UNIVERSITÀ DI PAVIA
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Referee Report - Ph.D. Thesis

Stochastic quantization and Osterwalder-Schrader axioms for quantum field theory models

submitted by Azam Jahandideh

The thesis submitted by Azam Jahandideh, is a mathematically rigorous and conceptually original work in the realm of constructive Quantum Field Theory (QFT) addressing the question whether, on $\mathbb{R}^2$, the $P(\Phi)_2$ model admits a measure which is reflection positive and invariant under the action of the Euclidean group.

The manuscript is well structured, with a clear and logical development, subdivided in six main chapters, including the introduction, and five appendices. In the following I shall comment on their content with the exception of the introduction. This provides a well-balanced and pedagogical overview of the main results as well as of the review of the literature. Overall, the introduction is well written and it serves the purpose of guiding the reader through the main body of the thesis. Concerning the core of this work:

Chapter 2 – It provides a succinct review of the main analytic and probabilistic tools used in this work. More precisely the author collects the technical background needed for the subsequent construction. The candidate displays a solid mastery of the geometry and analysis of the two–dimensional sphere, including spherical harmonics, spectral properties of the Laplace–Beltrami operator. This is complemented by a sufficiently detailed overview of all relevant function spaces, be it Sobolev, Bessel potential or Besov and Triebel–Lizorkin spaces. In addition Littlewood–Paley theory, and weighted spaces are discussed emphasizing the technical aspects which play a role in the following chapters. The probabilistic preliminaries, covering Gaussian measures, Wiener chaos, and stochastic PDEs, are standard but accurate, and this proves that the candidate is comfortable with the analytical framework underlying stochastic quantization.

At the level of presentation, the chapter is rather comprehensive and sufficiently self-contained, although all tools and concepts are discussed as if they are equally important. This slightly obscures the logical hierarchy among them. A more explicit guidance on which tools are central for the later arguments and which are included mainly for completeness could further improve readability. Nonetheless the material is well organized, mathematically sound, and it clearly demonstrates that the candidate has acquired the necessary technical competence and conceptual maturity to tackle advanced problems at the interface of analysis, probability, and constructive quantum field theory.

Chapter 3 – Chapter 3 is devoted to the construction of the finite-volume $P(\Phi)_2$ measure on the sphere and to the control of its ultraviolet limit. The candidate displays a very good mastery of constructive field theory techniques in low dimension, in particular the use of Wick ordering, spectral cutoffs, and Nelson-type hypercontractivity estimates. These are employed to establish existence, uniform integrability, and convergence of the regularized measures. The analysis is technically sound and it reflects a good understanding of how probabilistic and functional-analytic tools interact in the construction of non-Gaussian measures that are absolutely continuous with respect to that of the free field. The introduction of auxiliary, source-dependent measures is well motivated in view of their later role in the analysis of Osterwalder–Schrader properties.

From a stylistic perspective, the chapter is dense and technically driven, and, as it happens in may parts of this work, the broader strategy of the construction is not detailed in advance. It could be highlighted more explicitly to guide the reader through the sequence of estimates. Some arguments rely heavily on previously established results, which are correctly used but occasionally not fully contextualized. Nevertheless, the presentation is rigorous and ultimately quite effective. To summarize the content of Chapter 3 convincingly demonstrates that the candidate has mastered the core techniques of constructive $\mathrm{P}(\Phi)_2$ theory in finite volume, as well as the control of the ultraviolet limit that underpins the rest of the thesis.

Chapter 4 – Chapter 4 is devoted to the analysis of the stochastic quantization subordinated to the finite-volume measures introduced earlier. The candidate displayes a solid mastery of stochastic PDE techniques, ranging from the Da Prato–Debussche decomposition to fixed point arguments in suitable function spaces, and energy estimates. Both local and global well-posedness results are discussed with care, and the hurdles in switching from linear to nonlinear equations is handled in a conceptually clear way. The proof of invariance of the constructed measures under the associated Markov semigroups is the core of the chapter and it shows a mature understanding of the interplay between SPDE dynamics and invariant measures.

The chapter is technically demanding and largely self-contained, which reflects the candidate's ability to work independently with advanced analytical tools. At the same time, the exposition occasionally prioritizes technical details over a high-level discussion of the underlying strategy. A brief roadmap at the beginning of some sections could further improve accessibility. Overall, Chapter 4 is a convincing evidence of the candidate's mastery of stochastic quantization methods and of their effective use in a rigorous constructive quantum field theory setting.

Chapter 5 – Chapter 5 addresses the problem of tacking the infinite-volume limit, which plays a crucial technical role in the overall construction. The candidate develops uniform a priori bounds for the stochastic dynamics and he uses them effectively to establish tightness of the family of push-forward measures. The arguments rely on a careful combination of energy estimates and probabilistic compactness methods. This chapter shows that the candidate is able to handle limiting procedures and she understands well how stochastic estimates translate into structural properties of the underlying families of measures.

From an expository perspective, the chapter is focused and efficient, though, once more, the motivations behind some technical choices could occasionally be made more explicit to help the reader appreciate their role in the global strategy. Nevertheless, the results are clearly stated and they are rigorously proved. Overall, Chapter 5 convincingly demonstrates the candidate's ability to apply stochastic and analytical techniques in a coherent and effective manner.

Chapter 6 – Chapter 6 is devoted to the verification of the Osterwalder–Schrader axioms for the infinite-volume measures constructed in the previous chapters. The candidate successfully establishes integrability and regularity properties, she proves reflection positivity, and she shows invariance under translations, rotations, and reflections. The arguments combine probabilistic inputs, geometric considerations, and symmetry properties of the finite-volume measures in a technically competent and conceptually clear way. In particular, the transfer of symmetries from the sphere to the plane via stereographic projection is handled with care displaying a good understanding of the underlying geometric structure.

The chapter is dense and technically involved, as expected given the nature of the results, and at times it presupposes familiarity with the Osterwalder–Schrader framework. A slightly more expanded discussion of the logical interdependence of the axioms and of the limitations of the construction, notably concerning clustering and uniqueness, could further improve the presentation. Nonetheless, Chapter 6 represents a strong evidence of the candidate's ability to verify deep structural properties of Euclidean quantum field theories and it completes effectively the thesis.

In addition to the main chapters, the thesis includes 5 appendices, the last three devoted to the proofs of technical lemmas while the aim of the first two is to discuss succinctly Bessel potentials and stochastic estimates.

Concluding remarks – To summarize, the thesis of Azam Jahandideh is a solid and original contribution to the mathematical formulation of constructive Quantum Field Theory. The author has displayed advanced technical skills and a very good understanding of the probabilistic and analytical structures necessary to prove the main results of this work. The novel material is certainly more than enough to be awarded with a PhD degree in mathematics. The candidate contribution is clear and it highlights good scientific maturity.

Although the thesis is built on the results of [37] in the bibliography, it is by no means a simple rephrasing of that work. Several aspects that were only sketched or treated briefly in [37], *e.g.*, the detailed analysis of the stochastic quantization equations, the global-in-time well-posedness, and the tightness and infinite-volume arguments, are here developed in full detail and they are integrated into a coherent and self-contained construction.

The only aspect which is not fully positive is the English presentation which could benefit from a considerable revision. The author's style is generally understandable but not always fluent. The use of long sentences and of wrong articles appear more frequently than advisable in a scientific dissertation and a careful linguistic revision would significantly improve the readability of the work, without altering its scientific content.

In conclusion I recommend that the candidate be **admitted to the final defence**. A careful linguistic revision of the English text is, however, advised before the final submission, in order to improve the fluency and readability of the manuscript. This recommendation does not diminish the high scientific quality as well as the value of the work.

Pavia, li 15/12/2025
Prof. Claudio Dappiaggi