REFEREE'S REPORT FOR CONFERRING THE TITLE OF A HABILITATED DOCTOR TO DR. RADOSŁAW SZWEDEK

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Dr. Radosław Szwedek is a graduate of the Adam Mickiewicz University (UAM). He obtained the degree of a doctor in Mathematics at UAM in 1996. The advisor of his doctoral dissertation entitled 'Interpolacja operatorów i miar niezwartości' was Prof. Dr. Mieczysław Mastyło, and his research area is functional analysis.

The summary, which Dr. Szwedek gives on his professional accomplishments, demonstrates that he by now has developed into an outstanding academic personality with overall impressing skills.

During the period 2006 – 2019 he has published fourteen articles in journals of high international reputation - five written by himself and nine together with four other coauthors (Grafakos, Mastyło, Mleczko, and Sánchez Pérez).

The thematic content of the work of Dr. Szwedek shows a considerably wide range, and in almost all his articles the mathematical power comes from the fascinating theory of interpolation theory in Banach spaces.

Most of the established results seem to fit into the following abstract question: Given an interpolation functor $\mathcal{F}$, for which operators T between pairs of Banach couples $\vec{A}$ and $\vec{B}$ such that $T|A_0 : A_0 \to B_0$ has property P_0 and $T|A_1 : A_1 \to B_1$ has property P_1, is it true that the interpolated operator $T|_{\mathcal{F}(\vec{A})} : \mathcal{F}(\vec{A}) \to \mathcal{F}(\vec{B})$ has property $P_{\mathcal{F}}$?

In fact, almost all the mathematical work done by Dr. Szwedek over the recent years, sums up to this general scheme.

Here two comments are important – the first one is that in most of Dr. Szwedek's contributions this scheme is applied to very concrete mathematical problems that have crystallized out over many years having a long history with many fundamental contributions done earlier. The second comment is that these concrete problems forces the author to work with very concrete functors $\mathcal{F}$, namely those functors which fit with the general environment. And since there are many different functors of relevance, each with a lot of complicated abstract background, the work of Dr. Szwedek requires extensive knowledge of interpolation theory and mathematical analysis as a whole.

In his summary Dr. Szwedek divides his scientific work into two groups. The first one, labeled by [A1-7], forms a cycle of interconnected topics under the title 'Metric entropy, approximation quantities, spectra of operators and interpolation of Banach spaces'. The second group is formed by a series of contributions (indicated by [S1-7]) on various interesting topics which are still (almost all) connected through the use of fundamental tools from interpolation theory.

In the following we are going to emphasize a few aspects of the scientific work of the candidate – mainly concentrating on the first set [A1-7] of articles.

EVALUATION OF THE SCIENTIFIC ACHIEVEMENTS

Vector measures and Banach lattices. The general problem discussed in [A5] is as follows. Assume that $(L_{p_0}(m_0), L_{p_1}(m_1))$ is an interpolation couple of spaces of integrable functions with respect to vector measures m_0 and m_1, respectively, and $\mathcal{F}$ a (concrete) interpolation functor. How to describe $\mathcal{F}(L_{p_0}(m_0), L_{p_1}(m_1))$ as a space of integrable functions with respect to an 'interpolated' vector measure constructed from m_0 and m_1?

Note that each order continuous Banach function lattice with weak order unit can be represented as a space of functions which are integrable with respect to a vector measure. In fact, this is one of the many reasons why this problem is delicate and has been around since the early days of interpolation theory.

[A5] gives an approach for a construction of interpolation spaces of spaces of integrable functions based on $\ell_\infty(\Gamma, L_1(\mu))$-valued vector measures. The underlying ideas seem to be entirely different to those existing in the literature. The article provides a remarkable constructive procedure which allows to identify interpolation spaces of a couple $(L_{p_0}(m_0), L_{p_1}(m_1))$ as spaces of integrable functions with respect to an interpolated vector measure.

Concrete applications are given in the cases of the Calderón interpolation method and the K-method for real interpolation due to Peetre.

Hardy spaces on non-simply connected domains. Interpolation of Hardy spaces $H_p(\mathbb{D})$ on the disc (or equivalently Hardy spaces $H_p(\mathbb{T})$ on the torus $\mathbb{T}$) has been investigated by many authors – with celebrated results given by Jones, Kisliakov, and Xu, among others.

The aim of [A4] (jointly with Mleczko) is to extend part of this work to the more involved situation of Hardy spaces $HX(\mathbb{G})$, where X is a Banach lattice on $\mathbb{T}$ and $\mathbb{G}$ a circular domain in the complex plane.

Based on the interpolation of direct sums of Banach spaces, the paper finds a highly non-trivial, but still very elegant, way to obtain interpolation theorems for such Hardy spaces – stronger than those known so far. More precisely, it turns out that new Fatou type theorems $HX(\mathbb{G}) = HX(\partial\mathbb{G})$ as well as the fundamental equality $\mathcal{F}(\bigoplus \vec{A}_i) = \bigoplus \mathcal{F}(\vec{A}_i)$, where $\mathcal{F}$ is an arbitrary functor and the $\vec{A}_i$'s are finitely many Banach couples, lead in a natural way to new interpolation theorems for Hardy spaces $HX(\mathbb{G})$.

The measure of non-compactness, entropy numbers and s-numbers. This topic in my opinion forms the core of the scientific work of Dr. Szwedek. It is – in one way or the other – mainly concerned with the following fundamental problem from the theory of interpolation of Banach spaces:

Assume that $\mathcal{F}$ is an interpolation functor, and (A_0, A_1) and (B_0, B_1) two Banach couples. When is it true that every operator from $\vec{A}$ into $\vec{B}$, such that $T| : A_0 \to B_0$ is compact, acts as a compact operator from $\mathcal{F}(\vec{A})$ into $\mathcal{F}(\vec{B})$?

Starting with Krasnoselskii the literature is full of remarkable partial results due to Calderón, Cobos, Cwikel, Edmunds, Hayakawa, Lions, Mastyło, Peetre, and Persson, among others. But a satisfying somewhat final solution is still missing. For example, for the complex method it is in general unknown whether $T : [A_0, A_1]_\theta \to [B_0, B_1]_\theta$ is compact provided both operators $T|A_0$ and $T|A_1$ are.

Today most modern approaches to this involved problem go through 'graduation' of compactness. Measures of compactness of sets M in metric spaces, or bounded operators T in Banach spaces, have a long history – with incredible influence on the development of functional analysis as a whole. Of special importance are the so-called entropy numbers $(\varepsilon_n(T))_{n\in\mathbb{N}}$ of an operator T, the limit $\beta(T) = \lim_{n\to\infty} \varepsilon_n(T)$, and the sequence of so-called dyadic entropy numbers defined by $e_n(T) = \varepsilon_{2^n}(T)$.

In one way or the other, much of the work done in $[A1, A2, A3, A6, A7]$ is devoted to the 'compactness problem on interpolation', and the strategy mainly is to estimate appropriate measures of compactness of $T : \mathcal{F}(A_0, A_1) \to \mathcal{F}(B_0, B_1)$ by the corresponding measures of compactness of $T|A_0$ and $T|A_1$.

The following result is a beautiful sample proved by Dr. Szewdek in [A2]:

$$\beta(T : [A_0, A_1]_\theta \to [B_0, B_1]_\theta) \leq C\beta(T : A_0 \to B_0)^{1-\theta}\beta(T : A_1 \to B_1)^\theta ,$$

provided (B_0, B_1) satisfies Calderón's approximation condition (CA). Clearly, this proves that $T : [A_0, A_1]_\theta \to [B_0, B_1]_\theta$ is compact, whenever (B_0, B_1) satisfies (CA) and either $T|A_0$ or $T|A_1$ is compact. It is important to note that, looking at the slightly more general approximation property $(CA)_0$, the techniques from $[A2]$ are strong enough to extend main results from a seminal article of Calderón.

Let us comment on the remarkable contributions collected in $[A1, A3, A6, A7]$. Given $T : (A_0, A_1) \to (B_0, B_1)$ and an interpolation functor $\mathcal{F}$, these articles mainly deal with the difficult and long-standing problem of how the entropy numbers (or related s-numbers in the sense of Pietsch) of the interpolated operator $T : \mathcal{F}(A_0, A_1) \to \mathcal{F}(B_0, B_1)$ behave in relation with the corresponding numbers generated by the operator T acting between the 'corners' of its pairs?

Non trivial contributions to this problem usually lead to valuable applications in various different fields of modern analysis, as e.g. in the theory of eigenvalue distribution of compact (or more generally Riesz-) operators.

Edmunds and Netrusov were the first who showed that 'reasonable' real interpolation of entropy numbers is impossible – and this was later understood in a much deeper way in the remarkable article [A3]. However, in the case of complex interpolation a full answer to the question whether an estimate like

$$e_{k_1k_2}(T : [A_0, A_1]_\theta \to [B_0, B_1]_\theta) \leq C\, e_{k_1}(T : A_0 \to B_0)^{1-\theta} e_{k_2}(T : A_1 \to B_1)^\theta$$

holds true for all possible $T : \vec{A} \to \vec{B}$ and $\theta \in [0, 1]$, over many years seemed out of the scope of current technical abilities.

Against this background, the situation changed dramatically when Dr. Szwedek (jointly with M. Mastyło) started to focus on this cycle of problems. We emphasize that the collection of fundamental knowledge given in [A1, A3, A6, A7] not only on the interpolation of entropy numbers but also on the interpolation of all sorts of related s-numbers (like approximation numbers, Gelfand or Kolmogorov numbers) belong to the very best results proved in the field.

To see one of the many examples, we mention the following interpolative entropy estimate under geometric interpolation of Hilbert spaces from [A6], which indeed is astonishing as well as beautiful:

$$\varepsilon_k(T : [H_0, H_1]_\theta \to [H_0, H_1]_\theta) \leq 72\, \varepsilon_k(T : H_0 \to H_0)^{1-\theta}\varepsilon_k(T : H_1 \to H_1)^\theta .$$

Spectra of operators. The celebrated Carl-Triebel inequality

$$\Big(\prod_{i=1}^{n} |\lambda_i(T)|\Big)^{1/n} \leq g_n(T) := \inf_k k^{1/2n}\varepsilon_k(T)$$

between the eigenvalues $\lambda_i(T)$ of a Riesz operator $T : X \to X$ and its Gelfand moduli g_n is one of the many inequalities which illustrate the surprisingly deep interplay of algebra and analysis, and Carl's fascinating inequality

$$|\lambda_k(T : X \to X)| \leq \sqrt{2} e_k(T : X \to X),$$

estimating single eigenvalues by dyadic entropy numbers, is then an almost immediate consequence.

The main motivation for the deep article [A3] is to study interpolation spaces X (of exponent θ) with respect to Banach couples (A_0, A_1) such that for any operator $T : (A_0, A_1) \to (A_0, A_1)$ we have

$$e_{m+n-1}(T : X \to X) \leq C\, e_m(T : A_0 \to A_0)^{1-\theta} e_n(T : A_1 \to A_1)^{\theta},$$

which in the case of positive results by Carl's inequality immediately leads to

$$|\lambda_k(T : X \to X)| \leq C\, e_k(T : A_0 \to A_0)^{1-\theta} e_k(T : A_1 \to A_1)^{\theta}.$$

The article [A3] encounters a deep interaction of two highly interesting theories – the theory of eigenvalue distribution of operators in Banach spaces and the theory of interpolation in Banach spaces. In fact, this article approaches in an abstract but beautiful way the spectral theory of operators in abstract interpolation spaces.

Entropy numbers and spectral moduli of operators are introduced, and an intimate relationship between them and eigenvalues of operators are proved. An interpolation version of the Carl-Triebel eigenvalue inequality is given, and based on all these tools interpolation estimates for single eigenvalues as well as for geometric means of absolute values of the first n eigenvalues of operators follow. Some of these estimates may be regarded as generalizations of the classical spectral radius formula. In my view all these contributions are highly valuable.

Let us go on with a brief discussion of the second set [S1-7] from the mathematical work of Dr. Szwedek (less detailed than in the case of [A1-7]).

DISCUSSION OF FURTHER RESEARCH CONTRIBUTIONS

Both papers [S1] and [S2] study interpolation of measures of non-compactness as well as weak non-compactness of operators T in Banach spaces. [S1] proves a highly non-trivial estimate for the measure $\beta(T) = \lim_{n\to\infty} \varepsilon_n(T)$ between abstract real interpolation spaces, depending on the fundamental function of the space generating the method. And [S2] gives an analog result for the measure of weak non-compactness.

The publication [S3] is a careful investigation of the important interpolative contruction $(\mathcal{A}, \mathcal{B})_\varphi$ due to Mastyło, where $\mathcal{A}$ and $\mathcal{B}$ are two Banach operator ideal in the sense of Pietsch and φ an appropriate non-negative function in the variables $t, s \in [0, \infty)$. For the power function $\varphi(s,t) = s^{1-\theta}t^{\theta}$ these operator ideals were intensively studied by Matter. The authors concentrate on the study of $(\Pi_p, \mathcal{L})_\varphi$, where Π_p stand for the ideal of all p-summing and $\mathcal{L}$ for all bounded operators in Banach spaces. These interpolative ideals are described in terms of factorization through abstract interpolation Lorentz spaces. Relationships between Banach ideals determined by Orlicz sequence spaces are shown together with a variant of Pisier's factorization theorem for $(p, 1)$- summing operators on $C(K)$-spaces. Applications to Schatten classes are given.

One of the hot topics in interpolation theory, with an incredible high impact on modern Fourier analysis, is the interpolation of multilinear operators on Banach spaces and quasi

Banach spaces. The article [S4], jointly written with Grafakos and Mastyło, shows that interpolation of multilinear operators can be lifted to multilinear operators from spaces generated by the minimal methods to spaces generated by the maximal methods of interpolation defined on a class of couples of compatible p-Banach spaces. Additionally, the multilinear interpolation theorem for operators on Calderón-Lozanovskii spaces between L_p-spaces with $0 < p \leq 1$, and as application interpolation theorems for multilinear operators on quasi-Banach Orlicz spaces are given.

The article [S5] constitutes a deep study of (multilinear and polynomial) Kahane-Salem-Zygmund inequalities, which were originally designed to estimate the expectation of the supremum norm of trigonometric Bernoulli polynomials and multilinear forms in several variables. These inequalities have numerous applications in many areas of modern analysis, and in fact recently various variants for multilinear operators and polynomials in arbitrary Banach spaces were proved.

In particular, Bayart in a seminal paper uses two different methods to extend the Salem-Kahane-Zygmund inequalities within ℓ_p-spaces. The first method is based on Khintchine-type inequalities for Rademacher processes, and the second relies on controlling increments of a Rademacher process in an Orlicz space, and in this case an entropy argument is used. [S5] is a deep analysis of this second approach within interpolation theory, and the outcome then are new abstract Kahane-Salem-Zygmund inequalities. Ideas from stochastic processes and interpolation are used to control increments of a Rademacher process in an Orlicz space via entropy integrals. To do this a general estimate for entropy integrals is proved. Various application show the usefulness of these new estimates.

An isometry on a separable complex Hilbert space H is said to be a conjugation whenever $C^2 = \mathrm{id}$, and an operator T on H is called C-symmetric whenever C is a conjugation on H which additionally satisfies that $T = CTC^*$. Complex symmetries are operators on H which are C-symmetric for some conjugation C. Defined by Garcia and Putinar in 2006, these operators form a surprisingly large class which was studied intensively by various authors during recent years. The article [S6] analyses interpolation properties of complex symmetric operators. Among others it is proved that, if $T \in \mathcal{L}(\vec{H})$ is C-symmetric on H_0 and H_1, then T is C-symmetric on $[\vec{H}]_\theta$ for every possible θ. Interesting applications are given to complex symmetric Toeplitz operators on weighted Hilbert spaces of analytic functions.

Finally, motivated by some current developments in information science, the article [S7] uses various types of integration (as e.g. Bartle-Dunford-Schwartz integration or Choquet type integration) with respect to vector-valued fuzzy measures, to define and study certain information measuring tools. Examples related to existing biometric tools as well as new measuring indices are given.

EVALUATION OF FURTHER ACADEMIC ACTIVITIES

In the period from 2007 to 2019 Dr. Szwedek has given thirteen talks at international mathematical conferences and three more talks at scientific institutions outside of Poznań. I myself attained several of these talks, and want to emphasize that he in each of these talks was perfectly prepared and the way he argued was indeed crystal clear.

In 2017 the candidate was a member of the organizing committee of the international conference 'Banach spaces and Operator Theory with Applications' at the AMU Poznań.

As an investigator of a research team he participated in three projects of high international reputation – two directed by Prof. Dr. M. Mastyło and the third one by Prof. Dr. L. Skrzypczak and Prof. Dr. H.G. Leopold (a joint project of Poland and Germany).

Dr. Szwedek wrote peer-reviews for several international journals of high standard (e.g. for Mathematische Nachrichten or the Journal of Approximation theory). Moreover, he was an (auxiliary) supervisor of one PhD-student at the AMU, and a reviewer of several diploma theses at his home university.

Dr. Szwedek is an highly experienced lecturer. Over years his personal teaching load (at least for German standards) seemed to be very high. He lectured for students of mathematics, computer science, biology, chemistry, geoecology, among others. In pure mathematics the list of different subjects he taught is long (functional analysis, complex analysis, measure theory, etc.).

For his scientific activities Dr. Szwedek has received two awards at the Adam Mickiewicz University Poznań (2007 and 2017).

FINAL CONCLUSION

As pointed out above, the research of Dr. Radosław Szwedek is outstanding. Today he in fact plays an important role, with an absolutely independent profile, of the by now internationally highly recognized Poznań-school on 'interpolation and operator theory in Banach spaces' – founded by Prof. Dr. M. Mastyło who continues the long tradition of functional analysis beginning with Prof. Dr. W. Orlicz.

The research problems Dr. Radosław Szwedek chooses are often long-standing and hence delicate, and his contributions are creative, original and substantial. His scientific activities identify him as a highly professional full academic personality.

I am fully convinced that Dr. Radosław Szwedek deserves to be conferred with the degree of a habilitated doctor in exact and natural sciences, in the discipline of mathematics.

Oldenburg, July 12, 2020

Prof. Dr. Andreas Defant