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Report on the thesis
Topics on Topological robotics
On topological complexity of Eilenberg-MacLane spaces and effective topological complexity
presented by Arturo Espinosa Baro

The topological complexity of a space (TC) is a homotopy invariant which was introduced by Farber in his topological approach to the motion planning problem in robotics. This invariant is closely related to the Lusternik-Schnirelmann category (cat) but has some specificities which have opened new exciting lines of research, some motivated by more theoretical questions and other motivated by the more applied flavor of this concept.

Arturo Espinosa's thesis constitutes a very important contribution in these two directions.

One of the main theoretical problems in this area is to understand the topological complexity $\mathrm{TC}(G)$ of an Eilenberg-MacLane space $X = K(G,1)$ in terms of algebraic characteristics of its fundamental group G. In the case of the LS-category, the answer is well known by an Eilenberg-Ganea theorem, which states that $\mathrm{cat}(G)$ coincides with the cohomological dimension of G, $\mathrm{cd}(G)$. In the case of TC the problem, which was proposed by Farber, is still open, has attracted many mathematicians and is notoriously difficult.

In this thesis, Arturo Espinosa develops a very interesting approach to this problem through the study of the sectional category of subgroup inclusions. This offers an unifying framework for the analysis since $\mathrm{cat}(G) = \mathrm{secat}(1 \hookrightarrow G)$ while $\mathrm{TC}(G) = \mathrm{secat}(\Delta(G) \hookrightarrow G \times G)$. Additionally, the more general results which are obtained can be apply to other related invariants such as Rudyak's sequential topological complexity and Cohen-Farber-Weinberger parametrized topological complexity. On this line of research, Arturo Espinosa has obtained many new, interesting, and nontrivial results, which are developed in Chapters 4, 5 and 6. Below I describe some of the main contributions on this topic.

Inspired by recent important works on $\mathrm{TC}(G)$ (e.g. Farber, Grant, Lupton and Oprea (2019); Farber and Mescher (2020)), Arturo Espinosa develops many powerful tools for the study of $\mathrm{secat}(H \hookrightarrow G)$ including, in particular, a characterization of this invariant in terms of the classifying space $E_{\langle H\rangle}G$ with respect to the family of subgroups of G generated by H, a Berstein-Schwarz class ω relative to H, a notion of essential classes and a spectral sequence which permits one to detect essential classes and obtain information on the height of the Berstein-Schwarz class ω. Among the numerous results obtained by using these tools, I would like to highlight the following ones (some of them requiring some very mild assumptions):

- Denoting by $\mathrm{cd}_{\langle \mathrm{H}\rangle}(\mathrm{G})$ the Bredon cohomological dimension of G relative to H, we have the following inequalities:

$$\mathrm{height}(\omega) \leq \mathrm{secat}(\mathrm{H} \hookrightarrow \mathrm{G}) \leq \mathrm{cd}_{\langle \mathrm{H}\rangle}(\mathrm{G}).$$

- The invariant $\mathrm{secat}(H \hookrightarrow G)$ coincides with its classical upperbound $\mathrm{cd(G)}$ if and only if $\mathrm{height}(\omega) = \mathrm{cd(G)}$.

- Denoting by H_x the isotropy group of $xH \in G/H$ with respect to the left action of H, we have

$$\mathrm{cd(G)} - \kappa_{\mathrm{G,H}} \leq \mathrm{secat}(\mathrm{H} \hookrightarrow \mathrm{G}) \leq \mathrm{cd(G)}$$

  where $\kappa_{G,H} = \sup\{\mathrm{cd(H_x)} \mid \mathrm{x} \in \mathrm{G} \setminus \mathrm{H}\}$.

This last result, which has been obtained through a deep analysis of the sprectral sequence aforementioned, generalizes both the Eilenberg-Ganea theorem (when H is trivial) and a result by Farber-Mescher on $\mathrm{TC}(G)$. It has very nice applications to related invariants such as Rudyak's sequential topological complexity and Cohen-Farber-Weinberger parametrized topological complexity. In particular, for $r \geq 2$, the r-sequential topological complexity $TC_r(G)$ (which coincides with TC when $r = 2$) of $X = K(G, 1)$ satisfies

$$r \cdot \mathrm{cd}(G) - k(G) \leq TC_r(G) \leq r \cdot \mathrm{cd}(G)$$

where $k(G) = \max\{\mathrm{cd(C(g))} \mid \mathrm{g} \in \mathrm{G} \setminus \{1\}\}$ and $C(g)$ denotes the centralizer of g.

Arturo Espinosa also introduces the Adamson cohomology of the pair (G, H) in the discussion of $\mathrm{secat}(H \hookrightarrow G)$. In particular, he introduces an Adamson canonical class which appears to be universal (as is the Berstein-Schwarz class when H is trivial), he establishes a characterization of Adamson cohomology in terms of "zero-divisors" of the usual cohomology, and proves that the Adamson cohomological dimension satisfies

$$\mathrm{cd([G:H])} \leq \mathrm{cd}_{\langle \mathrm{H}\rangle}(\mathrm{G}).$$

Again, when H is trivial, these numbers are equal and coincide with $\mathrm{cat}(G)$. Although examples are given to show that we can have a strict inequality $\mathrm{secat}(H \hookrightarrow G) < \mathrm{cd([G:H])}$, it is interesting to note that the set of essential classes, which are of particular importance in the discussion of the problem in consideration, can be identified as the image of the Adamson cohomology by a natural homomorphism.

The part on the study of $\mathrm{secat}(H \hookrightarrow G)$ closes with an interesting approach of this invariant in terms of $\mathcal{A}$-genus. This permits Arturo Espinosa to obtain a new characterization of $\mathrm{secat}(H \hookrightarrow G)$ and to establish some new properties. In particular, under some extra conditions of normality, some refinements of the inequality $\mathrm{secat}(H \hookrightarrow G) \leq \mathrm{cd}_{\langle \mathrm{H}\rangle}(\mathrm{G})$ are given.

In a last and independent part, Arturo Espinosa studies the notion of effective topological complexity. This invariant was introduced by Z. Blaszczyk and M. Kaluba in order to measure the complexity of motion planning algorithms which take advantage of the symmetries of the system in consideration, that is, of the action of a group G on the configuration space of the system. This line of research is clearly motivated by the more applied flavor of the concept of topological complexity. Arturo Espinosa introduces the notion of effective category and establishes several properties of these effective cat and TC. In particular, he determines sufficient conditions under which these invariants coincide with or are bounded by the usual cat and TC of the orbit space, and establishes a criterion for the 2-stage effective TC not to vanish. These results as well as the numerous examples discussed in this chapter constitute a significant progress in the understanding of this invariant.

Conclusion:

The results obtained in this thesis are clearly original and represent a very significant contribution to the area of topological robotics. They gave rise to four articles, one of them is already published in the Journal of Pure of Applied Algebra. It is worth mentioning that one of these articles is in collaboration with leading experts of the field, who were not at all involved in the supervision of the candidate. By the deepness of the analysis realized in this work and by the new results obtained, Arturo Espinosa has undoubtedly demonstrated his deep knowledge of the subject, his ability in developing original solutions to mathematical problems and in manipulating nontrivial objects and techniques. The text is well organized, shows a strong mathematical maturity and includes all the formal requirements. For all these reasons, I consider that the doctoral dissertation presented by Arturo Espinosa Baro completely meets the conditions specified in Art. 187. of The Law on Higher Education and Science. Considering the higher international standards, I think that Arturo Espinosa Baro deserves to be awarded with a doctoral degree in the discipline of Mathematics and I recommend that this dissertation receives a distinction.

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