of group G
$\underline{B}G$	Classifying space for proper G-bundles
$\mathbb{C}P^n$	Complex projective space of dimension n
(C_*, d_*)	Chain complex with differential d_*
(C^*, d^*)	Cochain complex with differential d^*
cat(X)	Lusternik-Schnirelmann category of X
cat$^{G,k}(X)$	k-stage Lusternik-Schnirelmann category of X
cat$^{G,\infty}(X)$	Effective Lusternik-Schnirelmann category of X
cd(G)	Cohomological dimension of G
cd(ϕ)	Cohomological dimension of group homomorphism ϕ
cd$[G : H]$	Adamson cohomological dimension of G with respect to H
cd$_{\mathcal{F}}X$	Bredon cohomological dimension of X with respect to the family $\mathcal{F}$
cd(X)	Cohomological dimension of X
dim(X)	(Geometric) dimension of X
$\Delta_{r,G}$	*r-th* iterated diagonal subgroup of G^r
$\daleth(X)$	Saturated diagonal of $X \times X$
Ext$^n_R(M, N)$	*n-th* extension group of M and N
Ext$^n_{(G,H)}(M, N)$	*n-th* relative extension group of M and N
EG	Total space of group G
$EG \times_G X$	Homotopy quotient of EG and X with respect to G
$E_{\mathcal{F}}G$	Classifying space of G with respect to family $\mathcal{F}$

$\underline{E}G$	Classifying space for proper actions of G
$\{E_r^{*,*}, d_r\}$	(Cohomological) spectral sequence
$E_r^{p,q}$	(p,q)-element in the r-th page of the spectral sequence
$\mathcal{F}in$	Family of finite subgroups
G_2	Exceptional Lie group G_2
G/H	Lateral cosets space of G with respect to H
$G \times_Q G$	Fiber product of G over Q
$\mathrm{gd}(G)$	Geometric dimension of group G
$\underline{\mathrm{genus}}(G)$	Proper genus of the group G
$\mathrm{GL}(n, \mathbb{F})$	General linear group of dimension n over $\mathbb{F}$
$G_k(\mathbb{F}^n)$	k-dim Grassmannian over $\mathbb{F}^n$
$H \leqslant G$	H subgroup of G
$H \trianglelefteq G$	H normal subgroup of G
$\langle H \rangle$	Family of subgroups generated by H
$H_n(C_*, d_*)$	n dimensional homology group of chain complex C.
$H^n(C^*, d^*)$	n dimensional cohomology group of chain complex C.
$H^n(G, A)$	n dimensional cohomology group of G with coefficients in A.
$H^n(G; A)$	n dimensional cohomology group of X with (local) coefficients in A.
$H^n_G(X; M)$	n dimensional Borel cohomology group of X with respect to G and coefficients in $\underline{M}$.
$H^n_{\mathcal{F}}(X, \underline{M})$	n dimensional Bredon cohomology group of X with respect to family $\mathcal{F}$ and coefficients in $\underline{M}$.
$H^n([G : H], A)$	n dimensional Adamson cohomology group of G with respect to H and coefficients in A.
$\mathrm{Hom}_R(\cdot, \cdot)$	Hom bifunctor in the category of R-modules
$H \rtimes K$	Semidirect product of H and K
$\mathrm{Ind}_H^G(M)$	Induction module of M with respect to H
$\ker f$	Kernel of homomorphism f
$K(G, 1)$	Eilenberg MacLane space of group G
$l_{\mathcal{A},h^*,I}(X)$	$(\mathcal{A}, h^*, I)$-length of X
$M_n(\mathbb{F})$	$n \times n$-matrices with entries in $\mathbb{F}$
$O(n, \mathbb{F})$	Orthogonal group of degree n over $\mathbb{F}$
$\mathrm{Or}_{\mathcal{F}}G$	Orbit category of G with respect to the family $\mathcal{F}$
PX	Free path space of X
P_*X	Based path space of X
$\mathcal{P}_k(X)$	k-broken path space of X
$\mathrm{PO}(n)$	Projective orthogonal Lie group of degree n
$\mathrm{PU}(n)$	Projective unitary Lie group of degree n
$\pi_1(X)$	Fundamental group of X

$\pi_1 *_H \pi_2$	Free amalgamated product of groups π_1 and π_2 over H
$\mathbb{R}P^n$	Real projective space of dimension n
$\mathrm{Res}_H^G(M)$	Restriction module of M with respect to H
ρ_X	Orbit map of X under group action
S^n	n-dimensional sphere
$\mathrm{secat}(f)$	Sectional category of map f
$\mathrm{secat}(H \hookrightarrow G)$	Sectional category of subgroup inclusion of H into G
$\mathrm{SL}(n, \mathbb{F})$	Special linear group of dimension n over $\mathbb{F}$
$\mathrm{SO}(n)$	Special orthogonal Lie group of degree n
$\mathrm{Sp}(n, \mathbb{F})$	Symplectic group of degree $2n$ over $\mathbb{F}$
$\mathrm{Sp}(n)$	Compact symplectic group of degree n
$\mathrm{Spin}(n)$	Spin group of degree n
$\mathrm{SU}(n)$	Special unitary Lie group of degree n
Σ_g	Orientable surface of genus g
T^k	k-dimensional torus
$\mathrm{TC}(X)$	Topological complexity of X
$\mathrm{TC}_r(X)$	*r-th* sequential topological complexity of X
$\mathrm{TC}[p\colon E \to B]$	Parametrized topological complexity of p
$\mathrm{TC}^{G,k}(X)$	k-stage effective topological complexity of X
$\mathrm{TC}^{G,\infty}(X)$	Effective topological complexity of X
$\underline{\mathrm{TC}}(G)$	G-proper topological complexity of G
$\mathrm{U}(n)$	Unitary Lie group of degree n
$u \cup v$	Cup product of classes u and v
$V_k(\mathbb{F}^n)$	k-dim Stiefel manifold over $\mathbb{F}^n$
$\widetilde{X}$	Universal covering of X
$\widehat{X}$	Covering space of X
$\mathbb{Z}[G]$	Group ring of G
$\mathbb{Z}[X]$	Permutation module of G-space X

Alphabetical Index