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# Araki–Haag Detectors, Mourre Theory, and the Problem of Asymptotic Completeness in Algebraic Quantum Field Theory

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# Detektory Arakiego-Haaga, teoria Mourre'a i problem asymptotycznej zupełności w algebraicznej teorii pól kwantowych

Rozprawa doktorska złożona w celu spełnienia wymagań
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# Summary

Scattering theory describes the asymptotic evolution of systems of interacting particles. A key concept in this area is asymptotic completeness, which asserts that every state can be decomposed into bound and scattering states. While asymptotic completeness is well-understood in non-relativistic quantum mechanics, it remains an open and challenging problem in local relativistic quantum field theory (QFT). In this dissertation, we adopt the axiomatic (model-independent) framework of algebraic QFT and the Haag–Ruelle scattering theory to investigate the problem of asymptotic completeness.

Modern proofs of asymptotic completeness in quantum mechanics rely on a Mourre estimate, propagation estimates, and the convergence of asymptotic observables, such as the asymptotic velocity. In QFT, Araki–Haag detectors, first introduced by Araki and Haag (1967) and later further developed by Buchholz (1990), are natural asymptotic observables. Controlling their convergence is an important prerequisite for asymptotic completeness in QFT.

We prove the convergence of Araki–Haag detectors on states of bounded energy that belong to the absolutely continuous part of the energy-momentum spectrum below the three-particle threshold. This result brings us closer to two-particle asymptotic completeness than the earlier work of Dybalski and Gérard (2014), who analysed the convergence of products of detectors with distinct velocities. Our proof shares similarities with proofs of the existence and completeness of wave operators in quantum mechanics. Notably, we apply Mourre's conjugate operator method to derive a local decay estimate, which marks the first application of Mourre's method in the relativistic QFT framework.

The conjugate operator method is a mathematical technique from spectral theory, which is based on a strictly positive commutator estimate. This method has been crucial to advance the spectral and scattering theory of quantum-mechanical many-body systems. Apart from proving the convergence of Araki–Haag detectors, as mentioned above, we also apply Mourre's method to derive a limiting absorption principle for the energy-momentum operators in relativistic QFT. The limiting absorption principle allows us to reproduce results on spectral properties of the energy-momentum operators, such as the absence of singular continuous spectrum.

## Podsumowanie

Teoria rozpraszania opisuje asymptotyczną ewolucję układów oddziałujących cząstek. Kluczowym pojęciem w tej dziedzinie jest asymptotyczna zupełność, która stwierdza, że każdy stan można rozłożyć na stany związane i rozproszeniowe. Chociaż asymptotyczna zupełność jest dobrze zrozumiana w nierelatywistycznej mechanice kwantowej, pozostaje ona otwartym i trudnym problemem w lokalnej relatywistycznej teorii pól kwantowych (QFT). W niniejszej dysertacji przyjmujemy aksjomatyczne (niezależne od modelu) ramy algebraicznej QFT oraz teorię rozpraszania Haaga-Ruelle'a w celu zbadania problemu asymptotycznej zupełności.

Współczesne dowody asymptotycznej zupełności w mechanice kwantowej opierają się na oszacowaniu Mourre'a, oszacowaniach propagacyjnych oraz zbieżności obserwabli asymptotycznych, takich jak asymptotyczna prędkość. W QFT detektory Arakiego-Haaga, po raz pierwszy wprowadzone przez Arakiego i Haaga (1967) i później badane przez Buchholza (1990), stanowią naturalne obserwable asymptotyczne. Ich zbieżność jest warunkiem koniecznym dla asymptotycznej zupełności w QFT.

W tej pracy dowodzimy zbieżności detektorów Arakiego-Haaga na stanach o ograniczonej energii, które należą do absolutnie ciągłej części spektrum energii i pędu poniżej progu trójcząstkowego. Wynik ten przybliża nas do asymptotycznej zupełności stanów dwucząstkowych w większym stopniu niż wcześniejsza praca Dybalskiego i Gérarda (2014), którzy analizowali zbieżność iloczynów detektorów o różnych prędkościach. Nasz dowód wykazuje podobieństwa do dowodów istnienia i zupełności operatorów falowych w mechanice kwantowej. W szczególności stosujemy metodę operatora sprzężonego Mourre'a do wyprowadzenia lokalnego oszacowania zaniku, co jest pierwszym zastosowaniem teorii Mourre'a w relatywistycznej QFT.

Metoda operatora sprzężonego jest techniką matematyczną z zakresu teorii spektralnej, opartą na ścisłej dodatniości pewnych komutatorów. Metoda ta odegrała kluczową rolę w rozwoju teorii spektralnej i teorii rozpraszania układów wielocząstkowych w mechanice kwantowej. W naszej pracy stosujemy metodę Mourre'a, aby udowodnić zbieżność detektorów Arakiego-Haaga, jak wspomniano wyżej, oraz wykazać regularność rezolwent (tzw. *limiting absorption principle*) dla operatorów energii i pędu w relatywistycznej QFT. W ten sposób odtworzyliśmy wyniki dotyczące własności spektralnych operatorów energii i pędu, takie jak brak osobliwego spektrum ciągłego.

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## Use of generative AI

ChatGPT was used in the language- and TeX-editing process of the dissertation and for translating the summary to Polish.

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# Chapter 1

# Introduction

## 1.1 Scattering Theory

The primary objective of scattering theory is to understand how systems of interacting particles evolve asymptotically. While the equations of motion that govern the evolution of quantum systems are often too complicated to solve them exactly, it is expected that, after a sufficiently long time, the motion of the particles simplifies as they separate. The central concept in mathematical scattering theory is asymptotic completeness, which asserts that every quantum state in the state space $\mathcal{H}$ is a linear combination of bound and scattering states.

**Definition 1.1.1.** A quantum system is **asymptotically complete** if the state space $\mathcal{H}$ decomposes into the subspaces of bound states ($\mathcal{H}_{\text{bound}}$) and scattering states ($\mathcal{H}_{\text{scat}}$):

$$\mathcal{H} = \mathcal{H}_{\text{bound}} \oplus \mathcal{H}_{\text{scat}}. \tag{1.1}$$

The precise definitions of bound and scattering states depend on the specific physical context. We are mainly interested in the problem of asymptotic completeness in local relativistic quantum field theory (QFT). However, to provide context for our findings in QFT, we first review key results from scattering theory in non-relativistic quantum mechanics.

### 1.1.1 Asymptotic Completeness in Quantum Mechanics

A non-relativistic many-body quantum system is expected to break up asymptotically into freely moving clusters of particles. In the simplest case of two particles, either the two particles move freely at large separations or they form a bound state. For many-body systems, additional scattering channels arise because the particles can form clusters. For example, in a three-particle system, an initial configuration where two particles form a bound state can evolve in the following ways: the system might break up into three freely moving particles, retain its initial configuration (elastic scattering), or result in a rearrangement where two different particles form a bound state.

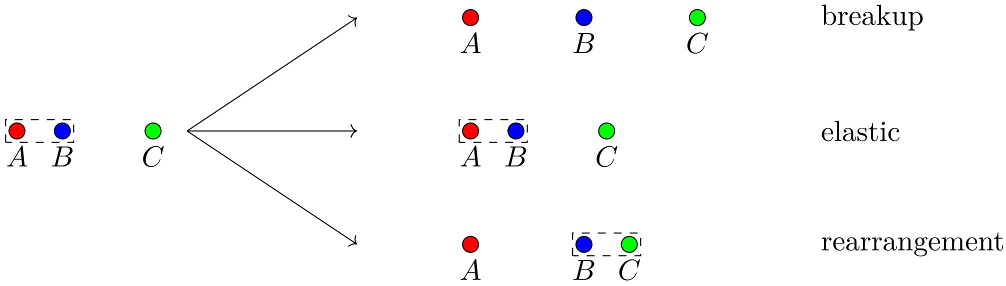


Figure 1.1: Scattering channels for an initial configuration of three particles, where two particles are in a bound state.

### Many-body Hamiltonians

A typical $N$-body Hamiltonian on the Hilbert space $L^2(\mathbb{R}^{dN})$ with pair-interaction potentials $V_{ij}$ has the following form:

$$H = -\sum_{i=1}^{N} \frac{1}{2m_i}\Delta_i + \sum_{1\leq i<j\leq N} V_{ij}(x_i - x_j), \tag{1.2}$$

where $m_i > 0$ is the mass of particle $i$ and $\Delta_i$ is the Laplacian on $L^2(\mathbb{R}^d)$.

Let $\mathcal{A}$ be the set of all **cluster decompositions** of $\{1,\dots,N\}$. For cluster decompositions $a, b \in \mathcal{A}$, we write $a \leq b$ if $a$ is finer than $b$; that is, for all $A \in a$, there is a $B \in b$ such that $A \subset B$. The **collision planes** $X_a$, $a \in \mathcal{A}$, are defined as the subspaces of the **configuration space** $X = \mathbb{R}^{dN}$ where particles from the same clusters collide:

$$X_a = \{x \in X \mid x_i = x_j \text{ if } (ij) \leq a\}. \tag{1.3}$$

Here, $(ij) \in \mathcal{A}$ is the cluster decomposition in which the particles $i$ and $j$ form a pair while the other particles are separated.

The Hamiltonian $H$ is invariant under translations. In scattering theory, it is often advantageous to introduce Jacobi coordinates to eliminate the translation-invariance by separating the centre-of-mass motion (see e.g. [Is23, Section 3.3]).

In the following, we introduce an abstract framework that encompasses Hamiltonians of the form (1.2), as well as more general Hamiltonians, such as those with centre-of-mass motion removed or those with additional confining potentials. The notation is adopted from [DG97, Section 5.1, Section 6.1].

Let $(\mathcal{A}, \leq)$ be a partially ordered set such that each two-element subset $\{a, b\} \subset \mathcal{A}$ has a least upper bound, denoted by $a \vee b$. The minimal and maximal element of $\mathcal{A}$ with respect to the order relation $\leq$ are denoted by $a_{\min}$ and $a_{\max}$, respectively. The configuration space $X$ is a Euclidean space (i.e. a finite-dimensional real vector space) equipped with a scalar product. The collision planes $\{X_a\}_{a\in\mathcal{A}}$ and the internal spaces $\{X^a\}_{a\in\mathcal{A}}$ are families of complementary subspaces (i.e. $X = X_a \oplus X^a$) satisfying the following properties:

- if $b \leq a$, then $X_a \subset X_b$ and $X^b \subset X^a$;
- $X_a \cap X_b = X_{a \vee b}$ and $X^a + X^b = X^{a \vee b}$;
- $X_{a_{\min}} = X$ and $X_{a_{\max}} = \bigcap_{a \in \mathcal{A}} X_a$.

The projections of $x \in X$ to $X_a$ and $X^a$ are the external coordinate $x_a$ and the internal coordinate $x^a$, respectively.

*Example.* Let $X = \mathbb{R}^{dN}$ be the Euclidean space equipped with the scalar product

$$x \cdot y = \sum_{i=1}^{N} m_i x_i \cdot y_i, \;\; x, y \in (\mathbb{R}^d)^N. \tag{1.4}$$

It is straightforward to verify that the collision planes $X_a$, as defined in (1.3), satisfy the stated properties if $X^a$ is taken to be the orthogonal complement of $X_a$ with respect to the scalar product (1.4).

A quantum-mechanical many-body system is described by the following Hamiltonian on the Hilbert space $\mathcal{H} = L^2(X)$:

$$H = -\frac{1}{2}\Delta + \sum_{a \in \mathcal{A}} V_a(x^a), \tag{1.5}$$

where $\Delta$ is the Laplacian on $L^2(X)$ and $V_a$ are (many-body) interaction potentials. The notation $V_a(x^a)$ should indicate that the potential $V_a$ is independent of the external coordinate $x_a$ (i.e. $V_a(x) = V_a(x + y_a)$ for all $y \in X_a$). For simplicity, we assume that the potentials are bounded and short-range:

$$|V_a(x^a)| \leq C \langle x^a \rangle^{-1-\varepsilon}, \;\; \varepsilon > 0. \tag{1.6}$$

The **cluster Hamiltonians**

$$H_a = -\frac{1}{2}\Delta + \sum_{b \leq a} V_b(x^b) \tag{1.7}$$

describe the motion of the particles when the particles belonging to different clusters are infinitely separated and thus have negligible interactions due to the short-range assumption. The internal motion of these particles is described by the **internal Hamiltonian**, which acts on the Hilbert space $L^2(X^a)$:

$$H^a = -\frac{1}{2}\Delta^a + \sum_{b \leq a} V_b(x^b), \tag{1.8}$$

where $\Delta^a$ is the Laplacian on $L^2(X^a)$. We denote by $P^a$ the projection onto the eigenspace of $H^a$, and we write $P_a = \mathbb{1} \otimes P^a$, where the tensor product is understood with respect to the factorisation $L^2(X) = L^2(X_a) \otimes L^2(X^a)$.

The internal Hamiltonian $H^a$ is also a many-body Hamiltonian defined on the configuration space $Y = X^a$, with collision planes $Y_b = X^a \cap X_b$. To simplify the analysis, we may replace $H$ with $H^{a_{\max}}$ to ensure that $X_{a_{\max}} = \{0\}$. This condition is desirable because if the space $X_{a_{\max}}$ is non-trivial, then $H$ is translation-invariant due to the property $V_a(x + y_{a_{\max}}) = V_a(x)$ for all $a \in \mathcal{A}$ and $y_{a_{\max}} \in X_{a_{\max}}$. This replacement corresponds to separating the centre-of-mass motion because

$$H = -\frac{1}{2}\Delta_{a_{\max}} + H^{a_{\max}}. \tag{1.9}$$

Henceforth, we assume that $X_{a_{\max}} = \{0\}$.

### Asymptotic Completeness

**Asymptotic clustering** occurs if the many-body system asymptotically breaks up into freely moving clusters of particles. Mathematically, this means that, for every state $\psi \in \mathcal{H}$, there exists a family $\{\psi_{a,\pm}\}_{a\in\mathcal{H}}$ of vectors, with $\psi_{a,\pm} \in P_a\mathcal{H}$, such that the following condition holds:

$$\lim_{t\to\pm\infty} \| \mathrm{e}^{-\mathrm{i}tH}\psi - \sum_{a\in\mathcal{A}} \mathrm{e}^{-\mathrm{i}tH_a}\psi_{a,\pm}\| = 0. \tag{1.10}$$

Motivated by this condition, we define the following **wave operators**:

$$\Omega_{a,\pm} = \lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}tH}\,\mathrm{e}^{-\mathrm{i}tH_a}P_a. \tag{1.11}$$

The ranges of the wave operators $\Omega_{a,\pm}$, $a \neq a_{\max}$, are the scattering states, while the range of $\Omega_{a_{\max},\pm} = P_{\mathrm{pp}}(H)$ describes the bound states (i.e. eigenstates of $H$). Asymptotic completeness asserts that the Hilbert space $\mathcal{H}$ can be decomposed into bound and scattering states.

**Definition 1.1.2.** The quantum-mechanical many-body system described by the Hamiltonian $H$ on the Hilbert space $\mathcal{H} = L^2(X)$ is **asymptotically complete** if

$$\mathcal{H} = \bigoplus_{a\in\mathcal{A}} \mathrm{ran}(\Omega_{a,\pm}). \tag{1.12}$$

It is easy to verify that asymptotic completeness is equivalent to asymptotic clustering assuming that the wave operators (1.11) exist.

An important asymptotic observable in modern proofs of asymptotic completeness is the **asymptotic velocity**:

$$P^+ = \lim_{t\to\infty} \mathrm{e}^{\mathrm{i}tH}\frac{x}{t}\,\mathrm{e}^{-\mathrm{i}tH}, \tag{1.13}$$

where the limit is understood in the strong resolvent sense. Once the existence of the asymptotic velocity is established, proving asymptotic completeness for short-range potentials becomes relatively straightforward [DG97, Section 6.7]. Notably, the asymptotic velocity exists for a large class of potentials, including very slowly decaying long-range potentials for which asymptotic completeness is known to fail [DG97, Section 6.6].

**History**

Asymptotic completeness in non-relativistic quantum mechanics is a well-understood chapter of mathematical physics, thanks to numerous advances throughout the 20th century. We briefly review the historical developments.

For two-particle systems (or, equivalently, for single-particle systems in an external potential), asymptotic completeness was established between the late 1950s and the 1960s. Early results on the unitarity of the scattering operator were provided by Kato [Ka57], Rosenblum [Ro57], and Kuroda [Ku59]. It is difficult to determine the first complete proof of asymptotic completeness because several mathematical physicists contributed to this problem around the same time. Among these contributions, we highlight Kato's seminal paper [Ka66], which introduced the concept of Kato smooth operators, and Lavine's work [La70a, La70b], which applied this concept to establish the existence and completeness of the wave operators. These methods remained highly influential over the past decades. Parts of our proof of Theorem 1.3.3 (from the paper [Kr24a]) are based on these techniques.

Asymptotic completeness for three-particle systems with short-range potentials was first proved by Faddeev [Fa65], whose technique was later improved by Ginibre and Moulin [GM74] and Thomas [Th75]. Faddeev's method, which was time-independent, involved the use of the so-called Faddeev equations. These equations relate the resolvent of the Hamiltonian $H$ to the resolvent of the cluster Hamiltonians $H_a$. However, Faddeev's time-independent technique had certain limitations: it only allowed for finitely many eigenvalues, required very short-range potentials, and its generalisation to systems with more than three particles remained difficult.

Several years after Faddeev's publication, Enss [En78, En79, En83, En84] initiated a program to analyse the phase-space propagation of quantum systems and the existence of asymptotic observables. This approach proved to be more fruitful, as Enss successfully established asymptotic completeness for three-particle systems with short- and long-range potentials, without relying on implicit assumptions about the spectrum of the Hamiltonian. This time-dependent framework ultimately led to a proof of asymptotic completeness for many-body short-range systems by Sigal and Soffer [SS87]. Subsequent simplified proofs were provided by Graf [Gr90] and Yafaev [Ya93]. Dereziński [De93] completed the picture by establishing asymptotic completeness for many-body long-range systems (for textbook expositions, see [DG97, Ya00, Is23]).

Time-dependent scattering theory relies on propagation estimates, which illustrate that the motion of quantum particles tends to be concentrated along classical trajectories. Several authors have extensively studied various aspects of these propagation estimates, in particular in the large-velocity and low-velocity regimes, as well as propagation in phase space [IK84, IK85, Si90, Sk91, HS91, Is94, HSS99]. One example of a phase-space propagation estimate, originally established by Graf [Gr90], takes the following form:

$$\int_1^t \left\| \mathbb{1}_{[0,\theta]}\left(\frac{|x|}{t}\right)\left(\frac{x_a}{t} - D_a\right) q_a\left(\frac{x}{t}\right)\chi(H)\,\mathrm{e}^{-\mathrm{i}tH}\psi\right\|^2 \frac{\mathrm{d}t}{t} \le C\|\psi\|^2, \tag{1.14}$$

where $\theta > 0$ is arbitrary, $a \in \mathcal{A}$ is a cluster decomposition, $q_a$ is a channel localisation operator, and $\chi \in C_c^\infty(\mathbb{R})$. This estimate demonstrates that the relative mean velocity $x_a/t$ between the clusters converges to the instantaneous inter-cluster velocity $D_a$ within the channel region $a$. For a simple proof of this propagation estimate, see [DG97, Proposition 6.6.3].

An important technical element in many of the aforementioned papers is a Mourre estimate, a strictly positive commutator estimate for the Hamiltonian $H$ with a conjugate operator $A$:

$$E(J)[H, \mathrm{i}A]E(J) \geq \theta E(J), \ \theta > 0, \tag{1.15}$$

where $E$ is the spectral measure of $H$ and $J \subset \mathbb{R}$ is a real subset that is separated from the eigenvalues and thresholds of $H$. (Thresholds are eigenvalues of the internal Hamiltonians $H^a$, $a \neq a_{\max}$.) This estimate plays a crucial role not only in establishing spectral properties of the Hamiltonian, such as the absence of singular continuous spectrum, but also in propagation estimates. For two- and three-body systems, a Mourre estimate was originally proved by Mourre [Mo81], and this result was later extended to many-body systems by Perry, Sigal, and Simon [PSS81] and Froese and Herbst [FH82]. In Section 1.4, we explain more details of Mourre's method.

**Dispersive Hamiltonians**

We conclude this subsection with some remarks on the scattering theory of dispersive many-body systems. Dispersive Hamiltonians are obtained by replacing the Laplacian in the Hamiltonian (1.5) with a more general dispersion relation $h(D)$:

$$H = h(D) + \sum_{a \in \mathcal{A}} V_a(x^a). \tag{1.16}$$

For two particles, proving asymptotic completeness works exactly as in the case of a quadratic dispersion relation (i.e. $h(D) = D^2/2$). However, for systems with three or more particles, asymptotic completeness is an open problem for dispersive Hamiltonians [DG97, p. 274] [Ya00, Section 11.5].

This is not merely a technical problem that could be solved by a simple adaptation of the existing proofs of asymptotic completeness. In the many-body case, establishing asymptotic completeness is difficult because the relative motion of the clusters and the internal motion of the particles within the clusters remain interdependent asymptotically. In contrast, if the dispersion relation is quadratic, the external and internal motions separate asymptotically. To explain this point, let us examine the classical equations of motion by considering the Hamilton function $\mathscr{H}_a$, which describes the asymptotic dynamics of the clusters in a cluster decomposition $a \in \mathcal{A}$:

$$\mathscr{H}_a(x, \xi) = h(\xi_a, \xi^a) + \sum_{b \leq a} V_b(x^b), \tag{1.17}$$

where $\xi_a$ is the inter-cluster momentum and $\xi^a$ the internal momentum of the particles within the clusters. We obtained $\mathscr{H}_a$ from the system's complete Hamilton function by suppressing the interaction between particles from different clusters. The Hamilton equations read as follows:

$$\dot{x}_a = \partial_{\xi_a} h(\xi_a, \xi^a), \qquad \dot{\xi}_a = 0, \tag{1.18}$$

$$\dot{x}^a = \partial_{\xi^a} h(\xi_a, \xi^a), \qquad \dot{\xi}^a = -\sum_{b \le a} \partial_{x^a} V_b(x^b). \tag{1.19}$$

From these equations, we conclude that the inter-cluster momentum $\xi_a$ is constant. The relative motion between the clusters is determined by integrating the Hamilton equation for $\dot{x}_a$:

$$x_a(t) = x_a(0) + \int_0^t \partial_{\xi_a} h(\xi_a, \xi^a(s))\,\mathrm{d}s. \tag{1.20}$$

We observe that, for general dispersion relations $h$, the relative motion of the clusters is influenced by the internal motion of the particles within the clusters. However, when $h(\xi) = \xi^2/2 = \xi_a^2/2 + (\xi^a)^2/2$, the velocity $\partial_{\xi_a} h(\xi) = \xi_a$ becomes independent of the internal momentum $\xi^a$.

The asymptotic interdependence of the external and internal motions significantly complicates the asymptotic dynamics in the dispersive case. In particular, the phase-space propagation estimate (1.14) cannot be expected to generalise to the dispersive case because $\partial_a h(D)$ is not the correct inter-cluster velocity; it fails to account for effects of the internal motion. Instead, the classical equations of motion suggest that a more realistic propagation estimate would take the following form:

$$\int_1^\infty \left\| \mathbb{1}_{[0,\theta]}\left(\frac{|x|}{t}\right)\left(\frac{x}{t} - \frac{1}{t}\int_0^t \mathrm{e}^{-\,\mathrm{i}sH_a} h'(D)\,\mathrm{e}^{\mathrm{i}sH_a}\,\mathrm{d}s\right) q_a\left(\frac{x}{t}\right)\chi(H)\,\mathrm{e}^{-\,\mathrm{i}tH}\psi_t \right\|^2 \frac{\mathrm{d}t}{t} \le C\|\psi\|^2, \tag{1.21}$$

where $\theta$, $q_a$, and $\chi$ are as in (1.14). Establishing (1.21) would certainly be a major step towards asymptotic completeness for dispersive many-body systems. As we discuss later (see Section 1.5), progress on this problem may also be of relevance in quantum field theory, where the dispersion relation of the particles is relativistic.

### 1.1.2 Asymptotic Completeness in Quantum Field Theory

In the previous subsection, we illustrated that the problem of asymptotic completeness is solved in non-relativistic quantum mechanics. In contrast, asymptotic completeness remains an open and challenging problem in local relativistic quantum field theory (QFT), with only a few partial results available for specific models. Notably, asymptotic completeness has been established in the low-coupling regime of the $P(\phi)_2$ model at the level of two [SZ76] and three particles [CD82]. The techniques employed in the cited papers are similar to the time-independent methods used in quantum mechanics, with the latter work being an adaptation of Faddeev's method to the QFT setting. More recent developments have led to the construction of asymptotically complete integrable models [Le08, Ta14], in which the scattering operator is specified from the outset. Asymptotic completeness has also been established for certain wedge-local models [DT11, DD23, BC24]. However, despite these impressive results, a complete understanding of asymptotic completeness in QFT remains elusive at the moment.

Proving asymptotic completeness in QFT is a difficult problem due to significant conceptual and technical challenges. One of the most notable differences to quantum mechanics is that QFT allows for processes that create or annihilate particles during scattering processes. A proof of asymptotic completeness would likely require prior knowledge of all types of particles present in a given QFT model. However, identifying these particles is itself a non-trivial problem because it is generally not possible to read out the particle content directly from the Lagrangian or equations of motions defining the model (e.g. additional bound states, such as solitons in $P(\phi)_2$ models, may emerge and contribute to the particle spectrum). Moreover, quantum field theories typically have a rich superselection structure, arising from inequivalent representations of the algebra of observables [DHR71, DHR74, BF82] (see [HM06] for a concise review). Inequivalent representations define different sectors, each labelled by some kind of a charge. The presence of charged particles poses additional difficulties, which we discuss further in Subsection 1.2.2.

Moreover, counterexamples to asymptotic completeness in QFT are known, which fall within the conventional frameworks of axiomatic quantum field theory as described in Subsection 1.2.1 below. These so-called generalised free fields [Gr61] contain unphysical states with too many degrees of freedom that do not allow a particle interpretation. A proof of asymptotic completeness would therefore require conditions that exclude these pathological states. Promising candidates for such conditions are phase-space compactness criteria [HS65, DL84, BP90] [Ha96, Section V.5].

On a positive note, progress on asymptotic completeness has been made in non-relativistic quantum field theory, which is considered to be an intermediate framework between quantum mechanics and local relativistic quantum field theory. For example, asymptotic completeness has been established for confined Pauli–Fierz Hamiltonians by Dereziński and Gérard [DG99], for the confined Nelson model by Ammari [Am00], for the $P(\phi)_2$ model in finite volume by Dereziński and Gérard [DG00], for Rayleigh and Compton scattering by Fröhlich, Griesemer, and Schlein [FGS02, FGS04], and for the translation-invariant Nelson model below the three-boson threshold by Dybalski and Møller [DM15]. A common feature of these papers is that the employed methods closely resemble the time-dependent techniques from quantum mechanics.

## 1.2 Algebraic Quantum Field Theory

Algebraic quantum field theory (AQFT) provides a mathematically rigorous framework for studying quantum field theories using the language of operator algebras. Developed in the mid-20th century by Haag, Kastler, and others [HK64], AQFT aims to describe quantum fields independently of specific models. Central to the framework is the notion of a local net of algebras: to each region of spacetime, one associates an algebra, whose elements represent observables measurable within that region. The axioms of AQFT, such as isotony, locality, and translation covariance (see below), reflect fundamental physical principles like causality and the independence of space-like separated observables.

### 1.2.1 Axioms

An algebraic quantum field theory is described by a $C^*$-algebra $\mathcal{A}$, generated by a net of von Neumann algebras $\{\mathcal{A}(O)\}_O$ on a common Hilbert space $\mathcal{H}$. The net is indexed by open and bounded spacetime regions $O \subset \mathbb{R}^d$ and is required to satisfy the following properties:

- Isotony: If $O_1 \subset O_2$, then $\mathcal{A}(O_1) \subset \mathcal{A}(O_2)$.
- Locality: If $O_1$ and $O_2$ are space-like separated, then elements from the corresponding algebras commute:

$$[\mathcal{A}(O_1), \mathcal{A}(O_2)] = 0. \tag{1.22}$$

- Translation covariance: A unitary representation $U : \mathbb{R}^d \to \mathfrak{B}(\mathcal{H})$ of the translation group $\mathbb{R}^d = \mathbb{R}^{1+s}$ exists that satisfies

$$U(x)\mathcal{A}(O)U(x)^* = \mathcal{A}(O + x), \ x \in \mathbb{R}^d. \tag{1.23}$$

  For an element $A \in \mathcal{A}$, we write $A(x)$ for the Heisenberg evolution $U(x)AU(x)^*$ and abbreviate $A(0, \mathbf{x})$ with $A(\mathbf{x})$.

- Vacuum vector: There is a translation-invariant vector $\Omega \in \mathcal{H}$ with norm $\|\Omega\| = 1$:

$$U(x)\Omega = \Omega, \ x \in \mathbb{R}^d. \tag{1.24}$$

  The vector $\Omega$ is the unique vector (up to a phase) that satisfies (1.24).

- Spectrum condition: The generators of the translation group $\mathbb{R}^d$ are the energy-momentum operators $P = (H, \mathbf{P})$ (i.e. $U(x) = \mathrm{e}^{-\mathrm{i}x\cdot P}$). The joint spectrum $\sigma(P)$ of the energy-momentum operators $P$ is contained in the forward light-cone $V_+$:

$$\sigma(P) \subset V_+ = \{(p_0, \mathbf{p}) \in \mathbb{R}^d \mid p_0 \geq |\mathbf{p}|\}. \tag{1.25}$$

To simplify the construction of scattering states, we assume the existence of an isolated mass shell of one-particle states in the energy-momentum spectrum, a condition typically satisfied by massive quantum field theory models. In the following, we always adopt the following assumption (see also Figure 1.2):

- Strong spectrum condition: The energy-momentum spectrum consists of the point 0, an isolated mass shell $H_m$, and the multi-particle spectrum $G_{2m}$, for an $m > 0$:

$$\{0\} \cup H_m \subset \sigma(P) \subset \{0\} \cup H_m \cup G_{2m}, \tag{1.26}$$

  where

$$H_m = \{(p_0, \mathbf{p}) \in \mathbb{R}^d \mid p_0 = \omega(\mathbf{p}) = \sqrt{m^2 + |\mathbf{p}|^2}\}, \tag{1.27}$$

$$G_{2m} = \{(p_0, \mathbf{p}) \in \mathbb{R}^d \mid p_0 \geq \sqrt{(2m)^2 + |\mathbf{p}|^2}\}. \tag{1.28}$$

If $E$ is the joint spectral measure of the energy-momentum operators $P$, then we define the one-particle space $\mathfrak{h}_m$ as the spectral subspace of $\mathcal{H}$ corresponding to the mass shell $H_m$:

$$\mathfrak{h}_m = E(H_m)\mathcal{H}. \tag{1.29}$$

States in $\mathfrak{h}_m$ are eigenstates of the relativistic mass operator $M = \sqrt{H^2 - |\mathbf{P}|^2}$ with eigenvalue $m$.

### 1.2.2 Haag–Ruelle Scattering Theory

Haag–Ruelle scattering theory provides a framework for defining scattering states in AQFT. Unlike conventional approaches to scattering theory in QFT, the Haag–Ruelle theory builds on fundamental principles such as locality, translation covariance, and the spectrum condition. This approach originated in the seminal works of Haag [Ha58] and Ruelle [Ru62]. More modern treatments of Haag–Ruelle scattering theory can be found in [BLOT90, Chapter 12] and [DG14a, DG14b].

#### Creation operators

In the first step of the construction, we define creation operators $B^* \in \mathcal{A}$, which create one-particle states from the vacuum $\Omega$ with good localisation properties. Due to the Reeh–Schlieder theorem, it is untenable to take $B^*$ from a local algebra $\mathcal{A}(O)$. Instead, we choose $B^*$ to be almost local.

**Definition 1.2.1.** Let $K_r$ be the double cone of radius $r > 0$. An element $A \in \mathcal{A}$ is **almost local** if a sequence $(A_r)$ of local operators $A_r \in \mathcal{A}(K_r)$ exists such that $A_r$ converges rapidly in norm to $A$ as $r \to \infty$ (i.e., for every $N \in \mathbb{N}$, $\|A - A_r\| \leq C_N r^{-N}$).

The energy-momentum transfer (or Arveson spectrum [Ar74]) of an element $A \in \mathcal{A}$ measures the change in energy-momentum of a state $\psi \in \mathcal{H}$ under the application of $A$.

**Proposition 1.2.2** ([DG14a, (2.4)])**.** *Let $A \in \mathcal{A}$. If $\Delta \subset \mathbb{R}^d$ is a Borel set, then*

$$AE(\Delta)\mathcal{H} \subset E(\overline{\Delta + \sigma_\alpha(A)})\mathcal{H}, \tag{1.30}$$

*where $\sigma_\alpha(A)$ is the* ***energy-momentum transfer*** *of $A$, defined as the support of the operator-valued distribution*

$$\mathbb{R}^d \ni p \mapsto \check{A}(p) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \mathrm{e}^{-\mathrm{i}p\cdot x} A(x)\,\mathrm{d}x. \tag{1.31}$$

The vector $B^*\Omega$ is a one-particle state with good localisation properties, provided that $B^*$ is almost local and its energy-momentum transfer is contained in a sufficiently small neighbourhood around a point of the mass shell $H_m$. This observation motivates the following definition:

**Definition 1.2.3.** An element $B^* \in \mathcal{A}$ is a **creation operator** if $B^*$ is almost local, its energy-momentum transfer $\sigma_\alpha(B^*)$ is a compact subset of the forward light cone $V_+$, and $\emptyset \neq \sigma_\alpha(B^*) \cap \sigma(P) \subset H_m$.

Let $B^*$ be a creation operator and $f_t = \mathrm{e}^{-\mathrm{i}t\omega(D_\mathbf{x})} f$ a Klein–Gordon wave packet, where $\omega(D_\mathbf{x}) = \sqrt{m^2 + |D_\mathbf{x}|^2}$ and $f \in L^2(\mathbb{R}^s)$ ($s = d-1$ is the space dimension). We define the following **Haag–Ruelle creation operators**, which compare the time evolution of the full Hamiltonian $H$ with that of a Klein–Gordon wave packet:

$$B_t^*[f_t] = \int_{\mathbb{R}^s} f_t(\mathbf{x}) B_t^*(\mathbf{x})\,\mathrm{d}\mathbf{x}, \quad B_t^*(\mathbf{x}) = U(t,\mathbf{x}) B^* U(t,\mathbf{x})^*. \tag{1.32}$$

### Scattering states

The following theorem is the main result of the Haag–Ruelle theory, which provides the construction of many-particle scattering states.

**Theorem 1.2.4** ([DG14b, Theorem 6.5])**.** *Let $B_1^*, \dots, B_n^*$ be creation operators and $f_1, \dots, f_n \in L^2(\mathbb{R}^s)$. The outgoing (out) and incoming (in) scattering states exist:*

$$\psi_1 \overset{\text{out}}{\times} \cdots \overset{\text{out}}{\times} \psi_n = \lim_{t\to\infty} B_{1,t}^*[f_{1,t}] \dots B_{n,t}^*[f_{n,t}]\Omega, \tag{1.33}$$

$$\psi_1 \overset{\text{in}}{\times} \cdots \overset{\text{in}}{\times} \psi_n = \lim_{t\to-\infty} B_{1,t}^*[f_{1,t}] \dots B_{n,t}^*[f_{n,t}]\Omega. \tag{1.34}$$

*The limits depend only on the one-particle states $\psi_i = B_i^*[f_i]\Omega \in \mathfrak{h}_m$.*

We define by $\mathcal{H}^{\text{out}}$ and $\mathcal{H}^{\text{in}}$ the Hilbert spaces spanned by all outgoing and incoming scattering states, respectively:

$$\mathcal{H}^{\text{out}} = \operatorname{span}\{\Omega, \psi_1 \overset{\text{out}}{\times} \dots \overset{\text{out}}{\times} \psi_n \mid \psi_1, \dots, \psi_n \in \mathfrak{h}_m, n \in \mathbb{N}\}, \tag{1.35}$$

$$\mathcal{H}^{\text{in}} = \operatorname{span}\{\Omega, \psi_1 \overset{\text{in}}{\times} \dots \overset{\text{in}}{\times} \psi_n \mid \psi_1, \dots, \psi_n \in \mathfrak{h}_m, n \in \mathbb{N}\}. \tag{1.36}$$

In this terminology, the vacuum vector and the one-particle states are treated as scattering states. However, we may also consider them as bound states because the vacuum vector and the one-particles states are eigenstates of the relativistic mass operator $M = \sqrt{H^2 - |\mathbf{P}|^2}$.

It can be shown that $\mathcal{H}^{\text{out}}$ and $\mathcal{H}^{\text{in}}$ can be identified in a natural way as (bosonic) Fock spaces over the one-particle space $\mathfrak{h}_m$. Specifically, let $\Gamma(\mathfrak{h}_m)$ be the symmetric Fock space over $\mathfrak{h}_m$ with Fock vacuum $\Omega_0$, and, for $\psi \in \mathfrak{h}_m$, let $a^*(\psi)$ be the Fock creation operator such that $a^*(\psi)\Omega_0 = \psi$. We define the **wave operators** $W^{\text{out}} : \Gamma(\mathfrak{h}_m) \to \mathcal{H}^{\text{out}}$ and $W^{\text{in}} : \Gamma(\mathfrak{h}_m) \to \mathcal{H}^{\text{in}}$ by the following relations:

$$W^{\text{out}}\Omega_0 = W^{\text{in}}\Omega_0 = \Omega, \tag{1.37}$$

$$W^{\text{out}}(a^*(\psi_1)\cdots a^*(\psi_n)\Omega_0) = \psi_1 \overset{\text{out}}{\times} \cdots \overset{\text{out}}{\times} \psi_n, \tag{1.38}$$

$$W^{\text{in}}(a^*(\psi_1)\cdots a^*(\psi_n)\Omega_0) = \psi_1 \overset{\text{in}}{\times} \cdots \overset{\text{in}}{\times} \psi_n. \tag{1.39}$$

It is not difficult to verify that the wave operators defined in this way commute with the representation of the translation group. Due to this natural identification, scattering states have a clear interpretation in terms of particle states.

### 1.2.3 Asymptotic Completeness

Asymptotic completeness asserts that the ranges of the wave operators $W^{\text{out}}$ and $W^{\text{in}}$, respectively, coincide with the Hilbert space $\mathcal{H}$.

**Definition 1.2.5.** An algebraic quantum field theory is **asymptotically complete** if

$$\mathcal{H} = \mathcal{H}^{\text{out}} = \mathcal{H}^{\text{in}}. \tag{1.40}$$

As explained in Subsection 1.1.2, proving asymptotic completeness in quantum field theory is a challenging problem. One complication, which is common to systems with infinitely many degrees of freedom, arises from the potential presence of charged particles. These charged particles are related to inequivalent representations of the algebra $\mathcal{A}$, as we mentioned in Subsection 1.1.2. Particles from superselection sectors with opposite charges can combine to form neutral pairs, which contribute additional states to the multi-particle spectrum of the vacuum sector. For simplicity, the construction of scattering states, which we presented above, was restricted to the vacuum sector. The Haag–Ruelle theory can be generalised to include charged particles [DHR74, BF82], which must be taken into account when attempting to establish asymptotic completeness in quantum field theories with non-trivial superselection structures.

#### Strategy for proving asymptotic completeness

Despite the significant challenges, which we encounter in quantum field theory, there exists a promising strategy for proving asymptotic completeness. This approach traces back to ideas from Buchholz [Bu86, Bu95] and Haag [Ha96, Chapter VI] and is based on the concept of particle detectors. The logical structure of the argument can be summarised in three main steps:

1. *Identification of particle detectors*: The first step is to identify observables in the algebra $\mathcal{A}$ that can be interpreted as particle detectors.

2. *Triggering by scattering states*: It must then be shown that these particle detectors are only triggered by scattering states. Mathematically, one has to demonstrate that any state from the orthogonal complements $(\mathcal{H}^{\text{out}})^{\perp}$ and $(\mathcal{H}^{\text{in}})^{\perp}$ of the scattering states lies in the kernel of any particle detector.

3. *Accessibility of quantum states through detectors*: Finally, it must be proved that every non-zero quantum state in $\mathcal{H}$ can trigger at least one particle detector.

If these three steps are accomplished, it follows that the orthogonal complements $(\mathcal{H}^{\text{out}})^{\perp}$ and $(\mathcal{H}^{\text{in}})^{\perp}$ are trivial. Consequently, every state in $\mathcal{H}$ is a scattering state, thereby establishing asymptotic completeness.

The appropriate observables, which serve as particle detectors, were identified by Araki and Haag [AH67]. They constructed asymptotic observables as asymptotic limits $(t \to \pm\infty)$ of the following sequences:

$$\mathrm{e}^{\mathrm{i}tH} \int_{\mathbb{R}^s} h\left(\frac{\mathbf{x}}{t}\right) (B^* B)(\mathbf{x}) \, \mathrm{d}\mathbf{x} \, \mathrm{e}^{-\mathrm{i}tH}, \tag{1.41}$$

where $h \in L^{\infty}(\mathbb{R}^s)$ is a velocity function and $B^*$ is a creation operator. These asymptotic observables, now known as Araki–Haag detectors, have a natural interpretation as particle counters. Later, Buchholz [Bu90] found an important uniform bound that facilitates the treatment of these detectors. We explain the details of the construction in Section 1.3.

Araki and Haag proved the convergence of Araki–Haag detectors on scattering states. However, to accomplish Step 2, we must prove the convergence on arbitrary states. This problem has been the focus of considerable interest among mathematical physicists since the publication of Araki and Haag's seminal paper [Bu86, Bu95, Po04, BS06]. However, only relatively recently Dybalski and Gérard [DG14a, DG14b] made significant progress in this area. They analysed the convergence of products of Araki–Haag detectors sensitive to particles with distinct velocities (i.e. products of detectors where the velocity functions have disjoint support) on arbitrary states of bounded energy. Their results relied on techniques from time-dependent quantum-mechanical scattering theory, such as large-velocity and phase-space propagation estimates. However, they could not establish the convergence of a single detector due to a missing low-velocity estimate.

In the paper [Kr24a], we built upon the framework developed by Dybalski and Gérard and successfully proved the convergence of a single Araki–Haag detector on states below the three-particle threshold. Furthermore, we demonstrated that the orthogonal complement of the scattering states (at the two-particle level) is mapped to 0 by Araki–Haag detectors. These results are described in Theorem 1.3.3 below and address Step 2 at the two-particle level.

At the moment, Step 3 remains open for future investigation. We anticipate that proving this step axiomatically is challenging because detailed knowledge of the superselection structure and particle content of a given model is likely required to establish the accessibility of quantum states through detectors.

Notably, there are approaches to Step 3 which rely on the existence of an energy-momentum tensor $T_{\mu\nu}(\mathbf{x})$ [Kr24a, Section 5.1]. Given that such a density for the energy-momentum operators exists, we may approach Step 3 as follows. If $\psi$ is a state of positive energy, then

$$0 < \langle \psi, H\psi \rangle = \langle \mathrm{e}^{-\mathrm{i}tH}\psi, H\,\mathrm{e}^{-\mathrm{i}tH}\psi \rangle = \langle \mathrm{e}^{-\mathrm{i}tH}\psi, \int_{\mathbb{R}^s} T_{00}(\mathbf{x})\,\mathrm{d}\mathbf{x}\,\mathrm{e}^{-\mathrm{i}tH}\psi \rangle, \tag{1.42}$$

where we used that *(i)* $\psi$ has positive energy, *(ii)* $H$ commutes with the time evolution $\mathrm{e}^{\mathrm{i}tH}$, and *(iii)* formally, the 00-component of the energy-momentum tensor integrates to the Hamiltonian. The observable on the right-hand side of (1.42) structurally resembles that of an approximating sequence of an Araki–Haag detector. If we could apply the convergence result of Theorem 1.3.3, it would follow that $\psi$ is triggered by an Araki–Haag detector, as the limit on the right-hand side of (1.42) would be non-zero.

The challenge, however, is that $T_{00}(\mathbf{x})$ does not have the canonical form $(B^*B)(\mathbf{x})$. To apply Theorem 1.3.3, we would need to write or approximate the energy-momentum tensor by a sum with operators of the form $B_i^*B_i$, where each $B_i^*$ is a creation operator. An approximation property of a similar type has been established by Dybalski [Dy08, Appendix D] for the free massive scalar field. It would be interesting to generalise this result to interacting quantum field theories.

We emphasise that this approach to establish Step 3 requires controlling the convergence of a single Araki–Haag detector or, through an iteration of the argument (1.42), the convergence of products of detectors sensitive to particles with the same velocity. The convergence of detectors sensitive to particles with disjoint velocity, as in the papers of Dybalski and Gérard, would not be sufficient. In this sense, Theorem 1.3.3 brings us closer to asymptotic completeness than this earlier result.

## 1.3 Araki–Haag Detectors

In this section, we formalise the notion of a particle detector. We review the results of Araki and Haag's seminal paper [AH67] and present our new result on the convergence of Araki–Haag detectors on arbitrary states.

### 1.3.1 Araki–Haag Formula

Araki and Haag [AH67] defined a detector $C$ as an almost local observable measuring deviations from the vacuum.

**Definition 1.3.1.** A **detector** is an almost local observable $C \in \mathcal{A}$ that is self-adjoint and annihilates the vacuum vector $\Omega$ (i.e. $C\Omega = 0$).

*Example.* Let $B \in \mathcal{A}$ be almost local. If $\sigma_\alpha(B) \cap V_+ = \emptyset$, where $V_+$ is the forward light cone, then $C = B^*B$ is a detector. The detectors of the form $B^*B$ generate a $*$-algebra $\mathcal{C}$, in which each element is a detector.

Let $C = B^*B$ be a detector as in the example. Araki and Haag analysed the Heisenberg evolution of the observable $C$. They observed that $C(t,\mathbf{x}) = U(t,\mathbf{x})CU(t,\mathbf{x})^*$ converges weakly to 0 as $t \to \infty$, which is a result of the dispersion of quantum states. Physically, this means that, after a sufficiently long time, information about the particle's localisation is lost. We integrate $C(t,\mathbf{x})$ over $\mathbb{R}^s$ to compensate for dispersion:

$$C(h;t) = \mathrm{e}^{\mathrm{i}tH} \int_{\mathbb{R}^s} h\left(\frac{\mathbf{x}}{t}\right)(B^*B)(\mathbf{x})\,\mathrm{d}\mathbf{x}\,\mathrm{e}^{-\,\mathrm{i}tH},\ h \in L^\infty(\mathbb{R}^s). \tag{1.43}$$

In [AH67], it was assumed that $h$ is compactly supported to ensure the convergence of the integral. However, later it was shown by Buchholz [Bu90] that the integral is well-defined on subspaces of finite energy thanks to the following uniform bound:

**Proposition 1.3.2** ([DG14a, Lemma 3.4])**.** *Let $B \in \mathcal{A}$ be almost local such that $\sigma_\alpha(B) \cap V_+ = \emptyset$. For every compact subset $\Delta \subset \mathbb{R}^d$ and $\psi \in \mathcal{H}$,*

$$\int_{\mathbb{R}^s} \|B(\mathbf{x})E(\Delta)\psi\|^2\,\mathrm{d}\mathbf{x} \le C_\Delta \|\psi\|^2. \tag{1.44}$$

Araki and Haag [AH67, Theorem 4] proved the following asymptotic formula (Araki–Haag formula), for regular scattering states $\phi, \psi \in \mathcal{H}^{\mathrm{out}}$ of bounded energy:

$$\lim_{t\to\infty} \langle \phi, C(h;t)\psi\rangle = (2\pi)^s \int_{\mathbb{R}^s} h(\nabla\omega(\mathbf{p}))\langle \mathbf{p}|B^*B|\mathbf{p}\rangle\langle \phi, a^*_{\mathrm{out}}(\mathbf{p})a_{\mathrm{out}}(\mathbf{p})\psi\rangle\,\mathrm{d}\mathbf{p}, \tag{1.45}$$

where $a^*_{\mathrm{out}}(\mathbf{p}) = W^{\mathrm{out}}a^*(\mathbf{p})(W^{\mathrm{out}})^*$. The asymptotic observable (1.45) is a Fock space number operator (i.e. a particle counter). The additional factor $h(\nabla\omega(\mathbf{p}))\langle \mathbf{p}|B^*B|\mathbf{p}\rangle$ is interpreted as the sensitivity of the counter to measure a particle of momentum $\mathbf{p}$. Specifically, $h$ is a velocity filter because particles with group velocity $\nabla\omega(\mathbf{p})$ outside the support of $h$ are not counted.

### 1.3.2 Convergence of Araki–Haag Detectors

The proof of the Araki–Haag formula follows from a relatively straightforward computation, which uses favourable properties of scattering states. However, extending the convergence result to arbitrary states appears to be significantly more difficult. In the paper [Kr24a], we managed to prove the following result.

**Theorem 1.3.3.** *Let $\Delta \subset \mathbb{R}^d$ be compact, $\psi \in E(\Delta)\mathcal{H}_{\mathrm{ac}}(P)$, and $B^*$ a creation operator. If $\overline{\Delta - \sigma_\alpha(B^*)} \cap \sigma(P) \subset H_m$, then, for every $h \in L^\infty(\mathbb{R}^s)$,*

$$C(h;t)\psi = \mathrm{e}^{\mathrm{i}tH} \int_{\mathbb{R}^s} h\left(\frac{\mathbf{x}}{t}\right)(B^*B)(\mathbf{x})\,\mathrm{d}\mathbf{x}\,\mathrm{e}^{-\,\mathrm{i}tH}\psi \tag{1.46}$$

*converges in $\mathcal{H}$ as $t \to \infty$. The limit is 0 if $\psi \in (\mathcal{H}^{\mathrm{out}})^\perp$.*

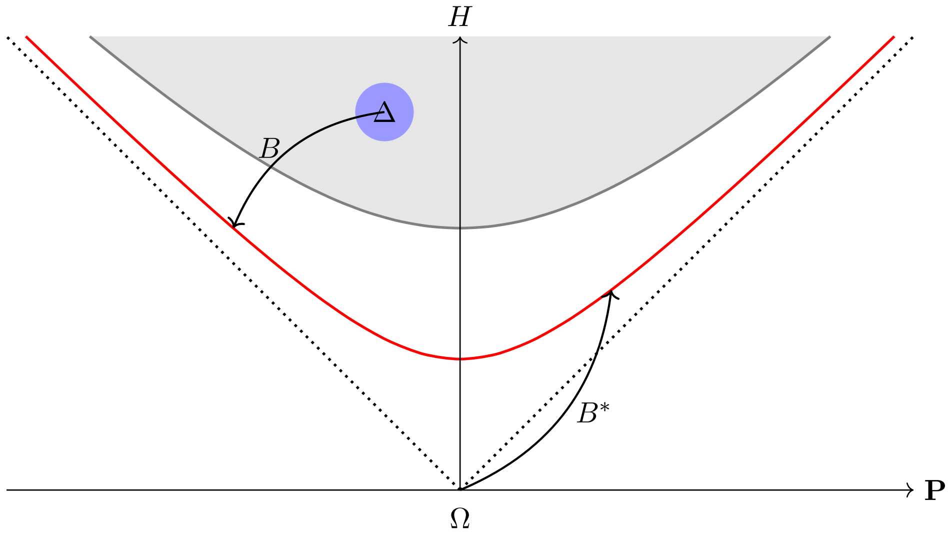


Figure 1.2: The energy-momentum spectrum contains an isolated mass shell (red line) and a continuous multi-particle spectrum (grey area) above the two-particle threshold (grey line). A creation operator $B^*$ maps the vacuum vector $\Omega$ to a one-particle state. In Theorem 1.3.3, we assume that $B^*$ is such that $\overline{\Delta - \sigma_\alpha(B^*)} \cap \sigma(P)$ is a subset of the mass shell.

Before presenting the main ideas of the proof, we discuss the assumptions of the theorem. The condition $\overline{\Delta - \sigma_\alpha(B^*)} \cap \sigma(P) \subset H_m$ effectively selects states from the multi-particle spectrum that lie below the three-particle threshold. If both the energy-momentum spectrum of $\psi$ and the energy-momentum transfer of $B^*$ were point-like, then, for creation operators $B^*$ and states $\psi$ below the three-particle threshold, $B\psi$ would either be zero or a one-particle state. However, because the energy-momentum spectrum of $\psi$ and the energy-momentum transfer of $B^*$ are not point-like but have a finite extension, it is possible for $B\psi$ to have a component in the multi-particle spectrum even when $\psi$ lies below the three-particle threshold.

Concerning the spectral assumption $\psi \in \mathcal{H}_{\mathrm{ac}}(P)$, we recall that the Hilbert space $\mathcal{H}$ can be decomposed into the pure point, absolutely continuous, and singular continuous spectral subspaces of the energy-momentum operators $P$:

$$\mathcal{H} = \mathcal{H}_{\mathrm{pp}}(P) \oplus \mathcal{H}_{\mathrm{ac}}(P) \oplus \mathcal{H}_{\mathrm{sc}}(P). \tag{1.47}$$

The pure point spectral subspace $\mathcal{H}_{\mathrm{pp}}(P)$ is spanned by the vacuum vector $\Omega$, whereas the singular continuous spectral subspace $\mathcal{H}_{\mathrm{sc}}(P)$ typically consists of eigenstates of the mass operator. These eigenstates correspond to mass shells in the energy-momentum spectrum, which constitute non-atomic Lebesgue null sets in $\mathbb{R}^d$. It is relatively straightforward to establish the convergence of Araki–Haag detectors on eigenstates of the mass operator (see [Kr24a, Proposition 3.3]). However, in general, $\mathcal{H}_{\mathrm{sc}}(P)$ may also contain exotic states related to the singular continuous spectrum of the mass operator, for which we are unable to control the convergence of Araki–Haag detectors.

We note that in asymptotically complete models, the singular continuous spectral subspace is trivial (i.e. $\mathcal{H}_{\mathrm{sc}}(P) = \{0\}$) because the energy-momentum operators $P$ are unitarily equivalent to those of the free field theory in this case. However, our convergence results also applies to some models which are not asymptotically complete, such as certain generalised free fields. This situation is analogous to the asymptotic velocity in quantum mechanics, whose convergence can be proved even when the potential decays too slowly for asymptotic completeness to hold (see Subsection 1.1.1).

*Main steps of the proof of Theorem 1.3.3.*

1. It is enough to establish the convergence of (1.46) for states $\psi \in (\mathcal{H}^{\mathrm{out}})^{\perp}$ because the convergence of (1.46) on scattering states has been proved in previous works [AH67] [DG14b, Proposition 7.1]. (A similar observation was made in [Dy18].) By an application of Proposition 1.3.2 and the Cauchy–Schwarz inequality (see [Kr24a, (3.10)]), it also suffices to prove that, for $\psi \in (\mathcal{H}^{\mathrm{out}})^{\perp}$,

$$\lim_{t\to\infty} \int_{\mathbb{R}^s} \langle \mathrm{e}^{-\mathrm{i}tH}\psi, (B^*B)(\mathbf{x})\, \mathrm{e}^{-\mathrm{i}tH}\psi\rangle \, \mathrm{d}\mathbf{x} = 0. \tag{1.48}$$

2. We utilise the assumption that $B\psi \in \mathfrak{h}_m$ is a one-particle state, along with the following lemma:

**Lemma 1.3.4** ([Kr24a, Lemma 3.5])**.** *If $\Delta \subset \mathbb{R}^d$ is compact and $\Delta \cap \sigma(P) \subset H_m$, then there exists a creation operator $B_2^*$ such that*

$$E(\Delta) = E(\Delta) \int_{\mathbb{R}^s} (B_2^* B_2)(\mathbf{y})\, \mathrm{d}\mathbf{y}\, E(\Delta). \tag{1.49}$$

This lemma is noteworthy because it demonstrates that one-particle states are accessible through Araki–Haag detectors. The proof of the lemma involves constructing a creation operator $B_2^*$ such that the wave function of the one-particle state $B_2^*\Omega$ in momentum space equals 1 on the projection of $\Delta$ onto the mass shell $H_m$. We apply this lemma by inserting a second Araki–Haag detector into (1.48):

$$\begin{aligned}
&\int_{\mathbb{R}^s} \langle \mathrm{e}^{-\mathrm{i}tH}\psi, (B^*B)(\mathbf{x})\, \mathrm{e}^{-\mathrm{i}tH}\psi\rangle \, \mathrm{d}\mathbf{x} \\
&= \int_{\mathbb{R}^s}\int_{\mathbb{R}^s} \langle \mathrm{e}^{-\mathrm{i}tH}\psi, B^*(\mathbf{x})(B_2^*B_2)(\mathbf{y})B(\mathbf{x})\, \mathrm{e}^{-\mathrm{i}tH}\psi\rangle \, \mathrm{d}\mathbf{x}\, \mathrm{d}\mathbf{y} \\
&= \int_{\mathbb{R}^s}\int_{\mathbb{R}^s} |\langle \mathrm{e}^{-\mathrm{i}tH}\psi, B^*(\mathbf{x})B_2^*(\mathbf{y})\Omega\rangle|^2 \, \mathrm{d}\mathbf{x}\, \mathrm{d}\mathbf{y} \\
&= \int_{\mathbb{R}^s}\int_{\mathbb{R}^s} |\langle \psi, \mathrm{e}^{\mathrm{i}t(H-\omega(D_{\mathbf{x}})-\omega(D_{\mathbf{y}}))} B^*(\mathbf{x})B_2^*(\mathbf{y})\Omega\rangle|^2 \, \mathrm{d}\mathbf{x}\, \mathrm{d}\mathbf{y}.
\end{aligned} \tag{1.50}$$

In the second step, we used the fact that applying two annihilation operators to a two-particle state results in the vacuum state. In the third step, we used that $\mathrm{e}^{\mathrm{i}t(\omega(D_{\mathbf{x}})+\omega(D_{\mathbf{y}}))}$

is an isometry on $L^2(\mathbb{R}^{2s})$. After a careful refinement of this argument, we can assume that $B^*$ and $B_2^*$ create one-particle states with distinct momenta (see [Kr24a, Proof of Theorem 3.4]).

3. The key idea is that

$$\varphi_t(\mathbf{x},\mathbf{y}) = \mathrm{e}^{\mathrm{i}t(H-\omega(D_\mathbf{x})-\omega(D_\mathbf{y}))} B^*(\mathbf{x}) B_2^*(\mathbf{y})\Omega \tag{1.51}$$

should converge to a scattering state as $t \to \infty$. Because $\psi \in (\mathcal{H}^{\mathrm{out}})^\perp$ is orthogonal to all scattering states, the limit of (1.50) should be 0. In fact, $\varphi_t$ converges pointwise to a scattering state as a consequence of Theorem 1.2.4. The technical challenge lies in proving the convergence of $\langle \psi, \varphi_t\rangle$ in the $L^2$-sense. In the following, it is convenient to formulate (1.51) in relative coordinates:

$$\mathbf{u} = \mathbf{x} - \mathbf{y},\ \mathbf{v} = \frac{1}{2}(\mathbf{x}+\mathbf{y}). \tag{1.52}$$

4. We establish the convergence of $\langle \psi, \varphi_t\rangle$ by verifying the Cauchy property through Cook's method (i.e. we demonstrate that the $t$-derivative is integrable):

$$\langle \psi, \varphi_{t_2}\rangle - \langle \psi, \varphi_{t_1}\rangle = \int_{t_1}^{t_2} \mathrm{e}^{-\mathrm{i}\tau(\omega(\frac{1}{2}D_\mathbf{v}+D_\mathbf{u})+\omega(\frac{1}{2}D_\mathbf{v}-D_\mathbf{u}))} \langle \psi, \mathrm{e}^{\mathrm{i}\tau H}\,\mathrm{e}^{-\mathrm{i}\mathbf{v}\cdot\mathbf{P}}\phi(\mathbf{u})\rangle\,\mathrm{d}\tau. \tag{1.53}$$

Here, $\phi$ is a function that involves a commutator of two creation operators:

$$\phi(\mathbf{u}) = \mathrm{e}^{-\frac{\mathrm{i}}{2}\mathbf{u}\cdot\mathbf{P}}[\tilde{B}_1^*, B_2^*(-\mathbf{u})]\Omega, \tag{1.54}$$

where $\tilde{B}_1^* = \int_{\mathbb{R}^d} g(x)B^*(x)\,\mathrm{d}x$ for a Schwartz function $g \in \mathcal{S}(\mathbb{R}^d)$ such that $\hat{g}(p) = p_0 - \omega(\mathbf{p})$ on the energy-momentum transfer $\sigma_\alpha(B^*)$. The commutator in $\phi$ is obtained from the $t$-derivative (see [Kr24a, Proof of Theorem 4.1, Step (ii)]). We observe that the function $\phi$ decays rapidly in the relative coordinate $\mathbf{u}$. This follows from the locality axiom and the fact that creation operators are almost local (see [Kr24a, Lemma 2.2]).

5. To proceed, we take the Fourier transform $\mathcal{F}_{\mathbf{v}\to\mathbf{p}}$ in the total variable $\mathbf{v}$. We must then prove the convergence to 0 as $t_1, t_2 \to \infty$ of the following expression:

$$\begin{aligned}
&\int_{\mathbb{R}^s}\int_{\mathbb{R}^s}\left|\int_{t_1}^{t_2} \mathrm{e}^{-\mathrm{i}\tau(\omega(\frac{1}{2}\mathbf{p}+D_\mathbf{u})+\omega(\frac{1}{2}\mathbf{p}-D_\mathbf{u}))}\mathcal{F}_{\mathbf{v}\to\mathbf{p}}\langle \psi, \mathrm{e}^{\mathrm{i}\tau H}\,\mathrm{e}^{-\mathrm{i}\mathbf{v}\cdot\mathbf{P}}\phi(\mathbf{u})\rangle\,\mathrm{d}\tau\right|^2 \mathrm{d}\mathbf{u}\,\mathrm{d}\mathbf{p}\\
&= \int_{\mathbb{R}^s} \sup_{\|f\|_{L^2}=1}\left|\int_{t_1}^{t_2}\int_{\mathbb{R}^s} \overline{f(\mathbf{u})}\,\mathrm{e}^{-\mathrm{i}\tau\omega_\mathbf{p}(D_\mathbf{u})}\mathcal{F}_{\mathbf{v}\to\mathbf{p}}\langle \psi, \mathrm{e}^{\mathrm{i}\tau H}\,\mathrm{e}^{-\mathrm{i}\mathbf{v}\cdot\mathbf{P}}\phi(\mathbf{u})\rangle\,\mathrm{d}\mathbf{u}\,\mathrm{d}\tau\right|^2 \mathrm{d}\mathbf{p}.
\end{aligned} \tag{1.55}$$

6. In (1.55), we insert a one through $1 = \langle A_\mathbf{p}\rangle^{-\nu}\langle A_\mathbf{p}\rangle^{\nu}$, where $\nu > 1/2$. Here, $A_\mathbf{p}$ is a modified generator of dilations. Its exact form, which is given in [Kr24a, (4.21)], is not

so relevant at the moment except that $A_{\mathbf{p}}$ is relatively bounded with respect to $\mathbf{u}$. An application of the Cauchy–Schwarz inequality yields the following bound:

$$\int_{\mathbb{R}^s} \left( \sup_{\|f\|_{L^2}=1} \int_{t_1}^{t_2} \|\langle A_{\mathbf{p}} \rangle^{-\nu} \, \mathrm{e}^{\mathrm{i}\tau\omega_{\mathbf{p}}(D_{\mathbf{u}})} \chi_{\mathbf{p}} f\|_{L^2}^2 \, \mathrm{d}\tau \right) \times \left( \int_{t_1}^{t_2} \|\langle A_{\mathbf{p}} \rangle^{\nu} \mathcal{F}_{\mathbf{v}\to\mathbf{p}} \langle \psi, \mathrm{e}^{\mathrm{i}\tau H} \, \mathrm{e}^{-\,\mathrm{i}\mathbf{v}\cdot\mathbf{P}} \phi \rangle\|_{L^2}^2 \, \mathrm{d}\tau \right) \mathrm{d}\mathbf{p}, \tag{1.56}$$

where $\chi_{\mathbf{p}}$ is a projector that suppresses contributions from zero relative momentum ($D_{\mathbf{u}} = 0$). This projector can be included because $\phi$ contains two creation operators that create one-particle states with distinct momenta.

7. It remains to demonstrate that the first factor in (1.56) can be bounded uniformly in $t_1$, $t_2$, and the total momentum $\mathbf{p}$, and that the second factor, integrated over $\mathbf{p}$, converges to 0 as $t_1, t_2 \to \infty$. That the second factor converges to 0, is an application of the decay properties of $\phi$ (i.e. a consequence of the locality axiom) and the assumption $\psi \in \mathcal{H}_{\mathrm{ac}}(P)$ (see [Kr24a, (4.28)]). The first factor is bounded due to the following local decay estimate:

$$\int_{-\infty}^{\infty} \|\langle A_{\mathbf{p}} \rangle^{-\nu} \, \mathrm{e}^{\mathrm{i}\tau\omega_{\mathbf{p}}(D_{\mathbf{u}})} \chi_{\mathbf{p}} f\|_{L^2}^2 \, \mathrm{d}\tau \leq C \|f\|_{L^2}^2. \tag{1.57}$$

An efficient way of proving such a local decay estimate is through Mourre's conjugate operator method. We explain this method and its main results in the next section. Once the local decay estimate is established, the proof of the theorem is complete. □

The structure of the proof of Theorem 1.3.3 bears resemblances to proofs of the existence and completeness of two-body wave operators in quantum mechanics. For example, when applying Cook's method in quantum mechanics, one obtains a term involving the potential, which decays at infinity. In quantum field theory, however, there are no potentials. Instead, as demonstrated in Step 4, decay arises from the commutator of almost local observables.

Step 5, which involves taking the Fourier transform in the total variable $\mathbf{v}$, corresponds to the separation of the centre-of-mass motion in quantum mechanics.

The use of local decay estimates, as seen in Step 7, is also well-known in quantum mechanics (see [ABG96, Theorem 7.1.4]). This technique of proving the existence and completeness of wave operators traces back to the works of Kato [Ka66] and Lavine [La70a, La70b].

## 1.4 Mourre's Conjugate Operator Method

Mourre's conjugate operator method is a mathematical technique for analysing spectral properties of a self-adjoint operator $H : D(H) \to \mathcal{H}$ on a Hilbert space $\mathcal{H}$. The method is based on a strictly positive commutator estimate (the so-called Mourre estimate) with a conjugate operator $A$. Mourre's method is relatively well-known in non-relativistic quantum mechanics, where it crucial to advance the spectral and scattering theory of many-body Schrödinger operators (see Subsection 1.1.1). Our applications of Mourre's conjugate operator in relativistic quantum field theory, as presented in the papers [Kr24a, Kr24b], are new.

### 1.4.1 Mourre Estimate

The following regularity classes are relevant for defining commutators of unbounded operators. We denote by $R(z) = (H - z)^{-1}$ the resolvent of $H$ for $z \in \rho(H)$ in the resolvent set of $H$.

**Definition 1.4.1.** Let $A$ be a self-adjoint operator on $\mathcal{H}$, and let $k \in \mathbb{N} \cup \{\infty\}$. The space $C^k(A)$ consists of all self-adjoint operators $H$ such that, for a $z \in \rho(H)$, the map $t \mapsto \mathrm{e}^{\mathrm{i}tA} R(z)\, \mathrm{e}^{-\mathrm{i}tA}$ is $C^k$ in the strong operator topology.

If the map $t \mapsto \mathrm{e}^{\mathrm{i}tA} R(z)\, \mathrm{e}^{-\mathrm{i}tA}$ is $C^1$ in the strong operator topology, the derivative in $t = 0$ defines the commutator $[R(z), \mathrm{i}A]$ as a bounded operator on $\mathcal{H}$. Because $H \in C^1(A)$, the commutator $[R(z), \mathrm{i}A]$ can be expressed as

$$[R(z), \mathrm{i}A] = -R(z)[H, \mathrm{i}A]R(z), \tag{1.58}$$

which defines $[H, \mathrm{i}A]$ as a map from $D(H)^*$ to $D(H)$ [ABG96, Theorem 6.2.10]. Here, we consider the domain $D(H)$ as a Banach space equipped with the graph norm.

**Definition 1.4.2.** The operator $H$ obeys a **Mourre estimate** on an open and bounded set $J \subset \mathbb{R}$ if a self-adjoint operator $A$ (**conjugate operator**) exists such that $H \in C^1(A)$ and, for an $a > 0$,

$$E(J)[H, \mathrm{i}A]E(J) \geq aE(J), \tag{1.59}$$

where $E$ is the spectral measure of $H$.

### 1.4.2 Main Results of Mourre's Method

The three main results of the conjugate operator method are (1) the limiting absorption principle, which controls the resolvent $R(z) = (H - z)^{-1}$ as the resolvent parameter $z \in \rho(H)$ approaches the spectrum in a topology defined by the conjugate operator $A$, (2) the local decay estimate, which asserts that $\langle A \rangle^{-\nu} = (1 + |A|^2)^{-\nu/2}$ is a locally $H$-smooth operator for $\nu > 1/2$, (3) the absence of singular spectrum in regions where the Mourre estimate holds.

**Theorem 1.4.3** (Limiting absorption principle)**.** *Let $H \in C^2(A)$. If $H$ obeys a Mourre estimate on an open and bounded set $J \subset \mathbb{R}$ with conjugate operator $A$, then, for every compact subset $K \subset J$ and every $\nu > 1/2$,*

$$\sup_{\lambda \in K, \mu > 0} \| \langle A \rangle^{-\nu} (H - \lambda \mp \mathrm{i}\mu)^{-1} \langle A \rangle^{-\nu} \| < \infty. \tag{1.60}$$

**Theorem 1.4.4** (local decay estimate)**.** *Let $H \in C^2(A)$. If $H$ obeys a Mourre estimate on an open and bounded set $J \subset \mathbb{R}$ with conjugate operator $A$, then, for every compact subset $K \subset J$ and every $\nu > 1/2$,*

$$\int_{-\infty}^{\infty} \|\langle A\rangle^{-\nu} \mathrm{e}^{-\mathrm{i}tH} E(K)\psi\|^2 \,\mathrm{d}t \leq C\|\psi\|^2, \ \psi \in \mathcal{H}. \tag{1.61}$$

**Theorem 1.4.5** (absence of singular spectrum)**.** *Let $H \in C^2(A)$. If $H$ obeys a Mourre estimate on an open and bounded set $J \subset \mathbb{R}$, then the spectrum of $H$ in $J$ is purely absolutely continuous.*

The three main results of the conjugate operator method are interrelated. Both the limiting absorption principle (LAP) and the local decay estimate (LDE) imply that the spectrum of $H$ in $J$ is purely absolutely continuous [ABG96, Proposition 7.1.3] [RS78, Theorem XIII.23]. Moreover, the LDE can be derived from the LAP, and conversely, the LAP for the imaginary part of the resolvent holds if the LDE is established [ABG96, Proposition 7.1.1].

There are independent proofs for both the LAP and LDE starting from the Mourre estimate. Several proofs of the LAP under differing assumptions have been found. Proofs under basically optimal assumptions are given in [ABG96, Sections 7.3–7.5] and [Sa97]. Direct proofs of the LDE from the Mourre estimate are less common but can be achieved through the low-velocity estimate by Hunziker, Sigal, and Soffer [HSS99].

Interestingly, conjugate operators for a self-adjoint operator $H$ on a set $J \subset \mathbb{R}$ can be constructed under the assumption that the spectrum of $H$ is purely absolutely continuous and of constant multiplicity in $J$ [ABG96, Proposition 7.2.14].

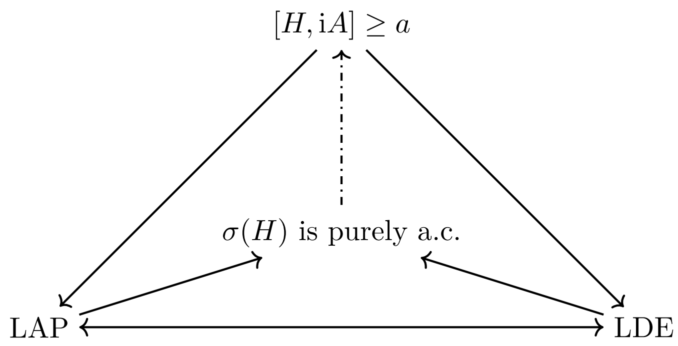


Figure 1.3: The main results of the Mourre method (LAP – limiting absorption principle, LDE – local decay estimate, a.c. – absolutely continuous) are interrelated.

### 1.4.3 Applications in Quantum Field Theory

We have already encountered an application of Mourre's conjugate operator method in the proof of the convergence of Araki–Haag detectors (Theorem 1.3.3). As we demonstrated in [Kr24b], Mourre's method can also be employed to deduce spectral properties

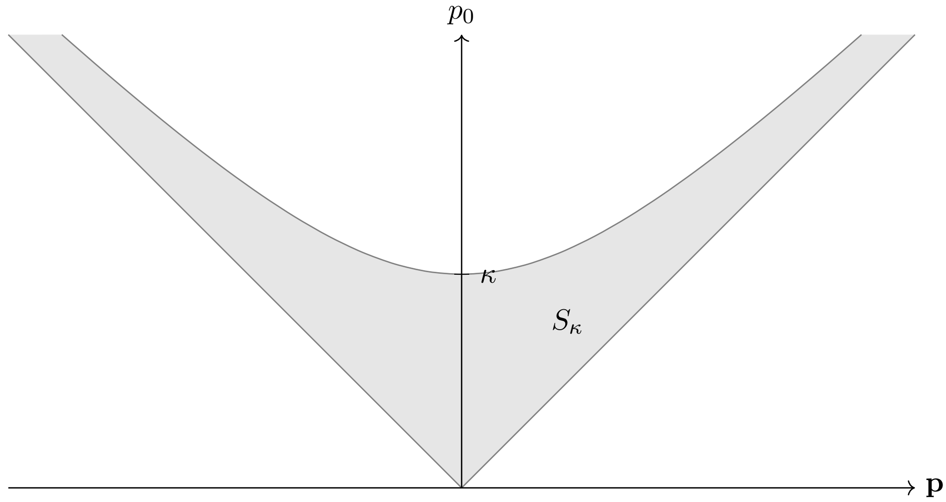


Figure 1.4: The set $S_\kappa \subset V_+$ is invariant under Lorentz boosts.

of the energy-momentum operators in relativistic quantum field theories. Specifically, the generators of Lorentz boosts can be used to construct conjugate operators.

**Theorem 1.4.6** ([Kr24b, Theorem 1.1])**.** *Let $U : \mathcal{P} \to \mathfrak{B}(\mathcal{H})$ be a strongly continuous unitary representation of the Poincaré group $\mathcal{P} = \mathcal{L} \ltimes \mathbb{R}^d$ on a Hilbert space $\mathcal{H}$, $P = (P_0, \mathbf{P})$ the generators of the translation subgroup $U|_{\mathbb{R}^d}$, $E$ the joint spectral measure of $P$, and $\mathbf{K}$ the generators of Lorentz boosts. Assume that the energy-momentum operators $P$ obey the spectrum condition. For $\kappa > 0$, define the following Lorentz-invariant sets:*

$$S_\kappa = \{\Lambda_1(t_1)\dots\Lambda_s(t_s)(p_0, \mathbf{0}) \mid t_1, \dots, t_s \in \mathbb{R}, p_0 \in [0, \kappa]\}^-, \tag{1.62}$$

*where $\Lambda_j(t_j)$ are the Lorentz boosts in the spatial direction $j \in \{1, \dots, s = d-1\}$ and $\{\dots\}^-$ denotes the closure in $\mathbb{R}^d$. For all compact subsets $I_0 \subset (\kappa, \infty)$ and $I_j \subset \mathbb{R}\backslash\{0\}$, for every $\nu > 1/2$,*

$$\sup_{\lambda \in I_0, \mu > 0} \|E(S_\kappa)\langle K_j\rangle^{-\nu}(P_0 - \lambda \mp \mathrm{i}\mu)^{-1}\langle K_j\rangle^{-\nu}E(S_\kappa)\| < \infty, \tag{1.63}$$

$$\sup_{\lambda \in I_j, \mu > 0} \|E(S_\kappa)\langle K_j\rangle^{-\nu}(P_j - \lambda \mp \mathrm{i}\mu)^{-1}\langle K_j\rangle^{-\nu}E(S_\kappa)\| < \infty. \tag{1.64}$$

The limiting absorption principles stated in the theorem are derived from a Mourre estimate. In the case of the momentum operators, the basic idea of the proof is particularly simple. Formally, the momentum operator $P_j$ and the generator $K_j$ of the Lorentz boost in the spatial direction $j$ satisfy the following commutation relation:

$$[P_j, \mathrm{i}K_j] = P_0. \tag{1.65}$$

By the spectrum condition, the energy operator $P_0$ is strictly positive on spectral subspaces of $P_j$ that are separated from 0. This gives the Mourre estimate for the momentum operator $P_j$:

$$E_j((a,\infty))[P_j, \mathrm{i}K_j]E_j((a,\infty)) = P_0 E_j((a,\infty)) \geq aE_j((a,\infty)),\ \ a > 0, \tag{1.66}$$

where $E_j$ is the spectral measure of $P_j$. To rigorously justify the commutation relation (1.65) and to apply the Mourre theory, it is necessary to demonstrate that $P_j \in C^2(K_j)$. This is the only non-trivial part of the argument and requires the introduction of the spectral projections $E(S_\kappa)$. These spectral projections ensure that $P_{0,\kappa} = P_0 E(S_\kappa)$ and $P_{j,\kappa} = P_j E(S_\kappa)$ are bounded relatively to each other and that $P_{0,\kappa}, P_{j,\kappa} \in C^\infty(K_j)$ [Kr24b, Proposition 4.2].

Notably, we also established a Mourre estimate for the energy operator $P_0$, but the construction of the conjugate operator is more complicated in this case. The following operator is similar to the generator of dilations in quantum mechanics, with the position operator replaced by the generator of Lorentz boosts:

$$A = \frac{1}{2}\left(P_1 K_1 + K_1 P_1\right). \tag{1.67}$$

Formally, the commutator of $P_0$ with i$A$ yields a positive operator:

$$[P_0, \mathrm{i}A] = P_1^2 \geq 0. \tag{1.68}$$

However, it is not possible to make $P_1^2$ strictly positive by restricting the commutation relation to spectral subspaces of $P_0$. The momentum of a state can always be zero, regardless of how large its energy is. To resolve this issue, we must restrict the commutation relation to the Lorentz-invariant set $S_\kappa$. This allows us to obtain a Mourre estimate on every open and bounded subset of $(\kappa, \infty)$ that is separated from $\kappa > 0$:

$$E_0((\kappa+\varepsilon,\infty))[P_{0,\kappa}, A_\kappa]E_0((\kappa+\varepsilon,\infty)) = P_1^2 E_0((\kappa+\varepsilon,\infty)) \geq aE((\kappa+\varepsilon,\infty)), \tag{1.69}$$

where $E_0$ is the spectral measure of $P_0$, $\varepsilon > 0$, $A_\kappa = AE(S_\kappa)$, and $a > 0$ depends on $\varepsilon$ (see also Figure 1.4). Moreover, we have $P_{0,\kappa} \in C^1(A_\kappa)$, but not necessarily $P_{0,\kappa} \in C^2(A_\kappa)$, which is required to apply the results of Mourre's method. To address this, a slight technical modification of the conjugate operator $A_\kappa$ is needed, where we replace $P_1$ with $F(P)$ for a suitable function $F$. For details, we refer to [Kr24b, Lemma 4.8].

The Mourre estimate for $P_0$ is an interesting result because the Hamiltonian is typically of rather abstract nature in quantum field theory (e.g. the Hamiltonian is renormalised through a limiting procedure or the Hamiltonian is defined axiomatically as the generator of time translations). Previously, a Mourre estimate was established for the spatially cut-off $P(\varphi)_2$ Hamiltonian by Dereziński and Gérard [DG00], but it remained an open problem whether a Mourre estimate can be proved for the Hamiltonian in the infinite-volume limit.

From Theorem 1.4.6, it follows that the spectra of the Hamiltonian and the momentum operators are purely absolutely continuous, except at the point 0 [Kr24b, Proposition 1.2]. This spectral result is well-known in relativistic quantum field theory. It was first proved by Maison [Ma68] through an application of Wigner's theorem. Our result not only reproduces Maison's result but also establishes a limiting absorption principle, which is a much stronger statement.

## 1.5 Conclusion and Outlook

Asymptotic completeness in local relativistic quantum field theory (QFT) is a challenging open problem due to several conceptual and technical difficulties. Nevertheless, there is a promising strategy for establishing asymptotic completeness that uses Araki–Haag detectors. We proved the convergence of Araki–Haag detectors on states $\psi \in \mathcal{H}_{\mathrm{ac}}(P)$ below the three-particle threshold, which is an important prerequisite for asymptotic completeness.

We demonstrated that Mourre's conjugate operator method, which is a well-known mathematical technique from non-relativistic quantum mechanics, is also relevant for the spectral and scattering theory of QFT. Specifically, we applied a local decay estimate to prove the convergence of Araki–Haag detectors (Theorem 1.3.3) and established a limiting absorption principle for the energy-momentum operators (Theorem 1.4.6). Both results were derived from a Mourre estimate.

We expect that further methods from many-body quantum mechanics could be adapted to the QFT setting. It would be interesting to explore whether insights from many-body quantum-mechanical scattering theory, especially the channel structure as described in Subsection 1.1.1, could be applied to prove the convergence of Araki–Haag detectors in regions above the three-particle threshold. This could be a promising direction for future research and could yield interesting correspondences between many-body quantum mechanics and quantum field theory. Progress in this direction may first require to solve the problem of asymptotic completeness for many-body dispersive Hamiltonians, particularly for those with a relativistic dispersion relation. A Mourre estimate in this case has been established with increasing generality by Dereziński [De90], Gérard [Ge91], and Damak [Da97]. A large-velocity estimate can be adapted from Graf's paper [Gr90, Theorem 4.1], while a low-velocity estimate can be derived from the Mourre estimate [HSS99]. The main obstacle to prove asymptotic completeness for dispersive Hamiltonians is to establish a suitable phase-space propagation estimate, as we discussed in Subsection 1.1.1. Among different possible approaches, the conjectured propagation estimate (1.21) seems to be the most promising at the moment.

Another interesting direction of future investigations is to close the gap between the convergence of Araki–Haag detectors and asymptotic completeness (Step 3 of the strategy described in Subsection 1.2.2). We anticipate it to be difficult to prove the accessibility of quantum states through detectors axiomatically because a detailed knowledge about the particle content may likely be necessary to achieve such a result. However, it could be more realistic to attempt a proof in the $P(\phi)_2$ or the massive Sine–Gordon model, where the particle spectrum is well-understood. As outlined in Subsection 1.2.2, this would require a detailed analysis of the energy-momentum tensor, which could be a realistic task in the Sine–Gordon model considering recent progress on its Minkowskian construction [BNR23, FC23]. Establishing the accessibility of quantum states through detectors in the Sine–Gordon model, in combination with Theorem 1.3.3, would lead to a proof of two-particle asymptotic completeness. To our knowledge, this would be a novel and significant result, considering the conceptual challenges in QFT.

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# Mourre Theory and Asymptotic Observables in Local Relativistic Quantum Field Theory

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**Abstract:** We prove the convergence of Araki–Haag detectors in any Haag–Kastler quantum field theory with an upper and lower mass gap. We cover the case of a single Araki–Haag detector on states of bounded energy, which are selected from the absolutely continuous part of the energy-momentum spectrum sufficiently close to the lower boundary of the multi-particle spectrum. These states essentially encompass those states in the multi-particle spectrum lying below the three-particle threshold. In our proof, we draw on insights from proofs of asymptotic completeness in quantum mechanics. Notably, we apply Mourre's conjugate operator method for the first time within the framework of Haag–Kastler quantum field theory. Furthermore, we discuss applications of our findings for the problem of asymptotic completeness in local relativistic quantum field theory.

## 1. Introduction

A fundamental task of scattering theory is to prove asymptotic completeness, which is important for interpreting quantum theories in terms of particles. In non-relativistic quantum mechanics, asymptotic completeness for $N$-particle Hamiltonians has been established through the works of Enss [En84], Sigal and Soffer [SS87], Graf [Gr90], Yafaev [Ya93], Dereziński [De93], and many others (see [DG97] for a textbook exposition). These classical results rely on the existence of asymptotic observables such as the asymptotic velocity.[1]

In local relativistic quantum field theory, however, asymptotic completeness remains an open problem, even at the level of two particles in massive theories. Asymptotic completeness has been proved only for few models, including integrable models [Le07] and the $P(\phi)_2$ model at the level of two [SZ76] and three particles [CD82].

[1] The cited papers are formulated mainly in time-dependent scattering theory. More recently, Skibsted [Sk23] provided a proof of asymptotic completeness for short-range interactions in time-independent scattering theory.

The reason for this gap lies in additional conceptual and technical difficulties in quantum field theory, as discussed in [DG14b]. Firstly, within the conventional Haag–Kastler framework, there are pathological counterexamples to asymptotic completeness (e.g. generalised free fields). Moreover, the infinite number of degrees of freedom in quantum field theory allows for a rich superselection structure and myriads of elusive charged particles. As a result, the vacuum sector accommodates not only neutral particles but also particle states containing oppositely charged pairs of particles. On the technical side, the understanding of dynamical properties of systems with relativistic dispersion relation is incomplete. Specifically, for $N \geq 3$, asymptotic completeness is an open problem for $N$-particle Hamiltonians with non-quadratic dispersion relation.

It is not surprising that experts in quantum field theory have focused on other properties than asymptotic completeness. A closely related problem is the convergence of asymptotic observables corresponding to the asymptotic velocity in quantum mechanics. The convergence of asymptotic observables is still a difficult problem in Haag–Kastler quantum field theory, where observables possibly evolve through a cascade of charged particles and pathological states. However, building on decades of previous research outlined below, we make substantial progress on this question.

Araki–Haag detectors have long ago been identified as natural asymptotic observables in quantum field theory. In their seminal paper, Araki and Haag [AH67] proved the convergence of these asymptotic observables on incoming scattering states ($\mathcal{H}^{\text{in}}$) and outgoing scattering states ($\mathcal{H}^{\text{out}}$) of bounded energy and interpreted them as particle counters.[2] However, the convergence of Araki–Haag detectors on arbitrary states of bounded energy has remained an open problem for decades despite continued interest related to various aspects of particles in quantum field theory [Bu86,Bu95,Po04,BS06] [Ha96, Section VI.1].

First convergence results of Araki–Haag detectors on arbitrary states of bounded energy have been obtained relatively recently by Dybalski and Gérard [DG14a,DG14b]. They managed to translate quantum mechanical methods such as large-velocity and phase-space propagation estimates to Haag–Kastler quantum field theory via a technically important uniform bound by Buchholz [Bu90]. Dybalski and Gérard covered products of two or more Araki–Haag detectors sensitive to particles with distinct velocities, but products of detectors sensitive to particles with coinciding velocities and the case of a single detector were not treated. The technical reason for this omission was a missing low-velocity propagation estimate, which is usually proved by Mourre's conjugate operator method [DG97, Theorem 4.13.1] [HSS99,Ri04].

Mourre's method is a powerful mathematical technique from spectral theory, which is based on a strictly positive commutator estimate. In the appendix, we provide a concise overview of some important results of this method. The conjugate operator method led to significant progress in the spectral and scattering theory of many-body Schrödinger operators, but it resisted so far any extension from quantum mechanics to Haag–Kastler quantum field theory. Through scattering theory, we manage to apply Mourre's method to quantum field theory. This allows us to prove the convergence of a single Araki–Haag detector on states of bounded energy sufficiently close to the lower boundary of the multi-particle spectrum.

To state our main result and explain the essence of our argument, we introduce some notation. Let $h \in C_c^\infty(\mathbb{R}^s)$ be a smooth compactly supported function and $B^*$ a creation

[2] Notably, Enss [En75] published a paper on Araki–Haag detectors before his celebrated proof of asymptotic completeness in three-particle quantum mechanical systems. Asymptotic observables play a central role in Enss' proof of asymptotic completeness.

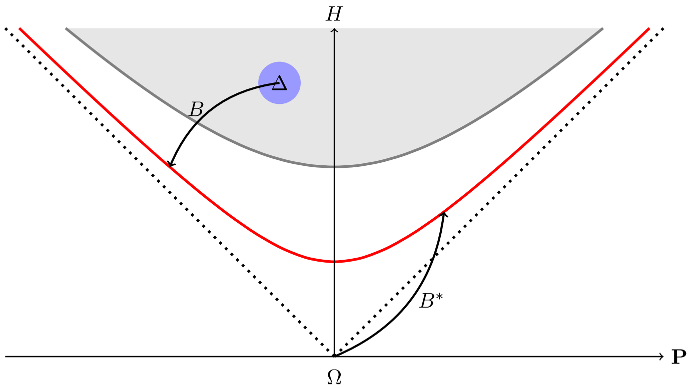


**Fig. 1.** The energy-momentum spectrum contains an isolated mass shell (red line) and a continuous multi-particle spectrum (grey area) above the two-particle threshold (grey line). A creation operator $B^*$ maps the vacuum vector $\Omega$ to a one-particle state. In Theorem 1.1, we assume that $B^*$ is such that $\overline{\Delta - \sigma_\alpha(B^*)} \cap \sigma(P)$ is a subset of the mass shell

operator (i.e. a bounded operator that creates one-particle states; see Sect. 2 for precise definitions). Araki–Haag detectors are asymptotic limits ($t \to \pm\infty$) of the following observable:

$$C(h,t) = \mathrm{e}^{\mathrm{i}tH} \int_{\mathbb{R}^s} h\left(\frac{\mathbf{x}}{t}\right)(B^*B)(\mathbf{x})\,\mathrm{d}\mathbf{x}\,\mathrm{e}^{-\mathrm{i}tH}. \tag{1.1}$$

It is possible to extend the above formula to $h \in L^\infty(\mathbb{R}^s)$ by Buchholz's uniform estimate [Bu90]. Let $\mathcal{H}_{\mathrm{ac}}(P)$ be the jointly absolutely continuous spectral subspace of the energy-momentum operator $P = (H, \mathbf{P})$. We denote the spectral measure of $P$ by $E$. Our main result is that $C(h,t)$ converges strongly on states $\psi \in \mathcal{H}_{\mathrm{ac}}(P)$ for which $B\psi$ is a one-particle state. We formulate the result for the limit $t \to \infty$ and outgoing scattering states. The result for the limit $t \to -\infty$ is analogous if $\mathcal{H}^{\mathrm{out}}$ is replaced by $\mathcal{H}^{\mathrm{in}}$.

**Theorem 1.1.** *Let $\Delta \subset \mathbb{R}^d$ be compact and $\psi \in E(\Delta)\mathcal{H} \cap \mathcal{H}_{\mathrm{ac}}(P)$ a state of bounded energy. If $B^*$ is a creation operator such that $B\psi$ is a one-particle state, then, for all $h \in L^\infty(\mathbb{R}^s)$, $C(h,t)\psi$ converges strongly in $\mathcal{H}$ as $t \to \infty$. If $\psi$ lies in the orthogonal complement of the scattering states $\mathcal{H}^{\mathrm{out}}$, then the limit is 0.*

Intuitively, the condition that $B\psi$ is a one-particle state selects states $\psi$ of the multi-particle spectrum below the three-particle threshold (see Fig. 1 and the comments preceding Theorem 3.4 for more details). We emphasise that the assumptions of the theorem exclude neither a non-trivial superselection structure nor pathological states with too many degrees of freedom.

Regarding the spectral assumption $\psi \in \mathcal{H}_{\mathrm{ac}}(P)$, we note that the Hilbert space $\mathcal{H}$ decomposes into the pure point, absolutely continuous, and singular continuous spectral subspace of $P$:

$$\mathcal{H} = \mathcal{H}_{\mathrm{pp}}(P) \oplus \mathcal{H}_{\mathrm{ac}}(P) \oplus \mathcal{H}_{\mathrm{sc}}(P). \tag{1.2}$$

Typically, the pure point spectral subspace $\mathcal{H}_{\mathrm{pp}}(P)$ is the span of the vacuum vector $\Omega$, and the singular continuous spectral subspace $\mathcal{H}_{\mathrm{sc}}(P)$ describes mass shells, isolated or embedded in the multi-particle spectrum. To prove the convergence of Araki–Haag detectors on eigenstates of the mass operator $M = \sqrt{H^2 - |\mathbf{P}|^2}$ is relatively simple (see Proposition 3.3). However, in general, $\mathcal{H}_{\mathrm{sc}}(P)$ may also contain exotic states for which we cannot prove the convergence of Araki–Haag detectors. In Lorentz covariant quantum field theories, these exotic states correspond to the singular continuous spectrum of the mass operator.

Theorem 1.1 promises to advance our understanding of two-particle asymptotic completeness in local relativistic quantum field theory. A quantum field theory is asymptotically complete in the two-particle region if $E(\Delta)\mathcal{H} = E(\Delta)\mathcal{H}^{\mathrm{out}} = E(\Delta)\mathcal{H}^{\mathrm{in}}$ for all $\Delta \subset \sigma(P)$ between the two- and three-particle threshold. To prove two-particle asymptotic completeness, we may adopt the following strategy: According to the last statement of the theorem, Araki–Haag detectors map the orthogonal complement of scattering states to 0. In physically relevant quantum field theories, quantum states should be accessible through experiments, implying that we should be able to construct a detector capable of detecting a given state $\psi \in \mathcal{H}$. Notably, we can indeed construct such detectors for one-particle states, as demonstrated in Lemma 3.5. However, establishing this property for states in the multi-particle spectrum requires additional assumptions.

**Corollary 1.2.** *Let $\Delta \subset \mathbb{R}^d$ be compact. If, for every $\psi \in E(\Delta)\mathcal{H}_{\mathrm{ac}}(P)$, a creation operator $B^*$ exists such that $B\psi$ is a one-particle state and*

$$\lim_{t\to\infty} \mathrm{e}^{\mathrm{i}tH} \int_{\mathbb{R}^s} (B^*B)(\mathbf{x})\, \mathrm{d}\mathbf{x}\, \mathrm{e}^{-\mathrm{i}tH} \psi \neq 0, \tag{1.3}$$

*then $E(\Delta)\mathcal{H}_{\mathrm{ac}}(P) = E(\Delta)\mathcal{H}^{\mathrm{out}}$.*

It is worth noting that there exist non-trivial models to which Theorem 1.1 applies. As discussed above, the assumption $\psi \in \mathcal{H}_{\mathrm{ac}}(P)$ should not be very restrictive, but it may be difficult to verify it in models. Notably, it is known from [SZ76] that the energy-momentum spectrum of the weakly coupled $P(\phi)_2$ model is absolutely continuous in the two-particle region. However, it is also known that this model is asymptotically complete in the two-particle region, and the convergence of Araki–Haag detectors on scattering states is already known from [AH67]. Models of interest for applying Theorem 1.1 are those where the spectral assumption holds true, but asymptotic completeness either is not established or fails. Among the simplest models, which belong to this class, are certain generalised free fields. We discuss a model with non-trivial $S$-matrix in Sect. 5.2.

In the remainder of the introduction, we outline the proof strategy of Theorem 1.1. The convergence of $C(h,t)$ on scattering states has been previously established in [AH67]. Hence, we focus on proving the convergence of $C(h,t)$ on states orthogonal to all scattering states, similarly as in [Dy18]. We formulate the convergence of $C(h,t)$ on the orthogonal complement of scattering states in Theorem 3.4. Consequently, Theorem 1.1 directly follows from Theorem 3.4 (see Sect. 3 for the proof of Theorem 1.1).

The first step in proving Theorem 3.4 is to reduce the convergence of a single detector to that of two detectors by introducing a second auxiliary detector (see Lemma 3.5). The convergence of two detectors sensitive to particles with distinct velocities (i.e. detectors for which the velocity functions $h_1$ and $h_2$ have disjoint support) has been analysed in [DG14a]. However, in our case, the supports of the velocity functions intersect. To proceed, a novel result in the form of an improved convergence property of Haag–Ruelle scattering states is required, which we establish in Theorem 4.1. Under the assumption that the momentum transfers of $B_1^*$ and $B_2^*$ are separated, we prove that, for every $\psi \in \mathcal{H}_{\mathrm{ac}}(P)$, the function

$$(\mathbf{x}, \mathbf{y}) \mapsto \langle \psi, \varphi_t(\mathbf{x}, \mathbf{y})\rangle = \mathrm{e}^{-\mathrm{i}t(\omega(D_{\mathbf{x}})+\omega(D_{\mathbf{y}}))} \langle \mathrm{e}^{-\mathrm{i}tH}\psi, B_1^*(\mathbf{x})B_2^*(\mathbf{y})\Omega\rangle \tag{1.4}$$

converges in $L^2(\mathbb{R}^{2s})$ as $t \to \infty$, where $\omega = \sqrt{m^2 + |\cdot|^2}$ is the relativistic dispersion relation and $D_{\mathbf{x}} = -\mathrm{i}\partial_{\mathbf{x}}$. Additionally, we demonstrate that the limit is 0 if $\psi$ lies in the orthogonal complement of the scattering states.

The proof strategy of Theorem 4.1 resembles the proof of Lavine's Theorem for the existence and completeness of two-particle wave operators in quantum mechanics [ABG96, Theorem 7.1.4] [Am09, Proposition 7.2]. In the first step, we apply Cook's method and reformulate the resulting expression in relative coordinates ($\mathbf{u} = \mathbf{x} - \mathbf{y}$, $\mathbf{v} = (\mathbf{x} + \mathbf{y})/2$). We prove that it is sufficient to establish the $L^2(\mathbb{R}^{2s})$-convergence of the following function to 0 as $t \to \infty$:

$$(\mathbf{u}, \mathbf{v}) \mapsto \int_t^\infty \mathrm{e}^{-\mathrm{i}\tau(\omega(\frac{1}{2}D_{\mathbf{v}} + D_{\mathbf{u}}) + \omega(\frac{1}{2}D_{\mathbf{v}} - D_{\mathbf{u}}))} \langle \psi, \mathrm{e}^{\mathrm{i}\tau H}\, \mathrm{e}^{-\mathrm{i}\mathbf{v}\cdot\mathbf{P}} \phi(\mathbf{u})\rangle\, \mathrm{d}\tau, \tag{1.5}$$

where $\phi$ is a Hilbert space-valued Schwartz function. In the next step, we perform a fibre decomposition of (1.5) along the total momentum $D_{\mathbf{v}}$ by taking the Fourier transformation. This step is similar to removing the centre-of-mass motion in many-body problems. We denote the Fourier variable corresponding to $\mathbf{v}$ by $\mathbf{p}$, and we arrive at the following bound for the $L^2$-norm of (1.5):

$$\begin{aligned} \int_{K_{\mathrm{tot}}} &\left( \sup_{\|f\|_{L^2}=1} \int_t^\infty \|\langle A_{\mathbf{p}}\rangle^{-\nu}\, \mathrm{e}^{\mathrm{i}\tau\omega_{\mathbf{p}}(D_{\mathbf{u}})} \chi f\|_{L^2}^2\, \mathrm{d}\tau \right) \\ &\times \left( \int_t^\infty \|\langle A_{\mathbf{p}}\rangle^{\nu} \mathcal{F}_{\mathbf{v}\to\mathbf{p}} \langle \psi, \mathrm{e}^{\mathrm{i}\tau H}\, \mathrm{e}^{-\mathrm{i}\mathbf{v}\cdot\mathbf{P}} \phi\rangle\|_{L^2}^2\, \mathrm{d}\tau \right) \mathrm{d}\mathbf{p}, \end{aligned} \tag{1.6}$$

where $K_{\mathrm{tot}}$ is a compact set, $\langle\cdot\rangle = \sqrt{1 + |\cdot|^2}$ denotes the Japanese bracket, $A_{\mathbf{p}}$ is a modified dilation operator (see (4.21)),

$$\omega_{\mathbf{p}}(D_{\mathbf{u}}) = \omega(\mathbf{p}/2 - D_{\mathbf{u}}) + \omega(\mathbf{p}/2 + D_{\mathbf{u}}) \tag{1.7}$$

is a pseudo-differential operator that corresponds to the energy of two free particles with relativistic dispersion relation and total momentum $\mathbf{p}$, $\chi$ is a cut-off that projects out contributions with vanishing relative momentum, and $\mathcal{F}$ denotes the Fourier transformation.

Applying techniques from Mourre's conjugate operator method, we prove that the first factor in brackets in (1.6) is uniformly bounded in $t$. The second factor in brackets converges to 0 as a consequence of the microcausality axiom. This crucial step combines methods from quantum mechanics (Mourre theory) with concepts from quantum field theory (microcausality).

The paper is structured as follows: In Sect. 2, we summarise the assumptions of the paper and recall relevant facts from Haag–Ruelle scattering theory. We introduce Araki–Haag detectors in Sect. 3, where we also prove the convergence of Araki–Haag detectors on the orthogonal complement of scattering states (Theorem 3.4). In Sect. 4, we analyse the convergence of the function (1.4). In Sect. 5, we discuss applications of Theorem 1.1 for a potential proof of asymptotic completeness and the applicability of our results to models. Moreover, we provide an outlook for further research directions. In Appendix A, we review key results of Mourre's conjugate operator method, which are relevant for the paper, and develop the notion of locally smooth operators for a family of commuting self-adjoint operators.

## 2. Haag–Ruelle Scattering Theory

In this paper, we work within the framework of Haag–Kastler quantum field theory. We summarise our assumptions and notation in Sect. 2.1. Following this, we revisit some well-known facts from Haag–Ruelle scattering theory. We refer to [DG14b, Section 2.2, Section 6] for more details.

*2.1. Assumptions and notation.* Observables are described by a net of $C^*$-algebras $\{\mathcal{A}(O)\}_{O\in\mathcal{J}}$, where $\mathcal{J}$ is the set of all bounded open subsets of $\mathbb{R}^d = \mathbb{R}^{1+s}$ (with $d$ representing the spacetime dimension and $s = d-1$ the spatial dimension). If $O_1 \subset O_2$, then $\mathcal{A}(O_1) \subset \mathcal{A}(O_2)$ (isotony), and if $O_1$ is contained in the causal complement of $O_2$, then $[\mathcal{A}(O_1), \mathcal{A}(O_2)] = \{0\}$ (microcausality). The global algebra $\mathcal{A}$ is the inductive $C^*$-limit of the net $\{\mathcal{A}(O)\}_{O\in\mathcal{J}}$. Moreover, a morphism $\alpha : \mathbb{R}^d \to \mathrm{Aut}(\mathcal{A})$ from the translation group $\mathbb{R}^d$ to the automorphism group $\mathrm{Aut}(\mathcal{A})$ of $\mathcal{A}$ exists such that, for $O \in \mathcal{J}$ and $x \in \mathbb{R}^d$, $\alpha_x\mathcal{A}(O) = \mathcal{A}(O+x)$.

We assume that a vacuum state $\rho_0 : \mathcal{A} \to \mathbb{C}$ invariant under the action of $\alpha$ exists. The GNS triple of $\rho_0$ is denoted by $(\pi_0, \mathcal{H}, \Omega)$, where $\pi_0 : \mathcal{A} \to \mathfrak{B}(\mathcal{H})$ is a representation of $\mathcal{A}$ on the Hilbert space $\mathcal{H}$ and $\Omega$ is the (cyclic) vacuum vector. Moreover, we impose the following assumptions: The spacetime translations are unitarily implemented on $\mathcal{H}$, that is, a strongly continuous unitary representation $U : \mathbb{R}^d \to \mathcal{B}(\mathcal{H})$ exists such that, for $A \in \mathcal{A}$ and $x \in \mathbb{R}^d$, $\pi_0(\alpha_x A) = U(x)\pi_0(A)U(x)^*$. Up to a multiplicative constant, $\Omega$ is the unique vector such that, for all $x \in \mathbb{R}^d$, $U(x)\Omega = \Omega$. The representation $U$ satisfies the strong spectrum condition, that is, if $\sigma(P)$ denotes the joint spectrum of the energy-momentum operator $P = (H, \mathbf{P})$ (i.e. the generators of $U$), then $\{0\} \cup H_m \subset \sigma(P) \subset \{0\} \cup H_m \cup G_{2m}$, where

$$H_m = \{(p_0, \mathbf{p}) \in \mathbb{R}^d \mid p_0 = \omega(\mathbf{p}) = \sqrt{m^2 + |\mathbf{p}|^2}\}, \tag{2.1}$$

$$G_{2m} = \{(p_0, \mathbf{p}) \in \mathbb{R}^d \mid p_0 \geq \sqrt{(2m)^2 + |\mathbf{p}|^2}\} \tag{2.2}$$

are the mass hyperboloid of mass $m > 0$ and the multi-particle spectrum, respectively. For simplicity, we assume that the mass eigenspace $\mathfrak{h}_m = E(H_m)\mathcal{H}$ of the mass operator $M = \sqrt{H^2 - |\mathbf{P}|^2}$ corresponds to a single spinless particle, where $E$ is the spectral measure of $P$.

The local observable algebras $\mathcal{R}(O)$, $O \in \mathcal{J}$, are von Neumann algebras generated by $\pi_0(\mathcal{A}(O))$, and the von Neumann algebra $\mathcal{R}$ generated by $\pi_0(\mathcal{A})$ is the global observable algebra.[3] For $A \in \mathcal{R}$ and $x = (x_0, \mathbf{x}) \in \mathbb{R}^{1+s}$, we write $A(x) = U(x)AU(x)^*$ and abbreviate $A(0, \mathbf{x})$ by $A(\mathbf{x})$. If $f \in \mathcal{S}(\mathbb{R}^d)$ is a Schwartz function, we define $A(f) = \int_{\mathbb{R}^d} f(x)A(x)\,\mathrm{d}x$, where the integral is defined in the weak sense. Similarly, for a Schwartz function $f \in \mathcal{S}(\mathbb{R}^s)$, we write $A[f] = \int_{\mathbb{R}^s} f(\mathbf{x})A(\mathbf{x})\,\mathrm{d}\mathbf{x}$.

*2.2. Haag–Ruelle creation operators.* We explain how to define creation operators $B^* \in \mathcal{R}$ such that $B^*\Omega \in \mathfrak{h}_m$ is a one-particle state with good localisation properties. It is untenable to take $B^*$ from a local observable algebra. Instead, we choose $B^*$ to be almost local, that is, $B^*$ is essentially localised in a double cone, and its norm outside a double cone decays rapidly.

[3] Under our assumptions, it can be shown that $\mathcal{R} = \mathfrak{B}(\mathcal{H})$ [Ar99, Theorem 4.6].

**Definition 2.1.** Let $K_r$ be the double cone of radius $r > 0$. An element $A \in \mathcal{R}$ is almost local if a sequence $(A_r)$ of local operators $A_r \in \mathcal{R}(K_r)$ exists such that $A_r$ converges rapidly in norm to $A$ as $r \to \infty$ (i.e., for every $N \in \mathbb{N}$, $\|A - A_r\| \leq C_N r^{-N}$).

According to the microcausality axiom, the commutator of two observables localised in space-like separated regions vanishes. The commutator of two almost local observables does not necessarily vanish, but its norm decays rapidly with increasing space-like separation.

**Lemma 2.2.** *If $A_1, A_2 \in \mathcal{R}$ are almost local, then, for every $N \in \mathbb{N}$, a constant $C_N$ exists such that $\|[A_1, A_2(\mathbf{x})]\| \leq C_N \langle \mathbf{x} \rangle^{-N}$.*

The lemma is a direct consequence of Definition 2.1. Next, we introduce the energy-momentum transfer of an element $A \in \mathcal{R}$, which characterises the change in energy-momentum of a state upon the action of $A$.

**Definition 2.3.** The energy-momentum transfer (or Arveson spectrum) $\sigma_\alpha(A)$ of an element $A \in \mathcal{R}$ is the support of the operator-valued distribution

$$\mathbb{R}^d \ni p \mapsto \check{A}(p) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \mathrm{e}^{-\mathrm{i}p\cdot x} A(x)\,\mathrm{d}x, \tag{2.3}$$

where $p \cdot x = p_0 x_0 - \mathbf{p} \cdot \mathbf{x}$ denotes the Minkowski product. Moreover, the momentum transfer of $A$ is $\pi_{\mathbf{P}}(\sigma_\alpha(A)) = \{\mathbf{p} \in \mathbb{R}^s \mid \exists p_0 \in \mathbb{R} \colon (p_0, \mathbf{p}) \in \sigma_\alpha(A)\}$.

We list the following key properties of the energy-momentum transfer: If $A^*$ denotes the adjoint of $A \in \mathcal{R}$, then $\sigma_\alpha(A^*) = -\sigma_\alpha(A)$. If $x \in \mathbb{R}^d$, then $\sigma_\alpha(A(x)) = \sigma_\alpha(A)$. For a Schwartz function $f \in \mathcal{S}(\mathbb{R}^d)$, it holds that $\sigma_\alpha(A(f)) \subset \mathrm{supp}(\hat{f}) \cap \sigma_\alpha(A)$, where $\hat{f}$ is the Fourier transform of $f$. Furthermore, if $\sigma_\alpha(A)$ is compact and $\hat{f} \in C_c^\infty(\mathbb{R}^d)$ satisfies $\hat{f} = 1$ on $\sigma_\alpha(A)$, then $A = A(f)$. Similarly, if $\pi_{\mathbf{P}}(\sigma_\alpha(A))$ is compact and $\hat{f} \in C_c^\infty(\mathbb{R}^s)$ satisfies $\hat{f} = 1$ on $\pi_{\mathbf{P}}(\sigma_\alpha(A))$, then $A = A[f]$.

The following proposition justifies the name energy-momentum transfer for the set $\sigma_\alpha(A)$.

**Proposition 2.4** ([DG14a, (2.4)]). *If $\Delta \subset \mathbb{R}^d$ is a Borel set, then*

$$A E(\Delta)\mathcal{H} \subset E(\overline{\Delta + \sigma_\alpha(A)})\mathcal{H}. \tag{2.4}$$

We see that $B^*\Omega$ is a one-particle state with good localisation properties if $B^*$ is almost local and has energy-momentum transfer contained in a sufficiently small set that intersects the mass hyperboloid $H_m$. This motivates the following definition:

**Definition 2.5.** An element $B^* \in \mathcal{R}$ is a creation operator if $B^*$ is almost local, its energy-momentum transfer $\sigma_\alpha(B^*)$ is a compact subset of the closed forward light cone $V_+ = \{p \in \mathbb{R}^d \mid p^0 \geq |\mathbf{p}|\}$, and $\emptyset \neq \sigma_\alpha(B^*) \cap \sigma(P) \subset H_m$.

In scattering theory, typically the time evolution in the distant past and far future of the (interacting) Hamiltonian $H$ is compared with the time evolution of a simpler (free) system. In Haag–Ruelle scattering theory, we compare the time evolution generated by $H$ with the evolution of Klein–Gordon wave packets by forming time-dependent Haag–Ruelle creation operators:

$$B_t^*[f_t] = \int_{\mathbb{R}^s} f_t(\mathbf{x}) B_t^*(\mathbf{x})\,\mathrm{d}\mathbf{x}, \quad B_t^*(\mathbf{x}) = U(t, \mathbf{x}) B^* U(t, \mathbf{x})^*, \;\; f_t = \mathrm{e}^{-\mathrm{i}t\omega(D_{\mathbf{x}})} f, \tag{2.5}$$

where $B^*$ is a creation operator, $f \in \mathcal{S}(\mathbb{R}^s)$ is a Schwartz function with compact support in Fourier space, and $\omega(D_{\mathbf{x}}) = \sqrt{m^2 + |D_{\mathbf{x}}|^2}$ is the relativistic dispersion relation (i.e. $f_t$ is a regular positive energy solution of the Klein–Gordon equation). The definition of Haag–Ruelle creation operators can be extended to $f \in L^2(\mathbb{R}^s)$. To define this extension, the following estimate is required, which relies on a uniform bound by Buchholz [Bu90].

**Proposition 2.6** ([DG14a, Lemma 3.4]). *Let $B \in \mathcal{R}$ be almost local such that $\sigma_\alpha(B) \cap V_+ = \emptyset$. For every compact subset $\Delta \subset \mathbb{R}^d$, a constant $C_\Delta$ exists such that, for every $\psi \in \mathcal{H}$,*

$$\int_{\mathbb{R}^s} \|B(\mathbf{x})E(\Delta)\psi\|^2 \,\mathrm{d}\mathbf{x} \leq C_\Delta \|\psi\|^2. \tag{2.6}$$

The following two corollaries are consequences of Propositions 2.4 and 2.6. The first corollary defines $B^*[f]$ on states of bounded energy for $f \in L^2(\mathbb{R}^s)$.

**Corollary 2.7** ([DG14b, Lemma 6.4]). *Let $B^*$ be a creation operator. For every compact subset $\Delta \subset \mathbb{R}^d$, a constant $C_\Delta$ exists such that $\|B[f]E(\Delta)\| \leq C_\Delta \|f\|_{L^2}$ and $\|B^*[f]E(\Delta)\| \leq C_\Delta \|f\|_{L^2}$.*

**Corollary 2.8** ([Bu74, Lemma 4]). *Let $B_1^*, \ldots, B_n^*$ be creation operators. A constant $C < \infty$ exists such that, for all $f \in L^2(\mathbb{R}^{ns})$,*

$$\left\| \int_{\mathbb{R}^{ns}} f(\mathbf{x}_1, \ldots, \mathbf{x}_n) B_1^*(\mathbf{x}_1) \ldots B_n^*(\mathbf{x}_n)\Omega \,\mathrm{d}\mathbf{x}_1 \ldots \mathrm{d}\mathbf{x}_n \right\| \leq C\|f\|_{L^2}. \tag{2.7}$$

*2.3. Scattering states.* If a Haag–Ruelle creation operator $B_t^*[f_t]$ is applied to the vacuum vector $\Omega$, the interacting and free time evolution exactly cancel each other, and we obtain a time-independent one-particle state.

**Lemma 2.9.** *Let $B^*$ be a creation operator and $f \in L^2(\mathbb{R}^s)$. The one-particle state $B_t^*[f_t]\Omega = \hat{f}(\mathbf{P})B^*\Omega \in \mathfrak{h}_m$ is independent of $t$.*

The next theorem provides the construction of multi-particle scattering states. We state the theorem only for outgoing scattering states (i.e. for $t \to \infty$). The results for incoming scattering states (i.e. $t \to -\infty$) are similar.

**Theorem 2.10.** *Let $B_1^*, \ldots, B_n^*$ be creation operators and $f_1, \ldots, f_n \in L^2(\mathbb{R}^s)$. The scattering states*

$$\psi_1 \overset{\text{out}}{\times} \cdots \overset{\text{out}}{\times} \psi_n = \lim_{t \to \infty} B_{1,t}^*[f_{1,t}] \ldots B_{n,t}^*[f_{n,t}]\Omega \tag{2.8}$$

*exist and depend only on the one-particle states $\psi_i = B_i^*[f_i]\Omega$. Moreover, scattering states have the following properties:*

1. *For $x \in \mathbb{R}^d$,*

$$U(x)(\psi_1 \overset{\text{out}}{\times} \cdots \overset{\text{out}}{\times} \psi_n) = U(x)\psi_1 \overset{\text{out}}{\times} \cdots \overset{\text{out}}{\times} U(x)\psi_n. \tag{2.9}$$

2. *If* $\phi_1 \overset{\text{out}}{\times} \cdots \overset{\text{out}}{\times} \phi_m$ *is another scattering state, then*

$$\langle \psi_1 \overset{\text{out}}{\times} \cdots \overset{\text{out}}{\times} \psi_n, \phi_1 \overset{\text{out}}{\times} \cdots \overset{\text{out}}{\times} \phi_m \rangle = \delta_{nm} \sum_{\sigma \in S_n} \langle \psi_1, \phi_{\sigma(1)} \rangle \ldots \langle \psi_n, \phi_{\sigma(n)} \rangle, \quad (2.10)$$

*where* $S_n$ *is the group of permutations of* $n$ *elements.*

*Remark.* In [DG14b, Theorem 6.5], the above theorem is stated for Klein–Gordon wave packets $f_1, \ldots, f_n \in L^2(\mathbb{R}^s)$ with disjoint velocity support. The span of $f_1 \otimes \cdots \otimes f_n$, where $f_1, \ldots, f_n$ have disjoint velocity support, is dense in $L^2(\mathbb{R}^{ns})$. By Corollary 2.8, the theorem extends to arbitrary families of Klein–Gordon wave packets.

Let $\Gamma(\mathfrak{h}_m)$ be the symmetric Fock space over the one-particle space $\mathfrak{h}_m = E(H_m)\mathcal{H}$ with Fock vacuum $\Omega_0$, and, for $\psi \in \mathfrak{h}_m$, let $a^*(\psi)$ be the Fock creation operator such that $a^*(\psi)\Omega_0 = \psi$. Moreover, let $\mathcal{H}^{\text{out}}$ be the Hilbert space generated by scattering states:

$$\mathcal{H}^{\text{out}} = \text{span}\{\Omega, \psi_1 \overset{\text{out}}{\times} \ldots \overset{\text{out}}{\times} \psi_n \mid \psi_1, \ldots, \psi_n \in \mathfrak{h}_m, n \in \mathbb{N}\}. \quad (2.11)$$

We define the isometric wave operator $W^{\text{out}} : \Gamma(\mathfrak{h}_m) \to \mathcal{H}^{\text{out}}$ by the following relations:

$$W^{\text{out}}\Omega_0 = \Omega, \quad (2.12)$$

$$W^{\text{out}}(a^*(\psi_1) \cdots a^*(\psi_n)\Omega_0) = \psi_1 \overset{\text{out}}{\times} \cdots \overset{\text{out}}{\times} \psi_n. \quad (2.13)$$

In the following, we write $a^*_{\text{out}}(\psi) = W^{\text{out}} a^*(\psi)(W^{\text{out}})^*$.

*2.4. Carleman functions.* A weakly measurable Hilbert space-valued function $\varphi : \mathbb{R}^{ns} \to \mathcal{H}$ is a Carleman function if its Carleman norm

$$\begin{aligned}\|\varphi\|_{\text{C}} &= \sup_{\|\psi\|_{\mathcal{H}}=1} \left( \int_{\mathbb{R}^{ns}} |\langle \psi, \varphi(\mathbf{x}_1, \ldots, \mathbf{x}_n) \rangle|^2 \, \mathrm{d}\mathbf{x}_1 \ldots \mathrm{d}\mathbf{x}_n \right)^{\frac{1}{2}} \\ &= \sup_{\|f\|_{L^2}=1} \left\| \int_{\mathbb{R}^{ns}} f(\mathbf{x}_1, \ldots, \mathbf{x}_n)\varphi(\mathbf{x}_1, \ldots, \mathbf{x}_n) \, \mathrm{d}\mathbf{x}_1 \ldots \mathrm{d}\mathbf{x}_n \right\| \end{aligned} \quad (2.14)$$

is finite.[4] Let $B_1^*, \ldots, B_n^*$ be creation operators. By Corollary 2.8, $(\mathbf{x}_1, \ldots, \mathbf{x}_n) \mapsto \varphi_0(\mathbf{x}_1, \ldots, \mathbf{x}_n) = B_1^*(\mathbf{x}_1) \ldots B_n^*(\mathbf{x}_n)\Omega$ is a Carleman function. In particular, for every $\psi \in \mathcal{H}$, $\langle \psi, \varphi_0 \rangle \in L^2(\mathbb{R}^{ns})$. We define $\varphi_t$ to be the Carleman function that obeys, for every $\psi \in \mathcal{H}$, the following identity:

$$\langle \psi, \varphi_t \rangle = \mathrm{e}^{-\mathrm{i}t(\omega(D_{\mathbf{x}_1}) + \cdots + \omega(D_{\mathbf{x}_n}))} \langle \mathrm{e}^{-\mathrm{i}tH}\psi, \varphi_0 \rangle. \quad (2.15)$$

Clearly, $\|\varphi_t\|_{\text{C}} = \|\varphi_0\|_{\text{C}}$. Moreover, if $f_1, \ldots, f_n \in \mathcal{S}(\mathbb{R}^s)$ are Schwartz functions which satisfy $\hat{f}_i = 1$ on the momentum transfer $\pi_{\mathbf{P}}(\sigma_\alpha(B_i^*))$, then

$$\varphi_t(\mathbf{x}_1, \ldots, \mathbf{x}_n) = B_{1,t}^*[f_{1,t}^{\mathbf{x}_1}] \ldots B_{n,t}^*[f_{n,t}^{\mathbf{x}_n}]\Omega, \quad (2.16)$$

[4] In [HS78, p. 63], such functions are called *bounded* Carleman functions in the sense that $\varphi$ defines a kernel of a bounded operator mapping $L^2(\mathbb{R}^{ns})$ into $\mathcal{H}$.

where $f_i^{\mathbf{x}_i} = f_i(\cdot - \mathbf{x}_i)$. By Theorem 2.10, for every $\mathbf{x}_1, \ldots, \mathbf{x}_n \in \mathbb{R}^s$, $\varphi_t(\mathbf{x}_1, \ldots, \mathbf{x}_n)$ converges in $\mathcal{H}$ as $t \to \infty$ to the scattering state

$$\varphi_+(\mathbf{x}_1, \ldots, \mathbf{x}_n) = a^*_{\text{out}}(B_1^*(\mathbf{x}_1)\Omega) \ldots a^*_{\text{out}}(B_n^*(\mathbf{x}_n)\Omega)\Omega, \tag{2.17}$$

and $\|\varphi_+\|_C \le \|\varphi_0\|_C$ by Fatou's lemma. In particular, for every $\psi \in \mathcal{H}$, we have $\langle \psi, \varphi_+ \rangle \in L^2(\mathbb{R}^{ns})$.

## 3. Araki–Haag Detectors

We introduce Araki–Haag detectors in Sect. 3.1, where we review convergence results of Araki–Haag detectors on scattering states. Subsequently, in Sect. 3.2, we discuss the convergence of single Araki–Haag detectors. Specifically, we prove our main result (Theorem 1.1).

*3.1. Araki–Haag formula.* A detector $C$ is an almost local observable measuring deviations from the vacuum with $C(x) = U(x)CU(x)^*$ representing a measurement in the neighbourhood of the spacetime point $x \in \mathbb{R}^d$.

**Definition 3.1.** A self-adjoint element $C \in \mathcal{R}$ is a detector if $C$ is almost local and $C\Omega = 0$.

*Example.* Let $B \in \mathcal{R}$ be almost local and denote the closed forward light cone by $V_+$. If $\sigma_\alpha(B) \cap V_+ = \emptyset$, then $C = B^*B$ is a detector. Indeed, $C$ is almost local and $B\Omega = 0$ by Proposition 2.4 and the spectrum condition. The detectors of the form $B^*B$ generate a $*$-algebra $\mathcal{C}$, where each element is itself a detector.

For an almost local element $A \in \mathcal{R}$ and regular scattering states $\phi, \psi \in \mathcal{H}^{\text{out}}$ of bounded energy in which no pair of particles has the same velocity, Araki and Haag [AH67, Theorem 2] proved the following asymptotic expansion as $t \to \infty$:

$$\begin{aligned}
&\langle \phi, A(t, \mathbf{x})\psi \rangle = \langle \Omega, A\Omega \rangle \langle \phi, \psi \rangle \\
&+ \int_{\mathbb{R}^s} \big( \langle \mathbf{p}|A(t,\mathbf{x})|\Omega\rangle \langle \phi, a^*_{\text{out}}(\mathbf{p})\psi\rangle + \langle \Omega|A(t,\mathbf{x})|\mathbf{p}\rangle \langle \phi, a_{\text{out}}(\mathbf{p})\psi\rangle \big)\, \mathrm{d}\mathbf{p} \\
&+ \int_{\mathbb{R}^s} \int_{\mathbb{R}^s} \langle \mathbf{q}|A(t,\mathbf{x})|\mathbf{p}\rangle \langle \phi, a^*_{\text{out}}(\mathbf{q}) a_{\text{out}}(\mathbf{p})\psi\rangle\, \mathrm{d}\mathbf{p}\, \mathrm{d}\mathbf{q} + R_{\phi,\psi,A}(t,\mathbf{x}),
\end{aligned} \tag{3.1}$$

where $R_{\phi,\psi,A}(t, \mathbf{x})$ is a remainder that decays rapidly in $t$ uniformly in $\mathbf{x} \in \mathbb{R}^s$. Here, we identify elements of the one-particle space $\mathfrak{h}_m$ with wave functions in $L^2(\mathbb{R}^s)$. A single-particle state with momentum $\mathbf{p}$ is denoted as $|\mathbf{p}\rangle$ with normalisation $\langle \mathbf{p}|\mathbf{q}\rangle = \delta(\mathbf{p} - \mathbf{q})$. Remember that, according to our assumptions, the one-particle space $\mathfrak{h}_m$ describes a single spinless particle (see [AH67] for the asymptotic expansion in the general case).

If $A = C$ is a detector, the first two terms of the asymptotic expansion (3.1) vanish, and the dominant contributions of $\langle \phi|C(t, \mathbf{x})|\psi\rangle$ as $t \to \infty$ arise from single-particle excitations. It can be shown that $\langle \phi, C(t, \mathbf{x})\psi\rangle$ converges to 0 with the rate $t^{-s}$ due to the dispersion of quantum states. To obtain a non-trivial limit as $t \to \infty$, we integrate the detector $C(t, \mathbf{x})$ over the entire space:

$$C(h, t) = \int_{\mathbb{R}^s} h\left(\frac{\mathbf{x}}{t}\right) C(t, \mathbf{x})\, \mathrm{d}\mathbf{x}, \;\; h \in C_c^\infty(\mathbb{R}^s). \tag{3.2}$$

Araki and Haag [AH67, Theorem 4] proved that, for scattering states $\phi, \psi \in \mathcal{H}^{\text{out}}$ as above,

$$\lim_{t\to\infty} \int_{\mathbb{R}^s} h\left(\frac{\mathbf{x}}{t}\right) \langle \phi, C(t,\mathbf{x})\psi\rangle \, \mathrm{d}\mathbf{x} = (2\pi)^s \int_{\mathbb{R}^s} h(\nabla\omega(\mathbf{p}))\langle \mathbf{p}|C|\mathbf{p}\rangle\langle\phi, a^*_{\text{out}}(\mathbf{p})a_{\text{out}}(\mathbf{p})\psi\rangle \, \mathrm{d}\mathbf{p}. \tag{3.3}$$

*Remark.* If $C \in \mathcal{C}$, we can extend the convergence result to all scattering states of bounded energy by Proposition 2.6.

The asymptotic observable (3.3) resembles the Fock space number operator (i.e. a particle counter). The additional factor $h(\nabla\omega(\mathbf{p}))\langle \mathbf{p}|C|\mathbf{p}\rangle$ is interpreted as the sensitivity of the counter to measure a particle of momentum $\mathbf{p}$. Specifically, $h$ is a velocity filter because particles with velocity $\nabla\omega(\mathbf{p})$ outside the support of $h$ are not counted. Henceforth, we refer to these asymptotic observables as Araki–Haag detectors. The formula (3.3) generalises to multiple detectors.

**Theorem 3.2** (Araki–Haag formula, [AH67, Theorem 5]). *Let $\phi, \psi \in \mathcal{H}^{\text{out}}$ be scattering states of bounded energy and $C_1, \dots, C_n \in \mathcal{C}$. If $h_1, \dots, h_n \in C_c^\infty(\mathbb{R}^s)$ have disjoint support, then*

$$\lim_{t\to\infty} \langle\phi, C_1(h_1,t)\dots C_n(h_n,t)\psi\rangle = \int_{\mathbb{R}^{ns}} h_1(\nabla\omega(\mathbf{p}_1))\dots h_n(\nabla\omega(\mathbf{p}_n))\Gamma(\mathbf{p}_1,\dots,\mathbf{p}_n) \times \langle\phi, a^*_{\text{out}}(\mathbf{p}_1)a_{\text{out}}(\mathbf{p}_1)\dots a^*_{\text{out}}(\mathbf{p}_n)a_{\text{out}}(\mathbf{p}_n)\psi\rangle \, \mathrm{d}\mathbf{p}_1\dots\mathrm{d}\mathbf{p}_n, \tag{3.4}$$

*where* $\Gamma(\mathbf{p}_1,\dots,\mathbf{p}_n) = (2\pi)^{ns}\langle\mathbf{p}_1|C_1|\mathbf{p}_1\rangle\dots\langle\mathbf{p}_n|C_n|\mathbf{p}_n\rangle$.

*3.2. Convergence of single Araki–Haag detectors.* To obtain particle detectors that are sensitive to particles of mass $m$, we choose $C = B^*B$, where $B^*$ is a creation operator. However, such detectors may also be sensitive to bound states of other masses as the following proposition illustrates.

**Proposition 3.3.** *Let $\Delta \subset \mathbb{R}^d$ be compact. If $\psi \in E(\Delta)\mathcal{H}$ is an eigenvector of the mass operator $M = \sqrt{H^2 - |\mathbf{P}|^2}$, then*

$$\int_{\mathbb{R}^s} \langle \mathrm{e}^{-\mathrm{i}tH}\psi, (B^*B)(\mathbf{x})\,\mathrm{e}^{-\mathrm{i}tH}\psi\rangle \, \mathrm{d}\mathbf{x} = \int_{\mathbb{R}^s} \langle\psi, (B^*B)(\mathbf{x})\psi\rangle\,\mathrm{d}\mathbf{x}. \tag{3.5}$$

*Proof.* If $m_b$ is an eigenvalue with eigenvector $\psi$, then $H\psi = (m_b^2 + |\mathbf{P}|^2)^{1/2}\psi = \omega_{m_b}(\mathbf{P})\psi$, and

$$\int_{\mathbb{R}^s} \langle \mathrm{e}^{-\mathrm{i}tH}\psi, (B^*B)(\mathbf{x})\,\mathrm{e}^{-\mathrm{i}tH}\psi\rangle\,\mathrm{d}\mathbf{x} = \langle\psi, \mathrm{e}^{\mathrm{i}t\omega_{m_b}(\mathbf{P})}\int_{\mathbb{R}^s}(B^*B)(\mathbf{x})\,\mathrm{d}\mathbf{x}\,\mathrm{e}^{-\mathrm{i}t\omega_{m_b}(\mathbf{P})}\psi\rangle = \int_{\mathbb{R}^s}\langle\psi, (B^*B)(\mathbf{x})\psi\rangle\,\mathrm{d}\mathbf{x}. \tag{3.6}$$

Observe that the translation-invariant operator $E(\Delta)\int_{\mathbb{R}^s}(B^*B)(\mathbf{x})\,\mathrm{d}\mathbf{x}E(\Delta)$, which is well-defined by Proposition 2.6, commutes with $\mathrm{e}^{\mathrm{i}t\omega_{m_b}(\mathbf{P})}$. □

We may exclude bound states of the mass operator by requiring that $\psi$ belongs to the jointly absolutely continuous spectral subspace $\mathcal{H}_{\mathrm{ac}}(P)$ of the energy-momentum operator $P$.[5] In fact, mass shells, isolated or embedded in the multi-particle spectrum, belong to the singular continuous subspace $\mathcal{H}_{\mathrm{sc}}(P)$. If $M\psi = m_b\psi$, then $\psi = E(H_{m_b})\psi$, and the mass hyperboloid $H_{m_b} \subset \mathbb{R}^d$ has Lebesgue measure 0. However, in general, $\mathcal{H}_{\mathrm{sc}}(P)$ may include exotic states that are not bound states of the mass operator. Under the additional assumption of Lorentz covariance, it can be shown that $\mathcal{H}_{\mathrm{ac}}(P) = \mathcal{H}_{\mathrm{ac}}(M)$ or, equivalently, $\mathcal{H}_{\mathrm{pp}}(P) \oplus \mathcal{H}_{\mathrm{sc}}(P) = \mathcal{H}_{\mathrm{pp}}(M) \oplus \mathcal{H}_{\mathrm{sc}}(M)$. This identity implies that the exotic states in $\mathcal{H}_{\mathrm{sc}}(P)$ correspond to the singular continuous spectrum of the mass operator.

We aim to extend the Araki–Haag formula to arbitrary states of bounded energy (i.e. to states that are not necessarily scattering states). We expect that Araki–Haag detectors do not detect a state that is orthogonal to all scattering states (i.e. the limit (3.3) should be 0 if $\psi$ is orthogonal to all scattering states). We manage to prove this expectation for detectors $C = B^*B$, where $B^*$ is a creation operator, and for states $\psi \in \mathcal{H}_{\mathrm{ac}}(P)$ such that $B\psi$ is a one-particle state.

Intuitively, the latter condition selects states of the multi-particle spectrum below the three-particle threshold. Specifically, if the energy-momentum spectrum of $\psi$ and the energy-momentum transfer of $B^*$ were point-like, then the condition that $B\psi$ is a one-particle state is always satisfied for creation operators $B^*$ and states $\psi$ below the three-particle threshold. However, due to the finite extension of the energy-momentum spectrum of $\psi$ and the energy-momentum transfer of $B^*$, it may happen that $B\psi$ has a component in the multi-particle spectrum for a state $\psi$ below the three-particle threshold. Also, there are creation operators $B^*$ and states $\psi$ above the three-particle threshold such that $B\psi$ is a one-particle state.

**Theorem 3.4.** *Let $\Delta \subset \mathbb{R}^d$ be compact, $\psi \in E(\Delta)\mathcal{H} \cap \mathcal{H}_{\mathrm{ac}}(P) \cap (\mathcal{H}^{\mathrm{out}})^{\perp}$, and $B^*$ a creation operator. If $\overline{\Delta - \sigma_\alpha(B^*)} \cap \sigma(P) \subset H_m$, then, for every $h \in L^\infty(\mathbb{R}^s)$,*

$$\lim_{t\to\infty} \mathrm{e}^{\mathrm{i}tH} \int_{\mathbb{R}^s} h\left(\frac{\mathbf{x}}{t}\right)(B^*B)(\mathbf{x})\,\mathrm{d}\mathbf{x}\,\mathrm{e}^{-\mathrm{i}tH}\psi = 0. \tag{3.7}$$

Before we present the proof, we explain that Theorem 3.4 implies our main result Theorem 1.1. This follows from the fact the convergence of Araki–Haag detectors on scattering states is already known and that it suffices to prove convergence separately on $\mathcal{H}^{\mathrm{out}}$ and $(\mathcal{H}^{\mathrm{out}})^{\perp}$, as in [Dy18].

*Proof of Theorem 1.1.* We decompose $\psi \in E(\Delta)\mathcal{H} \cap \mathcal{H}_{\mathrm{ac}}(P)$ into $\psi = \psi^{\mathrm{out}} + \psi^{\perp}$, where $\psi^{\mathrm{out}} \in \mathcal{H}^{\mathrm{out}}$ and $\psi^{\perp} \in (\mathcal{H}^{\mathrm{out}})^{\perp}$. The strong convergence of $C(h,t)\psi^{\mathrm{out}}$ in $\mathcal{H}$ was proved in [DG14b, Proposition 7.1]. The convergence of $C(h,t)\psi^{\perp}$ follows from Theorem 3.4. □

*Proof of Theorem 3.4.* The proof of the theorem is based on the following two results: the insertion of a second auxiliary detector (see Lemma 3.5 below) and the $L^2$-convergence of two-particle Haag–Ruelle scattering states (see Theorem 4.1 and Proposition 4.3 in Sect. 4).

We may assume that an $\varepsilon > 0$ exists such that $E(M \le 2m + \varepsilon)\psi = 0$, where $M = \sqrt{H^2 - |\mathbf{P}|^2}$ is the mass operator (i.e. $\psi$ lies above the two-particle threshold). Otherwise, we approximate $\psi$ by such elements. Moreover, we may assume that $\Delta$ is

[5] In [DG14a], a modified version of Araki–Haag detectors was proposed to circumvent the detection of bound states.

sufficiently small. Otherwise, we decompose $\Delta = \bigcup_i \Delta_i$ into finitely many sufficiently small compact sets $\Delta_i$ and prove the theorem for $\psi_i = E(\Delta_i)\psi$.

Let $\{\hat{g}_j\}_j \subset C_c^\infty(\mathbb{R}^d)$ be a locally finite smooth partition of unity. Because the Arveson spectrum of $B$ is compact, an $\hat{f} \in C_c^\infty(\mathbb{R}^d)$ exists such that $B = B(f) = \int_{\mathbb{R}^d} f(x)B(x)\,\mathrm{d}x$ and

$$B(f) = \sum_j B(f * g_j), \tag{3.8}$$

where only finitely many summands are non-zero. The Arveson spectrum of $B_j = B(f * g_j)$ is a subset of $\mathrm{supp}(\hat{f}\hat{g}_j) \cap \sigma_\alpha(B)$. Hence, by choosing an appropriate partition of unity, we may assume that, for every $j$, the Arveson spectrum of $B_j$ is sufficiently small. Observe that $B_j^*$ is not necessarily a creation operator because it may happen that $\sigma_\alpha(B_j^*) \cap \sigma(P) = \emptyset$. It remains to prove that, for every $j$,

$$\lim_{t\to\infty} \mathrm{e}^{\mathrm{i}tH} \int_{\mathbb{R}^s} h\left(\frac{\mathbf{x}}{t}\right)(B^* B_j)(\mathbf{x})\,\mathrm{d}\mathbf{x}\,\mathrm{e}^{-\mathrm{i}tH}\psi = 0. \tag{3.9}$$

Under the assumptions of the theorem, it occurs that either $\emptyset \neq \overline{\Delta - \sigma_\alpha(B_j^*)} \cap \sigma(P) \subset H_m$ or $\overline{\Delta - \sigma_\alpha(B_j^*)} \cap \sigma(P) = \emptyset$. If the latter is true, then $B_j\psi = 0$ and (3.9) is trivial. Thus, we may assume $\emptyset \neq \overline{\Delta - \sigma_\alpha(B_j^*)} \cap \sigma(P) \subset H_m$. We distinguish between the following three cases:

1. $\emptyset \neq \sigma_\alpha(B_j^*) \cap \sigma(P) \subset H_m$ and the sets $\pi_{\mathbf{P}}(\overline{\Delta - \sigma_\alpha(B_j^*)})$ and $\pi_{\mathbf{P}}(\sigma_\alpha(B_j^*))$ overlap.
2. $\emptyset \neq \sigma_\alpha(B_j^*) \cap \sigma(P) \subset H_m$ and the sets $\pi_{\mathbf{P}}(\overline{\Delta - \sigma_\alpha(B_j^*)})$ and $\pi_{\mathbf{P}}(\sigma_\alpha(B_j^*))$ are separated.
3. $\sigma_\alpha(B_j^*) \cap \sigma(P) = \emptyset$.

The list is exhaustive because $\sigma_\alpha(B_j^*) \subset \sigma_\alpha(B^*)$ and $\sigma_\alpha(B^*) \cap \sigma(P) \subset H_m$. We can exclude Case 1 if $\Delta$ lies above the two-particle threshold and $\Delta$ as well as $\sigma_\alpha(B_j)$ are sufficiently small (this can be assumed by the arguments above). For the proof of this claim, note that two vectors on the mass shell add up to a vector above the two-particle threshold only if the two vectors are distinct.

For the remaining two cases, note that if $\tilde{\Delta}$ is the closure of $\overline{\Delta - \sigma_\alpha(B_j^*)} + \sigma_\alpha(B^*)$, then we obtain the following estimate from the Cauchy–Schwarz inequality:

$$\begin{aligned}
&\| \mathrm{e}^{\mathrm{i}tH} \int_{\mathbb{R}^s} h\left(\frac{\mathbf{x}}{t}\right)(B^* B_j)(\mathbf{x})\,\mathrm{d}\mathbf{x}\,\mathrm{e}^{-\mathrm{i}tH}\psi \| \\
&= \sup_{\|\phi\|=1} |\langle \mathrm{e}^{-\mathrm{i}tH} E(\tilde{\Delta})\phi, \int_{\mathbb{R}^s} h\left(\frac{\mathbf{x}}{t}\right)(B^* B_j)(\mathbf{x})\,\mathrm{d}\mathbf{x}\;\mathrm{e}^{-\mathrm{i}tH}\psi\rangle| \\
&\leq \|h\|_{L^\infty} \sup_{\|\phi\|=1} \left(\int_{\mathbb{R}^s} \|B(\mathbf{x})\,\mathrm{e}^{-\mathrm{i}tH} E(\tilde{\Delta})\phi\|^2\,\mathrm{d}\mathbf{x}\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^s} \|B_j(\mathbf{x})\,\mathrm{e}^{-\mathrm{i}tH}\psi\|^2\,\mathrm{d}\mathbf{x}\right)^{\frac{1}{2}}.
\end{aligned} \tag{3.10}$$

It suffices to prove that the second factor in brackets converges to 0 because the first factor in brackets is bounded by Proposition 2.6. We observe that, for every $(t, \mathbf{x}) \in \mathbb{R}^d$,

$$B_j(\mathbf{x})\,\mathrm{e}^{-\mathrm{i}tH}\psi = E(\overline{\Delta - \sigma_\alpha(B_j^*)})B_j(\mathbf{x})\,\mathrm{e}^{-\mathrm{i}tH}\psi. \tag{3.11}$$

By Lemma 3.5, a creation operator $C^*$ exists such that the Arveson spectrum $\sigma_\alpha(C^*)$ lies in an open neighbourhood of $\overline{\Delta - \sigma_\alpha(B_j^*)}$ and, for every $t \in \mathbb{R}$,

$$\int_{\mathbb{R}^s} \|B_j(\mathbf{x})\, \mathrm{e}^{-\mathrm{i}tH}\psi\|^2\, \mathrm{d}\mathbf{x} = \int_{\mathbb{R}^s}\int_{\mathbb{R}^s} |\langle \psi, \mathrm{e}^{\mathrm{i}tH} B_j^*(\mathbf{x}) C^*(\mathbf{y})\Omega\rangle|^2\, \mathrm{d}\mathbf{x}\, \mathrm{d}\mathbf{y}. \tag{3.12}$$

In Case 2, we can choose $C^*$ such that $\pi_{\mathbf{P}}(\sigma_\alpha(C^*))$ is separated from $\pi_{\mathbf{P}}(\sigma_\alpha(B_j^*))$. The operator $\mathrm{e}^{-\mathrm{i}t(\omega(D_{\mathbf{x}})+\omega(D_{\mathbf{y}}))}$ is an isometry on $L^2(\mathbb{R}^{2s})$; hence, the r.h.s. of (3.12) equals $\|\langle\psi, \varphi_t\rangle\|_{L^2}^2$, where

$$\varphi_t(\mathbf{x}, \mathbf{y}) = \mathrm{e}^{\mathrm{i}t(H-\omega(D_{\mathbf{x}})-\omega(D_{\mathbf{y}}))} B_j^*(\mathbf{x}) C^*(\mathbf{y})\Omega. \tag{3.13}$$

We refer to Sect. 2.4 for the definition of $\varphi_t$ as a Carleman function. In Case 2, $\langle\psi, \varphi_t\rangle$ converges in $L^2(\mathbb{R}^{2s})$ to $\langle\psi, \varphi_+\rangle$ as $t \to \infty$ by Theorem 4.1, and $\langle\psi, \varphi_+\rangle = 0$ because $\psi \in (\mathcal{H}^{\mathrm{out}})^\perp$. In Case 3, $\langle\psi, \varphi_t\rangle$ converges in $L^2(\mathbb{R}^{2s})$ to 0 as $t \to \infty$ by Proposition 4.3. □

The following lemma, which we applied in the above proof of Theorem 3.4, demonstrates that one-particle states are accessible through detectors. Specifically, for every one-particle state $\psi$, we construct an Araki–Haag detector that is triggered by this state. The main idea of the proof is to identify one-particle states $\psi \in \mathfrak{h}_m$ with wave functions in $L^2(\mathbb{R}^s)$ and to choose a creation operator $B^*$ such that the wave function of $B^*\Omega$ is 1 on a given compact set.

**Lemma 3.5.** *Let $\Delta \subset \mathbb{R}^d$ be compact such that $\Delta \cap \sigma(P) \subset H_m$. A creation operator $B^*$ exists that satisfies*

$$E(\Delta) = E(\Delta) \int_{\mathbb{R}^s} (B^*B)(\mathbf{x})\, \mathrm{d}\mathbf{x}\; E(\Delta). \tag{3.14}$$

*For every $\varepsilon > 0$, the creation operator $B^*$ can be chosen such that its Arveson spectrum $\sigma_\alpha(B^*)$ is contained in an $\varepsilon$-neighbourhood of $\Delta$.*

*Proof.* The strategy of the proof is to demonstrate that the r.h.s. of (3.14) is equal to the l.h.s. of (3.14) for a suitable creation operator $B^*$. It suffices to demonstrate that, for all $\phi, \psi \in E(\Delta)\mathcal{H}$,

$$\langle\phi, \psi\rangle = \int_{\mathbb{R}^s} \langle\phi, (B^*B)(\mathbf{x})\psi\rangle\, \mathrm{d}\mathbf{x}. \tag{3.15}$$

Because states in $E(\Delta)\mathcal{H}$ are one-particle states, it holds that, for every creation operator $B^*$,

$$B(\mathbf{x})\psi = \langle\Omega, B(\mathbf{x})\psi\rangle\Omega, \tag{3.16}$$

and similarly for $\phi$. The set of generalised momentum eigenvectors $\{|\mathbf{p}\rangle\}_{\mathbf{p}\in\mathbb{R}^s}$ introduced in Sect. 3.1 obeys a completeness relation in the one-particle space: $\int_{\mathbb{R}^s} |\mathbf{p}\rangle\langle\mathbf{p}|\, \mathrm{d}\mathbf{p}$ is the identity in $\mathfrak{h}_m$. We utilise (3.16) and this completeness relation to obtain the following identity:

$$\int_{\mathbb{R}^s} \langle \phi, (B^*B)(\mathbf{x})\psi \rangle \,\mathrm{d}\mathbf{x} = \int_{\mathbb{R}^s} \langle \phi, B^*(\mathbf{x})\Omega \rangle \langle B^*(\mathbf{x})\Omega, \psi \rangle \,\mathrm{d}\mathbf{x}$$
$$= \int_{\mathbb{R}^s} \langle \phi|\mathbf{p}\rangle \langle \mathbf{p}|\psi\rangle |\langle \mathbf{p}|B^*\Omega\rangle|^2 \,\mathrm{d}\mathbf{p}. \tag{3.17}$$

The distributions $\mathbf{p} \mapsto \langle \phi|\mathbf{p}\rangle$ and $\mathbf{p} \mapsto \langle \mathbf{p}|\psi\rangle$ are compactly supported with support contained in $\pi_{\mathbf{P}}(\Delta)$. Moreover, a creation operator $B^*$ exists such that $\mathbf{p} \mapsto |\langle \mathbf{p}|B^*\Omega\rangle|^2$ is smooth, $|\langle \mathbf{p}|B^*\Omega\rangle|^2 = 1$ on a given compact set, and $|\langle \mathbf{p}|B^*\Omega\rangle|^2 = 0$ on a slightly larger set [Ar99, Section 5.3 (a)]. It follows that, for every $\varepsilon > 0$, we can choose a creation operator $B^*$ in such a way that

$$\int_{\mathbb{R}^s} \langle \phi|\mathbf{p}\rangle \langle \mathbf{p}|\psi\rangle |\langle \mathbf{p}|B^*\Omega\rangle|^2 \,\mathrm{d}\mathbf{p} = \int_{\mathbb{R}^s} \langle \phi|\mathbf{p}\rangle \langle \mathbf{p}|\psi\rangle \,\mathrm{d}\mathbf{p} = \langle \phi, \psi \rangle \tag{3.18}$$

and $\sigma_\alpha(B^*)$ is contained in an $\varepsilon$-neighbourhood of $\Delta$. □

## 4. $L^2$-Convergence of Two-Particle Scattering States

In this section, we state and prove a new convergence result for Haag–Ruelle scattering states that we applied in the proof of Theorem 3.4. We fix two creation operators $B_1^*, B_2^*$. In the following, we use the same notation as in Sect. 2.4 for $\varphi_t$ (with $n = 2$), that is,

$$\varphi_t(\mathbf{x}, \mathbf{y}) = \mathrm{e}^{\mathrm{i}t(H-\omega(D_\mathbf{x})-\omega(D_\mathbf{y}))} B_1^*(\mathbf{x}) B_2^*(\mathbf{y})\Omega. \tag{4.1}$$

Remember that, for every $\psi \in \mathcal{H}$, $\langle \psi, \varphi_t \rangle \in L^2(\mathbb{R}^{2s})$ and $\langle \psi, \varphi_t \rangle$ converges pointwise to $\langle \psi, \varphi_+ \rangle \in L^2(\mathbb{R}^{2s})$. We improve pointwise convergence to convergence in $L^2(\mathbb{R}^{2s})$.

**Theorem 4.1.** *Let $B_1^*$, $B_2^*$ be two creation operators such that the momentum transfers $\pi_{\mathbf{P}}(\sigma_\alpha(B_1^*))$, $\pi_{\mathbf{P}}(\sigma_\alpha(B_2^*))$ are separated. For every $\psi \in \mathcal{H}_{\mathrm{ac}}(P)$, $\langle \psi, \varphi_t \rangle$ converges in $L^2(\mathbb{R}^{2s})$ to $\langle \psi, \varphi_+ \rangle$ as $t \to \infty$.*

*Proof.* (i) It suffices to prove the theorem for vectors $\psi$ from a dense subset $\mathcal{D} \subset \mathcal{H}_{\mathrm{ac}}(P)$. In fact, a vector $\psi \in \mathcal{H}_{\mathrm{ac}}(P)$ is approximated by a sequence $(\psi_n)_{n\in\mathbb{N}}$ in $\mathcal{D}$; hence,

$$\|\langle \psi, \varphi_t - \varphi_+ \rangle\|_{L^2} \le 2\,\|\psi - \psi_n\|_{\mathcal{H}}\, \|\varphi_0\|_{\mathrm{C}} + \|\langle \psi_n, \varphi_t - \varphi_+ \rangle\|_{L^2}\,. \tag{4.2}$$

On the r.h.s. of (4.2), we take the limit $t \to \infty$ and, subsequently, the limit $n \to \infty$. In the following, we choose $\mathcal{D} = \mathcal{M}(P)$ (see Definition A.13), that is, we assume that the Radon–Nikodym derivative $\rho_\psi$ of the spectral measure $\langle \psi, E(\cdot)\psi \rangle$ is a bounded function. The space $\mathcal{M}(P)$ is dense in $\mathcal{H}_{\mathrm{ac}}(P)$ by Lemma A.14. We denote by $|||\psi|||$ the $L^\infty$-norm of $\sqrt{\rho_\psi}$.

(ii) We prove by Cook's method that, for $\psi \in \mathcal{M}(P)$, $\langle \psi, \varphi_t \rangle$ is a Cauchy sequence in $L^2(\mathbb{R}^{2s})$:

$$\|\langle \psi, \varphi_{t_2} - \varphi_{t_1} \rangle\|_{L^2} = \left\| \int_{t_1}^{t_2} \partial_\tau \langle \psi, \varphi_\tau \rangle \,\mathrm{d}\tau \right\|_{L^2}. \tag{4.3}$$

We claim that the time derivative amounts to replacing the product of $B_1^*(\mathbf{x})$ and $B_2^*(\mathbf{y})$ by a commutator of two creation operators. In fact,

$$\partial_\tau \varphi_\tau(\mathbf{x}, \mathbf{y}) = \mathrm{i}\,\mathrm{e}^{\mathrm{i}\tau(H-\omega(D_\mathbf{x})-\omega(D_\mathbf{y}))}(H - \omega(D_\mathbf{x}) - \omega(D_\mathbf{y}))B_1^*(\mathbf{x})B_2^*(\mathbf{y})\Omega, \tag{4.4}$$

and, because $\omega(D_{\mathbf{y}})B_2^*(\mathbf{y})\Omega = \omega(\mathbf{P})B_2^*(\mathbf{y})\Omega = HB_2^*(\mathbf{y})\Omega$, we obtain

$$(H - \omega(D_{\mathbf{x}}) - \omega(D_{\mathbf{y}}))B_1^*(\mathbf{x})B_2^*(\mathbf{y})\Omega \\ = [H, B_1^*](\mathbf{x})B_2^*(\mathbf{y})\Omega - \omega(D_{\mathbf{x}})B_1^*(\mathbf{x})B_2^*(\mathbf{y})\Omega. \tag{4.5}$$

If we write $B_1^* = B_1^*(f)$ for a Schwartz function $f \in \mathcal{S}(\mathbb{R}^d)$ that satisfies $\hat{f} = 1$ on $\sigma_\alpha(B_1^*)$, then

$$[H, B_1^*](\mathbf{x}) = B_1^*(-D_0 f)(\mathbf{x}), \tag{4.6}$$

$$\omega(D_{\mathbf{x}})B_1^*(\mathbf{x}) = B_1^*(\omega(\mathbf{D})f)(\mathbf{x}). \tag{4.7}$$

We conclude that

$$\partial_\tau \varphi_\tau(\mathbf{x}, \mathbf{y}) = \mathrm{i}\, \mathrm{e}^{\mathrm{i}\tau(H-\omega(D_{\mathbf{x}})-\omega(D_{\mathbf{y}}))}\tilde{B}_1^*(\mathbf{x})B_2^*(\mathbf{y})\Omega \\ = \mathrm{i}\, \mathrm{e}^{\mathrm{i}\tau(H-\omega(D_{\mathbf{x}})-\omega(D_{\mathbf{y}}))}[\tilde{B}_1^*(\mathbf{x}), B_2^*(\mathbf{y})]\Omega, \tag{4.8}$$

where $\tilde{B}_1^* = B_1^*(g)$ and $g \in \mathcal{S}(\mathbb{R}^d)$ is any function such that $\hat{g}(p) = p_0 - \omega(\mathbf{p})$ on $\sigma_\alpha(B_1^*)$. To obtain the commutator, we used $\tilde{B}_1^*\Omega = (H - \omega(\mathbf{P}))B_1^*\Omega = 0$.

(iii) The sets $K_{\mathrm{tot}} = \pi_{\mathbf{P}}(\sigma_\alpha(B_1^*)) + \pi_{\mathbf{P}}(\sigma_\alpha(B_2^*))$ and $K_{\mathrm{rel}} = \pi_{\mathbf{P}}(\sigma_\alpha(B_1^*)) - \pi_{\mathbf{P}}(\sigma_\alpha(B_2^*))$ contain the total and relative momentum support of the function $(\mathbf{x}, \mathbf{y}) \mapsto [\tilde{B}_1^*(\mathbf{x}), B_2^*(\mathbf{y})]\Omega$, respectively, that is,

$$[\tilde{B}_1^*(\mathbf{x}), B_2^*(\mathbf{y})]\Omega \\ = \chi(D_{\mathbf{x}} + D_{\mathbf{y}} \in K_{\mathrm{tot}})\chi(D_{\mathbf{x}} - D_{\mathbf{y}} \in K_{\mathrm{rel}})[\tilde{B}_1^*(\mathbf{x}), B_2^*(\mathbf{y})]\Omega, \tag{4.9}$$

where $\chi$ denotes the characteristic function. By assumption on the Arveson spectra of $B_1^*$ and $B_2^*$, the sets $K_{\mathrm{tot}}$ and $K_{\mathrm{rel}}$ are compact and $K_{\mathrm{rel}}$ is separated from 0.

(iv) It is convenient to introduce relative coordinates:

$$\mathbf{u} = \mathbf{x} - \mathbf{y}, \qquad D_{\mathbf{u}} = \frac{1}{2}(D_{\mathbf{x}} - D_{\mathbf{y}}), \tag{4.10}$$

$$\mathbf{v} = \frac{1}{2}(\mathbf{x} + \mathbf{y}), \qquad D_{\mathbf{v}} = D_{\mathbf{x}} + D_{\mathbf{y}}. \tag{4.11}$$

If we formulate (4.3) in relative coordinates, we arrive at the following identity:

$$\|\langle \psi, \varphi_{t_2} - \varphi_{t_1} \rangle\|_{L^2}^2 \\ = \int_{\mathbb{R}^{2s}} \left| \int_{t_1}^{t_2} \mathrm{e}^{-\mathrm{i}\tau(\omega(\frac{1}{2}D_{\mathbf{v}}+D_{\mathbf{u}})+\omega(\frac{1}{2}D_{\mathbf{v}}-D_{\mathbf{u}}))} \langle \psi, \mathrm{e}^{\mathrm{i}\tau H}\, \mathrm{e}^{-\mathrm{i}\mathbf{v}\cdot\mathbf{P}}\phi(\mathbf{u})\rangle \,\mathrm{d}\tau \right|^2 \mathrm{d}\mathbf{u}\,\mathrm{d}\mathbf{v}, \tag{4.12}$$

where

$$\phi(\mathbf{u}) = \mathrm{e}^{-\frac{\mathrm{i}}{2}\mathbf{u}\cdot\mathbf{P}}[\tilde{B}_1^*, B_2^*(-\mathbf{u})]\Omega \tag{4.13}$$

is a Hilbert space-valued Schwartz function. The function $\phi$ is smooth because it has bounded energy-momentum, and $\phi$ decays rapidly because the commutator $[\tilde{B}_1^*, B_2^*(-\mathbf{u})]$ decays rapidly in norm by Lemma 2.2. Moreover, by (4.9),

$$\langle \psi, \mathrm{e}^{\mathrm{i}\tau H}\, \mathrm{e}^{-\mathrm{i}\mathbf{v}\cdot\mathbf{P}}\phi(\mathbf{u})\rangle \\ = \chi(D_{\mathbf{v}} \in K_{\mathrm{tot}})\chi(2D_{\mathbf{u}} \in K_{\mathrm{rel}})\langle \psi, \mathrm{e}^{\mathrm{i}\tau H}\, \mathrm{e}^{-\mathrm{i}\mathbf{v}\cdot\mathbf{P}}\phi(\mathbf{u})\rangle. \tag{4.14}$$

(v) We apply Plancherel's theorem in the **v**-integral (we denote the Fourier transformation by $\mathcal{F}$), and duality in the **u**-integral:

$$\int_{\mathbb{R}^s}\int_{\mathbb{R}^s}\left|\int_{t_1}^{t_2} \mathrm{e}^{-\mathrm{i}\tau(\omega(\frac{1}{2}\mathbf{p}+D_{\mathbf{u}})+\omega(\frac{1}{2}\mathbf{p}-D_{\mathbf{u}}))}\mathcal{F}_{\mathbf{v}\to\mathbf{p}}\langle\psi, \mathrm{e}^{\mathrm{i}\tau H}\,\mathrm{e}^{-\mathrm{i}\mathbf{v}\cdot\mathbf{P}}\phi(\mathbf{u})\rangle\,\mathrm{d}\tau\right|^2 \mathrm{d}\mathbf{u}\,\mathrm{d}\mathbf{p}$$
$$= \int_{\mathbb{R}^s}\sup_{\|f\|_{L^2}=1}\left|\int_{t_1}^{t_2}\int_{\mathbb{R}^s}\overline{f(\mathbf{u})}\,\mathrm{e}^{-\mathrm{i}\tau\omega_{\mathbf{p}}(D_{\mathbf{u}})}\mathcal{F}_{\mathbf{v}\to\mathbf{p}}\langle\psi, \mathrm{e}^{\mathrm{i}\tau H}\,\mathrm{e}^{-\mathrm{i}\mathbf{v}\cdot\mathbf{P}}\phi(\mathbf{u})\rangle\,\mathrm{d}\mathbf{u}\,\mathrm{d}\tau\right|^2 \mathrm{d}\mathbf{p}, \tag{4.15}$$

where we introduced, for $\mathbf{p}\in\mathbb{R}^s$, the following operator on $L^2(\mathbb{R}^s)$:

$$\omega_{\mathbf{p}}(D_{\mathbf{u}}) = \omega(\mathbf{p}/2 + D_{\mathbf{u}}) + \omega(\mathbf{p}/2 - D_{\mathbf{u}}). \tag{4.16}$$

The operator $\omega_{\mathbf{p}}(D_{\mathbf{u}})$ corresponds to the energy of two free particles with relativistic dispersion relation and total momentum **p**. Observe that the minimal value of the Fourier multiplier $\mathbf{q}\mapsto\omega_{\mathbf{p}}(\mathbf{q}) = \omega(\mathbf{p}/2+\mathbf{q}) + \omega(\mathbf{p}/2-\mathbf{q})$ is $2\omega(\mathbf{p}/2)$. This value is assumed if and only if $\mathbf{q} = 0$ (i.e. if and only if the relative momentum is 0).

(vi) The area of integration of the **p**-integral in (4.15) can be restricted to the total momentum support $K_{\mathrm{tot}}$ due to (4.14). Moreover, because the relative momentum support $K_{\mathrm{rel}}$ is separated from 0, an $\varepsilon > 0$ exists such that, for all $\mathbf{p}\in K_{\mathrm{tot}}$,

$$E_{\mathbf{p}}(I_{\mathbf{p},\varepsilon})\phi(\mathbf{u}) = \phi(\mathbf{u}), \tag{4.17}$$

where $E_{\mathbf{p}}$ is the spectral measure of $\omega_{\mathbf{p}}(D_{\mathbf{u}})$, $I_{\mathbf{p},\varepsilon} = [2\omega(\mathbf{p}/2)+\varepsilon, \beta]$, and

$$\beta = \sup_{\mathbf{p}\in K_{\mathrm{tot}}}\sup_{\mathbf{q}\in K_{\mathrm{rel}}}\omega_{\mathbf{p}}(\mathbf{q}) = \sup_{\mathbf{p}\in\pi_{\mathbf{P}}(\sigma_\alpha(B_1^*))}\sup_{\mathbf{q}\in\pi_{\mathbf{P}}(\sigma_\alpha(B_2^*))}(\omega(\mathbf{p})+\omega(\mathbf{q})) < \infty \tag{4.18}$$

is the maximal energy of two free relativistic particles with total momentum in $K_{\mathrm{tot}}$ and relative momentum in $K_{\mathrm{rel}}$.

(vii) Let $\theta\in C_c^\infty(0,\infty)$ satisfy $\theta = 1$ on $(\varepsilon/2, \beta+1)$. For $\mathbf{p}\in\mathbb{R}^s$, set $\theta_{\mathbf{p}}(\lambda) = \theta(\lambda - 2\omega_{\mathbf{p}}(\mathbf{p}/2))$ and

$$\mathbf{F}_{\mathbf{p}}(\mathbf{q}) = \theta_{\mathbf{p}}(\omega_{\mathbf{p}}(\mathbf{q}))\frac{\nabla\omega_{\mathbf{p}}(\mathbf{q})}{|\nabla\omega_{\mathbf{p}}(\mathbf{q})|^2}, \tag{4.19}$$

where

$$\nabla\omega_{\mathbf{p}}(\mathbf{q}) = \frac{\frac{1}{2}\mathbf{p}+\mathbf{q}}{\omega(\frac{1}{2}\mathbf{p}+\mathbf{q})} - \frac{\frac{1}{2}\mathbf{p}-\mathbf{q}}{\omega(\frac{1}{2}\mathbf{p}-\mathbf{q})}. \tag{4.20}$$

The function $\mathbf{F}_{\mathbf{p}}(\mathbf{q})$ is well-defined because $|\nabla\omega_{\mathbf{p}}(\mathbf{q})| \geq b > 0$ for all $\mathbf{q}\in\omega_{\mathbf{p}}^{-1}(\mathrm{supp}(\theta_{\mathbf{p}}))$. This follows from the fact that $\nabla\omega_{\mathbf{p}}(\mathbf{q}) = 0$ if and only if $\mathbf{q} = 0$ and 0 is separated from the set $\omega_{\mathbf{p}}^{-1}(\mathrm{supp}(\theta_{\mathbf{p}}))$. The optimal value of $b$ depends on $\mathbf{p}\in\mathbb{R}^s$, but $b > 0$ can be chosen independent of **p** as long as **p** ranges over compact subsets of $\mathbb{R}^s$. We define the following modified dilation operator in relative coordinates:

$$A_{\mathbf{p}} = \frac{1}{2}(\mathbf{F}_{\mathbf{p}}(D_{\mathbf{u}})\cdot\mathbf{u} + \mathbf{u}\cdot\mathbf{F}_{\mathbf{p}}(D_{\mathbf{u}})) = \mathbf{F}_{\mathbf{p}}(D_{\mathbf{u}})\cdot\mathbf{u} + \frac{\mathrm{i}}{2}(\nabla\cdot\mathbf{F}_{\mathbf{p}})(D_{\mathbf{u}}), \tag{4.21}$$

where $\mathbf{u}$ is the multiplication operator by the relative coordinate $\mathbf{u}$. The operator $A_{\mathbf{p}}$ is essentially self-adjoint on the Schwartz space $\mathcal{S}(\mathbb{R}^s)$ [ABG96, Lemma 7.6.4]. We denote its self-adjoint closure by the same symbol. The divergence of $\mathbf{F}_{\mathbf{p}}$ in (4.21) can be computed explicitly:

$$\nabla \cdot \mathbf{F}_{\mathbf{p}}(\mathbf{q}) = \theta'_{\mathbf{p}}(\omega_{\mathbf{p}}(\mathbf{q})) - \theta_{\mathbf{p}}(\omega_{\mathbf{p}}(\mathbf{q})) \frac{\Delta\omega_{\mathbf{p}}(\mathbf{q})}{|\nabla\omega_{\mathbf{p}}(\mathbf{q})|^2}, \tag{4.22}$$

where the Laplacian of $\omega_{\mathbf{p}}$ is a bounded function:

$$\Delta\omega_{\mathbf{p}}(\mathbf{q}) = \frac{s}{\omega(\frac{1}{2}\mathbf{p}+\mathbf{q})} + \frac{s}{\omega(\frac{1}{2}\mathbf{p}-\mathbf{q})} - \frac{|\frac{1}{2}\mathbf{p}+\mathbf{q}|^2}{\omega(\frac{1}{2}\mathbf{p}+\mathbf{q})^3} - \frac{|\frac{1}{2}\mathbf{p}-\mathbf{q}|^2}{\omega(\frac{1}{2}\mathbf{p}-\mathbf{q})^3}. \tag{4.23}$$

We take the operator $A_{\mathbf{p}}$ as a conjugate operator for $\omega_{\mathbf{p}}(D_{\mathbf{u}})$ because the commutator

$$[\omega_{\mathbf{p}}(D_{\mathbf{u}}), \mathrm{i}A_{\mathbf{p}}] = \theta_{\mathbf{p}}(\omega_{\mathbf{p}}(D_{\mathbf{u}})) \tag{4.24}$$

assumes a simple form. Formally, we obtain (4.24) by utilising the well-known commutation relation $[\omega_{\mathbf{p}}(D_{\mathbf{u}}), \mathrm{i}\mathbf{u}] = \nabla\omega_{\mathbf{p}}(D_{\mathbf{u}})$; for more details, we refer to the comments below the proof and the example subsequent to Definition A.3. We denote by $\langle\cdot\rangle$ the multiplication operator on $L^2(\mathbb{R}^s)$ that maps $f$ to $\mathbf{u} \mapsto \langle\mathbf{u}\rangle f(\mathbf{u})$. From (4.21), the boundedness of $\Delta\omega_{\mathbf{p}}$, and $|\nabla\omega_{\mathbf{p}}(\mathbf{q})| \geq b > 0$ for $\mathbf{q} \in \omega_{\mathbf{p}}^{-1}(\mathrm{supp}(\theta_{\mathbf{p}}))$, it follows that, for all $f \in \mathcal{S}(\mathbb{R}^s)$, $\|\langle A_{\mathbf{p}}\rangle\langle\cdot\rangle^{-1} f\| \leq \|\langle\cdot\rangle^{-1} f\| + \|A_{\mathbf{p}}\langle\cdot\rangle^{-1} f\| \leq C\|f\|$, where $C$ can be chosen independent of $\mathbf{p}$ as long as $\mathbf{p}$ ranges over compact subsets of $\mathbb{R}^s$. Thus, $\langle A_{\mathbf{p}}\rangle\langle\cdot\rangle^{-1}$ extends to a bounded operator, which is bounded by $C$. From interpolation (Lemma A.8 with $X = 1$), it follows that $\|\langle A_{\mathbf{p}}\rangle^{\nu}\langle\cdot\rangle^{-\nu}\| \leq C^{\nu}$ and $\|\langle A_{\mathbf{p}}\rangle^{\nu} f\| \leq C^{\nu}\|\langle\cdot\rangle^{\nu} f\|$ for $\nu \in [0, 1]$.

(viii) Let $\nu \in (1/2, 1]$. We insert $1 = \langle A_{\mathbf{p}}\rangle^{-\nu}\langle A_{\mathbf{p}}\rangle^{\nu}$ into (4.15) and apply the Cauchy–Schwarz inequality to arrive at the following bound of (4.15):

$$\int_{K_{\mathrm{tot}}} \left( \sup_{\|f\|_{L^2}=1} \int_{t_1}^{t_2} \|\langle A_{\mathbf{p}}\rangle^{-\nu} \, \mathrm{e}^{\mathrm{i}\tau\omega_{\mathbf{p}}(D_{\mathbf{u}})} E_{\mathbf{p}}(I_{\mathbf{p},\varepsilon}) f\|_{L^2}^2 \, \mathrm{d}\tau \right) \\ \times \left( \int_{t_1}^{t_2} \|\langle A_{\mathbf{p}}\rangle^{\nu} \mathcal{F}_{\mathbf{v}\to\mathbf{p}} \langle\psi, \mathrm{e}^{\mathrm{i}\tau H} \, \mathrm{e}^{-\mathrm{i}\mathbf{v}\cdot\mathbf{P}}\phi\rangle\|_{L^2}^2 \, \mathrm{d}\tau \right) \mathrm{d}\mathbf{p}. \tag{4.25}$$

By Lemma 4.2 below, for every $\mathbf{p} \in \mathbb{R}^s$, a constant $c(\mathbf{p}) < \infty$ exists such that

$$\sup_{\|f\|_{L^2}=1} \int_{-\infty}^{\infty} \|\langle A_{\mathbf{p}}\rangle^{-\nu} \, \mathrm{e}^{\mathrm{i}\tau\omega_{\mathbf{p}}(D_{\mathbf{u}})} E_{\mathbf{p}}(I_{\mathbf{p},\varepsilon}) f\|_{L^2}^2 \, \mathrm{d}\tau \leq c(\mathbf{p}), \tag{4.26}$$

and $\sup_{\mathbf{p}\in K_{\mathrm{tot}}} c(\mathbf{p}) < \infty$ because $K_{\mathrm{tot}}$ is compact. It remains to prove that

$$\int_{K_{\mathrm{tot}}} \int_{t_1}^{t_2} \|\langle A_{\mathbf{p}}\rangle^{\nu} \mathcal{F}_{\mathbf{v}\to\mathbf{p}} \langle\psi, \mathrm{e}^{\mathrm{i}\tau H} \, \mathrm{e}^{-\mathrm{i}\mathbf{v}\cdot\mathbf{P}}\phi\rangle\|_{L^2}^2 \, \mathrm{d}\tau \, \mathrm{d}\mathbf{p} \\ \leq C^{2\nu} \int_{\mathbb{R}^s} \int_{t_1}^{t_2} \|\langle\cdot\rangle^{\nu} \mathcal{F}_{\mathbf{v}\to\mathbf{p}} \langle\psi, \mathrm{e}^{\mathrm{i}\tau H} \, \mathrm{e}^{-\mathrm{i}\mathbf{v}\cdot\mathbf{P}}\phi\rangle\|_{L^2}^2 \, \mathrm{d}\tau \, \mathrm{d}\mathbf{p} \tag{4.27}$$

converges to 0 as $t_1, t_2 \to \infty$. For this, it suffices to observe that, by Proposition A.15,

$$\int_{-\infty}^{\infty}\int_{\mathbb{R}^s}\int_{\mathbb{R}^s} |\langle \mathbf{u}\rangle^{\nu}\langle\psi, \mathrm{e}^{\mathrm{i}\tau H}\,\mathrm{e}^{-\mathrm{i}\mathbf{v}\cdot\mathbf{P}}\phi(\mathbf{u})\rangle|^2\,\mathrm{d}\mathbf{u}\,\mathrm{d}\mathbf{v}\,\mathrm{d}\tau \\ \leq (2\pi)^d |||\psi|||^2 \int_{\mathbb{R}^s} \|\langle\mathbf{u}\rangle^{\nu}\phi(\mathbf{u})\|^2\,\mathrm{d}\mathbf{u} < \infty. \tag{4.28}$$

Note that the last integral is finite because $\|\phi(\mathbf{u})\|$ decays rapidly.

□

To complete the proof of Theorem 4.1, it remains to demonstrate (4.26). This bound can be derived using Mourre's conjugate operator method (see Section A.3). For a fixed $\mathbf{p} \in \mathbb{R}^s$, (4.26) directly follows from a Mourre estimate for $\omega_{\mathbf{p}}(D_{\mathbf{u}})$. Specifically, $\omega_{\mathbf{p}}(D_{\mathbf{u}})$ obeys a Mourre estimate with conjugate operator $A_{\mathbf{p}}$ on the open set $J_{\mathbf{p},\varepsilon} = (2\omega(\mathbf{p}/2) + \varepsilon/2, \beta + 1)$, which contains the compact interval $I_{\mathbf{p},\varepsilon}$. To prove this claim, we refer to the example subsequent to Definition A.3, that is, we must confirm the estimates (A.4) for $h = \omega_{\mathbf{p}}$. We computed the gradient and Laplacian of $\omega_{\mathbf{p}}$ in (4.20) and (4.23), respectively. The gradient $\nabla\omega_{\mathbf{p}}$ is bounded from below by a positive constant $b > 0$ for all $\mathbf{q} \in \omega_{\mathbf{p}}^{-1}(J_{\mathbf{p},\varepsilon})$, and $\Delta\omega_{\mathbf{p}}$ is a bounded function; thus, $|\Delta\omega_{\mathbf{p}}(\mathbf{q})| \leq \|\Delta\omega_{\mathbf{p}}\|_{\infty} b^{-2}|\nabla\omega_{\mathbf{p}}(\mathbf{q})|^2$. This implies $\omega_{\mathbf{p}}(D_{\mathbf{u}}) \in C^{\infty}(A_{\mathbf{p}})$ and the following Mourre estimate:

$$E_{\mathbf{p}}(J_{\mathbf{p},\varepsilon})[\omega_{\mathbf{p}}(D_{\mathbf{u}}), \mathrm{i}A_{\mathbf{p}}]E_{\mathbf{p}}(J_{\mathbf{p},\varepsilon}) = \theta_{\mathbf{p}}(\omega_{\mathbf{p}}(D_{\mathbf{u}}))E_{\mathbf{p}}(J_{\mathbf{p},\varepsilon}) = E_{\mathbf{p}}(J_{\mathbf{p},\varepsilon}). \tag{4.29}$$

The function $\theta$ was chosen such that $\theta_{\mathbf{p}} = 1$ on $J_{\mathbf{p},\varepsilon}$. By Proposition A.11, the Mourre estimate (4.29) yields (4.26) with

$$c(\mathbf{p}) = 8 \sup_{\lambda\in I_{\mathbf{p},\varepsilon},\mu\in(0,1)} \|\langle A_{\mathbf{p}}\rangle^{-\nu}\Im(\omega_{\mathbf{p}}(D_{\mathbf{u}}) - \lambda - \mathrm{i}\mu)^{-1}\langle A_{\mathbf{p}}\rangle^{-\nu}\| < \infty, \tag{4.30}$$

where $c(\mathbf{p})$ is finite for every $\mathbf{p} \in \mathbb{R}^s$ due to the limiting absorption principle (Theorem A.4). However, determining the dependence of $c(\mathbf{p})$ on the total momentum $\mathbf{p}$ seems to be non-trivial (i.e. verifying $\sup_{\mathbf{p}\in K_{\mathrm{tot}}} c(\mathbf{p}) < \infty$). In the proof of the following lemma, we demonstrate that $c(\mathbf{p})$ is bounded by a function that is continuous in $\mathbf{p}$.

**Lemma 4.2.** *Let $\mathbf{p} \in \mathbb{R}^s$ and $\varepsilon > 0$. Let $\omega_{\mathbf{p}}(D_{\mathbf{u}})$ and $A_{\mathbf{p}}$ be the operators defined in (4.16) and (4.21), respectively, and set $I_{\mathbf{p},\varepsilon} = [2\omega(\mathbf{p}/2) + \varepsilon, \beta]$ for a $\beta \in (2\omega(\mathbf{p}/2) + \varepsilon, \infty)$. For every $\nu > 1/2$, a constant $c(\mathbf{p})$ exists such that, for $f \in L^2(\mathbb{R}^s)$,*

$$\int_{-\infty}^{\infty} \|\langle A_{\mathbf{p}}\rangle^{-\nu}\,\mathrm{e}^{\mathrm{i}\tau\omega_{\mathbf{p}}(D_{\mathbf{u}})}E_{\mathbf{p}}(I_{\mathbf{p},\varepsilon})f\|_{L^2}^2\,\mathrm{d}\tau \leq c(\mathbf{p})\|f\|_{L^2}^2, \tag{4.31}$$

*where $\sup_{\mathbf{p}\in K} c(\mathbf{p}) < \infty$ for every compact set $K \subset \mathbb{R}^s$.*

*Proof.* We may assume $1/2 < \nu \leq 1$ because if (4.31) holds for a given $\nu = \nu_0$, then it obviously holds for all $\nu \geq \nu_0$. As argued above, it suffices to demonstrate that $c(\mathbf{p})$ as chosen in (4.30) is bounded by a continuous function. To obtain such a bound, we apply Proposition A.7 with $\lambda_0 = 0$, where we must verify a limiting absorption principle for the inverse operator $\omega_{\mathbf{p}}(D_{\mathbf{u}})^{-1}$. The inverse exists and is a bounded operator because $\omega_{\mathbf{p}}(D_{\mathbf{u}}) \geq 2m > 0$ is bounded from below by a positive constant.

Let $\tilde{E}_{\mathbf{p}}$ be the spectral measure of $\omega_{\mathbf{p}}(D_{\mathbf{u}})^{-1}$. The Mourre estimate (4.29) for $\omega_{\mathbf{p}}(D_{\mathbf{u}})$ on $J_{\mathbf{p},\varepsilon}$ implies a Mourre estimate for the inverse operator $\omega_{\mathbf{p}}(D_{\mathbf{u}})^{-1}$ on $\tilde{J}_{\mathbf{p},\varepsilon} = \{\lambda \in \mathbb{R} \mid \lambda^{-1} \in J_{\mathbf{p},\varepsilon}\}$:

$$\begin{aligned} &- \tilde{E}_{\mathbf{p}}(\tilde{J}_{\mathbf{p},\varepsilon})[\omega_{\mathbf{p}}(D_{\mathbf{u}})^{-1}, \mathrm{i}A_{\mathbf{p}}]\tilde{E}_{\mathbf{p}}(\tilde{J}_{\mathbf{p},\varepsilon}) \\ &\quad = \tilde{E}_{\mathbf{p}}(\tilde{J}_{\mathbf{p},\varepsilon})\omega_{\mathbf{p}}(D_{\mathbf{u}})^{-1}[\omega_{\mathbf{p}}(D_{\mathbf{u}}), \mathrm{i}A_{\mathbf{p}}]\omega_{\mathbf{p}}(D_{\mathbf{u}})^{-1}\tilde{E}_{\mathbf{p}}(\tilde{J}_{\mathbf{p},\varepsilon}) \\ &\quad \geq \frac{1}{(1+\beta)^2}\tilde{E}_{\mathbf{p}}(\tilde{J}_{\mathbf{p},\varepsilon}), \end{aligned} \tag{4.32}$$

where we applied (A.2) in the first step. We remark that $A_{\mathbf{p}}$ depends implicitly on $\varepsilon$ through $\theta$ and that $A_{\mathbf{p}}$ is only well-defined as long as $\varepsilon > 0$. Let $\tilde{I}_{\mathbf{p},\varepsilon} = \{\lambda \in \mathbb{R} \mid \lambda^{-1} \in I_{\mathbf{p},\varepsilon}\} \subset \tilde{J}_{\mathbf{p},\varepsilon}$. From Proposition A.7, we obtain the following estimate for $c(\mathbf{p})$:

$$\begin{aligned} \frac{1}{8}c(\mathbf{p}) &\leq \sup_{\lambda \in I_{\mathbf{p},\varepsilon}, \mu \in (0,1)} \|\langle A_{\mathbf{p}}\rangle^{-\nu}(\omega_{\mathbf{p}}(D_{\mathbf{u}}) - \lambda - \mathrm{i}\mu)^{-1}\langle A_{\mathbf{p}}\rangle^{-\nu}\| \\ &\leq \sup_{\lambda \in \tilde{I}_{\mathbf{p},\varepsilon}, \mu > 0} |\lambda| \left( |\lambda| + \frac{1}{|\lambda|} + \|\langle A_{\mathbf{p}}\rangle^{-\nu}(\omega_{\mathbf{p}}(D_{\mathbf{u}})^{-1} - \lambda + \mathrm{i}\mu)^{-1}\langle A_{\mathbf{p}}\rangle^{-\nu}\| \right) \\ &\quad \times \|\langle A_{\mathbf{p}}\rangle^{\nu}\omega_{\mathbf{p}}(D_{\mathbf{u}})^{-1}\langle A_{\mathbf{p}}\rangle^{-\nu}\|. \end{aligned} \tag{4.33}$$

The contributions of the supremum from $\mu \geq 1$ are uncritical; hence, we can restrict $\mu$ to (0, 1). In the remaining expression, we bound the factors individually: For $\lambda \in \tilde{I}_{\mathbf{p},\varepsilon}$, we have $|\lambda| \leq (2\omega_{\mathbf{p}}(\mathbf{p}/2) + \varepsilon)^{-1}$ and $|\lambda|^{-1} \leq \beta$. From Theorem A.5, we obtain the following estimate:

$$\begin{aligned} &\sup_{\lambda \in \tilde{I}_{\mathbf{p},\varepsilon}, \mu \in (0,1)} \|\langle A_{\mathbf{p}}\rangle^{-\nu}(\omega_{\mathbf{p}}(D_{\mathbf{u}})^{-1} - \lambda \mp \mathrm{i}\mu)^{-1}\langle A_{\mathbf{p}}\rangle^{-\nu}\| \\ &\quad \leq \left[ \left( \frac{4}{a\varepsilon_0} + \frac{c_3\varepsilon_0^{\nu}}{\nu} \right)^{\frac{1}{2}} + \frac{c_3\varepsilon_0^{\nu-\frac{1}{2}}}{\nu - \frac{1}{2}} \right]^2 \mathrm{e}^{c_3\varepsilon_0}, \end{aligned} \tag{4.34}$$

where

$$\begin{aligned} c_1 \equiv c_1(\mathbf{p}) &= \|\omega_{\mathbf{p}}(D_{\mathbf{u}})^{-1}\| + \|[\omega_{\mathbf{p}}(D_{\mathbf{u}})^{-1}, A_{\mathbf{p}}]\| \\ &\quad + (1 + 4(1+\beta)^2)\|[\omega_{\mathbf{p}}(D_{\mathbf{u}})^{-1}, A_{\mathbf{p}}]\|^2 \\ &\quad + \|[[\omega_{\mathbf{p}}(D_{\mathbf{u}})^{-1}, A_{\mathbf{p}}], A_{\mathbf{p}}]\| + \frac{1}{(1+\beta)^2} + \frac{1}{\beta} + \frac{1}{|2\omega(\mathbf{p}/2) + \varepsilon|} + 1, \end{aligned} \tag{4.35}$$

$$c_2 \equiv c_2(\mathbf{p}) = \left( \sqrt{2} + \frac{\sqrt{8c_1(\mathbf{p})}}{\delta(\mathbf{p})} \right)(1+\beta), \tag{4.36}$$

$$c_3 \equiv c_3(\mathbf{p}) = 4c_2(\mathbf{p}) + 2c_1(\mathbf{p})c_2(\mathbf{p})^2, \tag{4.37}$$

$$\varepsilon_0 \equiv \varepsilon_0(\mathbf{p}) = \min\left\{ \frac{\delta(\mathbf{p})}{4(1+\beta)c_1(\mathbf{p})}, \frac{\delta(\mathbf{p})^2}{16c_1(\mathbf{p})^2} \right\}, \tag{4.38}$$

and $\delta \equiv \delta(\mathbf{p}) > 0$ is a continuous function such that $\tilde{I}_{\mathbf{p},\varepsilon} + \delta(\mathbf{p}) \subset J_{\mathbf{p},\varepsilon}$. Moreover, by interpolation (Lemma A.8) and (A.24), it holds that, for $\nu \in [0, 1]$,

$$\|\langle A_{\mathbf{p}}\rangle^{\nu}\omega_{\mathbf{p}}(D_{\mathbf{u}})^{-1}\langle A_{\mathbf{p}}\rangle^{-\nu}\| \le \|\omega_{\mathbf{p}}(D_{\mathbf{u}})^{-1}\|^{1-\nu}\|\langle A_{\mathbf{p}}\rangle\omega_{\mathbf{p}}(D_{\mathbf{u}})^{-1}\langle A_{\mathbf{p}}\rangle^{-1}\|^{\nu}$$
$$\le \|\omega_{\mathbf{p}}(D_{\mathbf{u}})^{-1}\|^{1-\nu}(\|\omega_{\mathbf{p}}(D_{\mathbf{u}})^{-1}\| + \|[A_{\mathbf{p}}, \omega_{\mathbf{p}}(D_{\mathbf{u}})^{-1}](A_{\mathbf{p}} + \mathrm{i})^{-1}\|)^{\nu} < \infty, \quad (4.39)$$

where we used $\|\langle A_{\mathbf{p}}\rangle(A_{\mathbf{p}} + \mathrm{i})^{-1}\| = \|(A_{\mathbf{p}} + \mathrm{i})\langle A_{\mathbf{p}}\rangle^{-1}\| = 1$ in the second step. We observe that all expressions depend continuously on $\mathbf{p}$; hence, $c(\mathbf{p})$ is bounded by a continuous function. □

In the following proposition, we prove the statement of Theorem 4.1 under slightly modified assumptions. Specifically, we drop the requirement for the momentum transfers of $B_1^*$ and $B_2^*$ to be separated, but we assume $B_1^*\Omega = 0$. This assumption simplifies the proof because we obtain the commutator of the two almost local operators $B_1^*$, $B_2^*$ from $B_1^*\Omega = 0$ (see Step (ii) in the proof of Theorem 4.1).

**Proposition 4.3.** *If $B_1^*$, $B_2^*$ are almost local operators with compact Arveson spectrum such that $B_1^*\Omega = 0$, then, for every $\psi \in \mathcal{H}_{\mathrm{ac}}(P)$,*

$$\lim_{t\to\infty} \int_{\mathbb{R}^s}\int_{\mathbb{R}^s} |\langle\psi, \mathrm{e}^{\mathrm{i}tH} B_1^*(\mathbf{x})B_2^*(\mathbf{y})\Omega\rangle|^2 \,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y} = 0. \quad (4.40)$$

*Proof.* As in the proof of Theorem 4.1, it suffices to prove the proposition for $\psi \in \mathcal{M}(P)$. We obtain the following identity from the assumption $B_1^*\Omega = 0$:

$$\int_{\mathbb{R}^{2s}} |\langle\psi, \mathrm{e}^{\mathrm{i}tH} B_1^*(\mathbf{x})B_2^*(\mathbf{y})\Omega\rangle|^2 \,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}$$
$$= \int_{\mathbb{R}^{2s}} |\langle\psi, \mathrm{e}^{\mathrm{i}tH}\,\mathrm{e}^{-\mathrm{i}\mathbf{x}\cdot\mathbf{P}}[B_1^*, B_2^*(\mathbf{y})]\Omega\rangle|^2 \,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}; \quad (4.41)$$

hence, by Proposition A.15,

$$\int_{-\infty}^{\infty}\left(\int_{\mathbb{R}^{2s}} |\langle\psi, \mathrm{e}^{\mathrm{i}tH} B_1^*(\mathbf{x})B_2^*(\mathbf{y})\Omega\rangle|^2 \,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}\right)\mathrm{d}t$$
$$\le (2\pi)^d |||\psi|||^2 \int_{\mathbb{R}^s} \|[B_1^*, B_2^*(\mathbf{y})]\Omega\|^2 \,\mathrm{d}\mathbf{y} < \infty. \quad (4.42)$$

The last integral is finite by Lemma 2.2 because $B_1^*$ and $B_2^*$ are almost local. This implies that the $L^2(\mathbb{R}^{2s})$-valued function $g(t; \mathbf{x}, \mathbf{y}) = \langle\psi, \mathrm{e}^{\mathrm{i}tH} B_1^*(\mathbf{x})B_2^*(\mathbf{y})\Omega\rangle$ is B-convergent to 0 [ABHN11, Definition 4.1.1], that is, for every $\delta > 0$,

$$\lim_{t\to\infty} \frac{1}{\delta}\int_t^{t+\delta} g(\tau)\,\mathrm{d}\tau = 0. \quad (4.43)$$

In fact, the B-convergence of $g$ is a consequence of the following estimate:

$$\left\|\frac{1}{\delta}\int_t^{t+\delta} g(\tau)\,\mathrm{d}\tau\right\|_{L^2} \le \frac{1}{\sqrt{\delta}}\left(\int_t^{t+\delta} \|g(\tau)\|_{L^2}^2\,\mathrm{d}\tau\right)^{\frac{1}{2}} \xrightarrow{t\to\infty} 0. \quad (4.44)$$

To prove that $g$ also converges in $L^2(\mathbb{R}^{2s})$, we demonstrate that $g$ is slowly oscillating [ABHN11, Definition 4.2.1], that is, for every $\varepsilon > 0$, a $\delta > 0$ and $t_0 \ge 0$ exist such that $\|g(t) - g(s)\| \le \varepsilon$ whenever $s, t \ge t_0$ and $|t - s| \le \delta$. If $g$ is slowly oscillating and B-convergent to 0, then we deduce from a simple Tauberian theorem [ABHN11, Theorem 4.2.3] that $g$ converges in $L^2(\mathbb{R}^{2s})$ to 0. In order to estimate

$$\|g(t) - g(s)\|^2 = \int_{\mathbb{R}^{2s}} |\langle \psi, \mathrm{e}^{-\mathrm{i}\mathbf{x}\cdot\mathbf{P}}(\mathrm{e}^{\mathrm{i}tH} - \mathrm{e}^{\mathrm{i}sH})[B_1^*, B_2^*(\mathbf{y})]\Omega\rangle|^2 \,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}, \tag{4.45}$$

we apply Proposition A.15 to the family $(\mathbf{P}, H)$. We select $a = \{1, \ldots, s\} \subset \mathcal{N} = \{1, \ldots, s+1\}$ so that $(\mathbf{P}, H)_a = \mathbf{P}$. The vector $[B_1^*, B_2^*(\mathbf{y})]\Omega$ has bounded energy, that is, for a $p_0 \in \mathbb{R}$,

$$[B_1^*, B_2^*(\mathbf{y})]\Omega = E(H \le p_0)[B_1^*, B_2^*(\mathbf{y})]\Omega. \tag{4.46}$$

Taking (4.46) into account, (A.52) of Proposition A.15 yields the following estimate:

$$\begin{aligned}\|g(t) - g(s)\|^2 &\le (2\pi)^s 2p_0 |\!|\!|\psi|\!|\!|^2 \int_{\mathbb{R}^s} \|(\mathrm{e}^{\mathrm{i}tH} - \mathrm{e}^{\mathrm{i}sH})[B_1^*, B_2^*(\mathbf{y})]\Omega\|^2 \,\mathrm{d}\mathbf{y} \\ &\le (2\pi)^s 2p_0^3 |\!|\!|\psi|\!|\!|^2 |t-s|^2 \int_{\mathbb{R}^s} \|[B_1^*, B_2^*(\mathbf{y})]\Omega\|^2 \,\mathrm{d}\mathbf{y};\end{aligned} \tag{4.47}$$

hence, $g$ is slowly oscillating. □

## 5. Applications and Outlook

In this section, we discuss the relevance of Theorem 1.1 to the problem of asymptotic completeness in quantum field theory, its applicability to models, and its extension to spin systems. Furthermore, we outline a strategy for proving the convergence of Araki–Haag detectors in regions of the multi-particle spectrum above the three-particle threshold.

*5.1. Asymptotic completeness.* Our main result has potential implications for a proof of asymptotic completeness in local relativistic quantum field theory, which is a long-standing open problem, as discussed in Section 1. Theorem 1.1 alone does not imply asymptotic completeness because it also applies to models that are not asymptotically complete, such as certain generalised free fields. To bridge the gap from Theorem 1.1 to asymptotic completeness, an additional condition is necessary. We presented one such condition in Corollary 1.2. Another potential condition, which is easier to verify in models, could be that the Hamiltonian $H$ can be written as a space integral over a local energy density [Ha96, p. 278], represented schematically as

$$H = \int_{\mathbb{R}^s} T^{00}(\mathbf{x}) \,\mathrm{d}\mathbf{x}, \tag{5.1}$$

where $T^{00}(\mathbf{x})$ is the 00-component of the energy-momentum tensor (see Condition T in [Dy10] for an appropriate smearing of the energy-momentum tensor). For a non-zero state $\psi \in \mathcal{H}$ orthogonal to the vacuum vector, we then have

$$0 < \langle \psi, H\psi \rangle = \int_{\mathbb{R}^s} \langle \mathrm{e}^{-\mathrm{i}tH}\psi, T^{00}(\mathbf{x})\,\mathrm{e}^{-\mathrm{i}tH}\psi \rangle \,\mathrm{d}\mathbf{x}. \tag{5.2}$$

We expect that the r.h.s. converges to 0 as $t \to \infty$ if $\psi$ is not a scattering state, similar to Theorem 1.1, where $T^{00}(\mathbf{x})$ functions as the detector. This would provide a contradiction to $\langle \psi, H\psi \rangle > 0$, implying that all states are scattering states. However, $T^{00}(\mathbf{x})$ is not in the canonical form $(B^*B)(\mathbf{x})$ analysed in this paper, where $B^*$ is a creation operator. Notably, it has been shown that in the free massive scalar field theory, the energy-momentum tensor restricted to subspaces of bounded energy can be approximated by a sum of operators of the form $B^*B$, where $B$ is almost local and energy-decreasing [Dy08, Section D]. This result would make Theorem 1.1 applicable if it could be established that $B^*$ is a creation operator. Extending the results of [Dy08, Section D] to interacting models is an interesting direction towards proving asymptotic completeness in quantum field theory.

*5.2. Models – Free products of Borchers triples.* Essential for applying Theorem 1.1 to models is the assumption for the multi-particle spectrum to be absolutely continuous in the two-particle region. An interesting class of models, which meets this requirement, emerges from the free product constructions of Borchers triples by Longo, Tanimoto, and Ueda [LTU19]. A two-dimensional Borchers triple $\mathcal{B} = (\mathcal{M}, U, \Omega)$ comprises a von Neumann algebra $\mathcal{M}$, a unitary representation $U$ of $\mathbb{R}^2$ with joint spectrum in $V_+$, and a cyclic and separating vector $\Omega$ such that $\Omega$ is invariant under $U$ and $U(x)\mathcal{M}U(x)^* \subset \mathcal{M}$ for all $x \in W_R$, where $W_R$ is the right wedge. Typical examples of Borchers triples stem from Haag–Kastler nets with $\mathcal{M}$ being the wedge algebra generated by the local observable algebras $\mathcal{R}(O)$, $O \subset W_R$. Conversely, given a Borchers triple $(\mathcal{M}, U, \Omega)$, we can construct a local net by setting $\mathcal{R}(D_{a,b}) = U(a)\mathcal{M}U(a)^* \cap U(b)\mathcal{M}'U(b)^*$. Here, $D_{a,b} = (W_R + a) \cap (W_L + b)$, $a, b \in \mathbb{R}^2$, is a double cone, and $W_L$ is the left wedge. The resulting net satisfies microcausality, isotony, and Poincaré covariance [LTU19, Section 2.1.2].

Consider two identical copies $\mathcal{B}_1, \mathcal{B}_2$ of the Borchers triple corresponding to the two-dimensional free massive scalar field theory [LTU19, Section 5.1]. According to [LTU19, Proposition 5.1], the free product $\mathcal{B}_1 \star \mathcal{B}_2$ again forms a Borchers triple $\mathcal{B} = (\mathcal{M}, U, \Omega)$. Because $U$ is the free product of $U_1$ and $U_2$ (i.e., essentially, a direct sum involving only the unitary representations $U_1$ and $U_2$ of the free theory), the multi-particle spectrum of $U$ is absolutely continuous. Furthermore, the two-particle $S$-matrix of the free product model is non-trivial, yet asymptotic completeness fails [LTU19, Section 5.3].

We remark that the non-triviality of the local algebras arising from the free product construction was not proved in [LTU19]. If it could be established that the vacuum vector is cyclic for the local algebras, then the free product construction would present an interesting class of models where the convergence result of Theorem 1.1 was not known previously.

*5.3. Spin systems.* Theorem 1.1 can be extended to spin systems through the adapted Haag–Ruelle scattering theory developed by Bachmann, Dybalski, and Naaijkens [BDN16]. A spin system is a $C^*$-dynamical system $(\mathcal{A}, \tau)$, where $\tau$ is a unitary representation of the spacetime translation group. The algebra $\mathcal{A}$ is generated by a local net $\{\mathcal{A}(\Lambda)\}_\Lambda$, where $\Lambda \subset \mathbb{Z}^s$ is a bounded spatial region. The challenge in adapting the Haag–Ruelle scattering theory to spin systems is to define a suitable almost local algebra. By replacing the double cones $K_r$ in Definition 2.1 with open balls in $\mathbb{Z}^s$ of radius $r$, we obtain an algebra that is a priori not invariant under time translations. However, by utilising the Lieb–Robinson bound,

$$\|[\tau_t(A), B]\| \leq C_{A,B}\, \mathrm{e}^{\lambda(v_{\mathrm{LR}}t - d(A,B))}, \tag{5.3}$$

where $\lambda > 0$ is a constant, $v_{LR} > 0$ the Lieb–Robinson velocity, and $d(A, B)$ the distance between the localisation regions of the local observables $A$ and $B$, it is possible to define an almost local algebra that is invariant under spacetime translations and satisfies Lemma 2.2 [BDN16, Theorem 3.10].

The proof of Theorem 1.1 readily extends to spin systems with only minor adjustments, provided that the isolated mass shell in the energy-momentum spectrum is regular and pseudo-relativistic. These properties of the mass shell have been verified for the Ising model in a strong magnetic field [BDN16, Section 6].

To our knowledge, it is unknown whether the energy-momentum spectrum of the Ising model is absolutely continuous in the two-particle region. Thus far, the spectral analysis of spin systems has focused on perturbation theory of the energy spectrum [Po92, Ya06, NSY23, DFPR23]. Nevertheless, the cited papers showcase a diverse array of techniques available for investigating spectral properties of spin systems, which could potentially be extended to analyse the energy-momentum spectrum as well.

Analogous to quantum field theory, the adapted version of Theorem 1.1 could be utilised to prove two-particle asymptotic completeness in spin systems. Notably, Buchholz [Bu86] constructed an ideal local detector $C$ under assumptions which are typical for spin systems. If it were possible to verify that this detector has the canonical form $C = B^*B$, where $B^*$ is a creation operator, it would provide a proof of two-particle asymptotic completeness through Corollary 1.2.

*5.4. Convergence of Araki–Haag detectors in the many-particle region.* Extending the convergence of Araki–Haag detectors into regions of the multi-particle spectrum above the three-particle threshold presents an interesting direction for further research. Relevant for this problem is the $L^2$-convergence of many-body Haag–Ruelle scattering states:

$$\varphi_t(\mathbf{x}_1, \ldots, \mathbf{x}_n) = \mathrm{e}^{\mathrm{i}t(H-\omega(D_{\mathbf{x}_1})-\cdots-\omega(D_{\mathbf{x}_n}))} B_1^*(\mathbf{x}_1) \ldots B_n^*(\mathbf{x}_n)\Omega. \tag{5.4}$$

In Sect. 4, we established the convergence of $\langle \psi, \varphi_t \rangle$, $\psi \in \mathcal{H}_{\mathrm{ac}}(P)$, in $L^2(\mathbb{R}^{ns})$ for the case $n = 2$, but extending this proof to $n \geq 3$ requires new ideas. The main difficulty lies in Step (ii) of the proof, where differentiation w.r.t. $t$ yields commutators of creation operators. While for $n = 2$ the resulting expression decays in the relative coordinate $\mathbf{u}$, we obtain anisotropic decay for $n \geq 3$ (e.g. the expression does not decay if two or more particles stay close to each other while the remaining particles escape to infinity). This anisotropic decay resembles the behaviour encountered in the scattering theory of many-body quantum mechanical systems, where the many-body potential exhibits similar anisotropic decay. Physically, the anisotropic decay of the potential is associated with the formation of many-body bound states, which must be considered to describe the asymptotic evolution of the quantum system correctly.

A promising approach to adapt the proof of Theorem 4.1 to the many-body case is to transfer Yafaev's proof of asymptotic completeness [Ya93] to quantum field theory. Yafaev's technique is a natural extension of Lavine's theorem to the many-body case. His proof relies on certain radiation estimates, which trace back to Sommerfeld's radiation condition. In the context of quantum field theory, such radiation estimates could prove equally relevant.

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**Declarations**

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## A Mourre's Conjugate Operator Method

Mourre's conjugate operator method is a powerful tool to analyse spectral properties of a self-adjoint operator $H : D(H) \to \mathcal{H}$ on a Hilbert space $\mathcal{H}$ based on a strictly positive commutator estimate (the so-called Mourre estimate, see Section A.1). One of the main results of the conjugate operator method is the limiting absorption principle (see Section A.2), which controls the resolvent $R(z) = (H - z)^{-1}$ as the resolvent parameter $z \in \rho(H)$ approaches the spectrum. Closely related to the limiting absorption principle are locally smooth operators, which we discuss in Section A.3.

*A.1 Mourre estimate.* We introduce the following regularity classes, which are relevant for defining commutators.

**Definition A.1.** Let $A$ be a self-adjoint operator on $\mathcal{H}$ and let $k \in \mathbb{N} \cup \{\infty\}$. We denote by $C^k(A)$ the space of all self-adjoint operators $H$ such that, for a $z \in \rho(H)$, $t \mapsto \mathrm{e}^{\mathrm{i}tA} R(z)\, \mathrm{e}^{-\mathrm{i}tA}$ is a $C^k$-map in the strong operator topology.

By [ABG96, Lemma 6.2.1], if $z \in \rho(H)$ exists such that $t \mapsto \mathrm{e}^{\mathrm{i}tA} R(z)\, \mathrm{e}^{-\mathrm{i}tA}$ is a $C^k$-map, then $t \mapsto \mathrm{e}^{\mathrm{i}tA} R(z)\, \mathrm{e}^{-\mathrm{i}tA}$ is a $C^k$-map for all $z \in \rho(H)$. Moreover, if $H$ is bounded, $t \mapsto \mathrm{e}^{\mathrm{i}tA} R(z)\, \mathrm{e}^{-\mathrm{i}tA}$ is a $C^k$-map if and only if $t \mapsto \mathrm{e}^{\mathrm{i}tA} H\, \mathrm{e}^{-\mathrm{i}tA}$ is a $C^k$-map. If $H \in C^1(A)$ is bounded, we can define the commutator $[H, A]$ on $\mathcal{H}$ as the strong derivative of $t \mapsto \mathrm{i}\, \mathrm{e}^{\mathrm{i}tA} H\, \mathrm{e}^{-\mathrm{i}tA}$. If $H \in C^1(A)$ is unbounded, the sesquilinear form defined by $HA - AH$ on $D(A) \cap D(H)$ extends to $D(H)$.

**Proposition A.2** ([ABG96, Theorem 6.2.10 (b)]). *If $H \in C^1(A)$, then $D(A) \cap D(H)$ is a core for $H$, and the sesquilinear form*

$$(f, g) \mapsto \langle Hf, Ag\rangle - \langle Af, Hg\rangle, \ \ f, g \in D(A) \cap D(H), \tag{A.1}$$

*has a unique extension to a continuous sesquilinear form on $D(H)$, where $D(H)$ is equipped with the graph topology. If we denote the operator associated to the extended sesquilinear form by $[H, A]$, then the following identity holds on $\mathcal{H}$ in the form sense:*

$$[R(z), A] = -R(z)[H, A]R(z), \ \ z \in \rho(H). \tag{A.2}$$

Observe that $[R(z), A]$ is a bounded operator on $\mathcal{H}$ if $H \in C^1(A)$ and that $R(z)$ maps $\mathcal{H}$ into $D(H)$ and the dual space $D(H)'$ into $\mathcal{H}$. We are now able to formalise the notion of a strictly positive commutator estimate. The example following the definition is applied in Sect. 4.

**Definition A.3.** The operator $H$ obeys a Mourre estimate on an open bounded set $J \subset \mathbb{R}$ if a self-adjoint operator $A$ (conjugate operator) exists such that $H \in C^1(A)$ and, for an $a > 0$,

$$E(J)[H, \mathrm{i}A]E(J) \geq aE(J), \tag{A.3}$$

where $E$ is the spectral measure of $H$.

*Example.* ([ABG96, Lemma 7.6.4]) Let $\mathcal{H} = L^2(\mathbb{R}^n)$, and define $H = h(D)$, where $h : \mathbb{R}^n \to \mathbb{R}$ is a Borel function and $D = -\mathrm{i}\partial$. Suppose $J \subset \mathbb{R}$ is an open set such that $\Omega = h^{-1}(J)$ is also open in $\mathbb{R}^n$, and $h$ belongs to $C^2$ on a neighbourhood of the closure of $\Omega$. Assume that a constant $c > 0$ exists such that, for $x \in \Omega$,

$$|\nabla h(x)| \geq c, \ |\Delta h(x)| \leq c^{-1}|\nabla h(x)|^2. \tag{A.4}$$

Consider $\theta \in C_c^\infty(J)$ to be real-valued, and let $F$ be defined as $F(x) = \theta(h(x))|\nabla h(x)|^{-2}\nabla h(x)$ for $x \in \Omega$, and $F(x) = 0$ otherwise. The modified dilation operator

$$A = \frac{1}{2}(F(D) \cdot X + X \cdot F(D)) \tag{A.5}$$

is essentially self-adjoint on $\mathcal{S}(\mathbb{R}^n)$ and $H \in C^\infty(A)$. Moreover, for all $k \geq 1$, $\mathrm{ad}^k_{-\mathrm{i}A}(H) = \theta_{k-1}(H)$ are bounded operators on $\mathcal{H}$, where $\theta_k(\lambda) = [\theta(\lambda)\partial_\lambda]^k\theta(\lambda)$. In particular,

$$E(J_\theta)[H, \mathrm{i}A]E(J_\theta) = \theta(H)E(J_\theta) = E(J_\theta), \tag{A.6}$$

where $J_\theta = \{\lambda \in \mathbb{R} \mid \theta(\lambda) = 1\}$ (i.e. $H$ obeys a Mourre estimate on every open subset of $J_\theta$).

*A.2 Limiting absorption principle.* The limiting absorption principle extends, for $\nu > 1/2$, the holomorphic resolvent function $\mathbb{C}_\pm \ni z \mapsto \langle A\rangle^{-\nu}R(z)\langle A\rangle^{-\nu}$ to a continuous function on $\mathbb{C}_\pm \cup J$ if $H$ obeys a Mourre estimate on $J$ with conjugate operator $A$. The limiting absorption principle has been proved under different assumptions. To our knowledge, the optimal assumptions are those in [ABG96, Theorem 7.4.1] if $H$ has a spectral gap (i.e. $\sigma(H) \neq \mathbb{R}$) and [Sa97, Theorem 0.1] if $H$ does not have a spectral gap. We prefer to cite the limiting absorption principle under less optimal assumptions, which are sufficient for our purposes and avoid introducing Besov spaces associated to a $C_0$-group.

**Theorem A.4** (Limiting absorption principle, [Ge08, Theorem 1]). *Let $H \in C^2(A)$. If $H$ obeys a Mourre estimate on $J$, then, for every compact interval $I \subset J$ and every $\nu > 1/2$,*

$$\sup_{\lambda \in I, \mu > 0} \|\langle A\rangle^{-\nu}(H - \lambda \mp \mathrm{i}\mu)^{-1}\langle A\rangle^{-\nu}\| < \infty. \tag{A.7}$$

In our application, it is relevant to determine the dependence of the bound (A.7) on the parameter $a$, the operators $A$, $H$, and the sets $I$, $J$. It is easier to determine this dependence if $H$ is a bounded operator.

**Theorem A.5** *Let $H \in C^2(A)$ be a bounded operator. If $H$ obeys a Mourre estimate on $J$, then, for every compact interval $I = [\alpha, \beta] \subset J$ such that $I + [-\delta, \delta] \subset J$ for a $\delta > 0$, and every $\nu > 1/2$,*

$$\sup_{\lambda \in I, \mu \in (0,1)} \|\langle A\rangle^{-\nu}(H - \lambda \mp i\mu)^{-1}\langle A\rangle^{-\nu}\| \leq \left[\left(\frac{4}{a\varepsilon_0} + \frac{c_3\varepsilon_0^{\nu}}{\nu}\right)^{\frac{1}{2}} + \frac{c_3\varepsilon_0^{\nu-\frac{1}{2}}}{\nu - \frac{1}{2}}\right]^2 e^{c_3\varepsilon_0}, \tag{A.8}$$

*where*

$$c_1 = \|H\| + \|[H, A]\| + \left(1 + \frac{4}{a}\right)\|[H, A]\|^2 + \|[[H, A], A]\| + a + |\alpha| + |\beta| + 1, \tag{A.9}$$

$$c_2 = \sqrt{\frac{2}{a}} + \frac{1}{\delta}\sqrt{\frac{8c_1}{a}}, \tag{A.10}$$

$$c_3 = 4c_2 + 2c_1c_2^2, \tag{A.11}$$

$$\varepsilon_0 = \min\left\{\frac{\sqrt{a}\delta}{4c_1}, \frac{\delta^2}{16c_1^2}\right\}. \tag{A.12}$$

*Proof.* The proof relies on a clever approximation $G_\varepsilon^{\pm}(\lambda, \mu)$ of the resolvent $R(\lambda \pm i\mu)$ such that, formally, $G_\varepsilon^{\pm}(\lambda, \mu) \to R(\lambda \pm i\mu)$ as $\varepsilon \downarrow 0$. A differential inequality ensures that $\langle A\rangle^{-\nu}G_\varepsilon^{\pm}(\lambda, \mu)\langle A\rangle^{-\nu}$ remains bounded as $\mu \downarrow 0$ and $\varepsilon \downarrow 0$. A complete proof of the theorem is provided in [Am09, Theorem 6.3]. We sketch the idea of the proof to obtain the bound (A.8). For $\lambda \in I$ and $\mu \in (0, 1)$, we define the operator-valued function

$$\Phi_\varepsilon = \langle \varepsilon A\rangle^{-1}\langle A\rangle^{-\nu}G_\varepsilon^{\pm}(\lambda, \mu)\langle A\rangle^{-\nu}\langle \varepsilon A\rangle^{-1}, \tag{A.13}$$

where $G_\varepsilon^{\pm}(\lambda, \mu)$ is the inverse of $H - \lambda \mp i\mu \mp i\varepsilon[H, iA]$. For $\varepsilon \in (0, \varepsilon_0)$, the existence of the inverse is a consequence of the Mourre estimate. We must show that $\Phi_\varepsilon$ remains bounded as $\varepsilon \downarrow 0$ and $\mu \downarrow 0$. This is achieved by proving the following differential inequality [Am09, (6.56)]:

$$\|\Phi_\varepsilon'\| \leq c_3\varepsilon^{\nu-1} + c_3\varepsilon^{\nu-\frac{3}{2}}\|\Phi_\varepsilon\|^{\frac{1}{2}} + c_3\|\Phi_\varepsilon\|. \tag{A.14}$$

Moreover, $\|\Phi_{\varepsilon_0}\| \leq 4/(a\varepsilon_0)$ [Am09, p. 279]. We complete the proof by Lemma A.6 below. □

**Lemma A.6** (Method of differential inequality, [Am09, Section 6.2.1]). *Let $\varepsilon_0 > 0$ and $(0, \varepsilon_0) \ni \varepsilon \mapsto \Phi_\varepsilon$ be a continuously differentiable $\mathfrak{B}(\mathcal{H})$-valued function. If $\Phi_\varepsilon$ is a solution of the differential inequality*

$$\|\Phi_\varepsilon'\| \leq \theta_1(\varepsilon) + \theta_2(\varepsilon)\|\Phi_\varepsilon\|^{\frac{1}{2}} + \gamma\|\Phi_\varepsilon\|, \tag{A.15}$$

*where* $\theta_k : (0, \varepsilon_0) \to [0, \infty)$ *satisfy* $\int_0^{\varepsilon_0} \theta_k(\varepsilon)\, \mathrm{d}\varepsilon < \infty$, $k \in \{1, 2\}$, *then* $\Phi_\varepsilon$ *is bounded as follows:*

$$\|\Phi_\varepsilon\| \leq \left[ \left( \|\Phi_{\varepsilon_0}\| + \int_0^{\varepsilon_0} \theta_1(\varepsilon')\, \mathrm{d}\varepsilon' \right)^{\frac{1}{2}} + \int_0^{\varepsilon_0} \theta_2(\varepsilon')\, \mathrm{d}\varepsilon' \right]^2 \mathrm{e}^{\gamma \varepsilon_0}. \tag{A.16}$$

If $H$ has a spectral gap, it is straightforward to obtain an explicit estimate for (A.7) from Theorem A.5.

**Proposition A.7.** *Let* $H \in C^2(A)$ *have a spectral gap. Select* $\lambda_0 \in \mathbb{R}\backslash\sigma(H)$ *and define* $R = (H - \lambda_0)^{-1}$. *If H obeys a Mourre estimate on* $J \subset \sigma(H)$, *then, for every compact interval* $I \subset J$ *and every* $1/2 < \nu \leq 1$,

$$\begin{aligned} &\sup_{\lambda \in I, \mu \in (0,1)} \|\langle A\rangle^{-\nu}(H - \lambda \mp \mathrm{i}\mu)^{-1}\langle A\rangle^{-\nu}\| \\ &\leq \sup_{\lambda \in \tilde{I}, \mu > 0} |\lambda| \left( |\lambda| + \frac{1}{|\lambda|} + \|\langle A\rangle^{-\nu}(R - \lambda \pm \mathrm{i}\mu)^{-1}\langle A\rangle^{-\nu}\| \right) \|\langle A\rangle^{\nu} R \langle A\rangle^{-\nu}\| < \infty, \end{aligned} \tag{A.17}$$

*where* $\tilde{I} = \{(\lambda - \lambda_0)^{-1} \mid \lambda \in I\}$.

*Proof.* The main steps of the proof are the same as in the proof of [ABG96, Theorem 7.4.1]. The resolvents of $H$ and $R$ are related as follows:

$$(H - \lambda \mp \mathrm{i}\mu)^{-1} = -(\lambda - \lambda_0 \pm \mathrm{i}\mu)^{-1}[R - (\lambda - \lambda_0 \pm \mathrm{i}\mu)^{-1}]^{-1} R. \tag{A.18}$$

This identity entails the following estimate:

$$\begin{aligned} &\sup_{\lambda \in I, \mu \in (0,1)} \|\langle A\rangle^{-\nu}(H - \lambda \mp \mathrm{i}\mu)^{-1}\langle A\rangle^{-\nu}\| \\ &\quad \leq \sup_{\lambda \in I, \mu \in (0,1)} |\lambda - \lambda_0 \pm \mathrm{i}\mu|^{-1} \|\langle A\rangle^{-\nu}(R - (\lambda - \lambda_0 \pm \mathrm{i}\mu)^{-1})^{-1}\langle A\rangle^{-\nu}\| \\ &\qquad \times \|\langle A\rangle^{\nu} R \langle A\rangle^{-\nu}\|. \end{aligned} \tag{A.19}$$

For $z \in \rho(R)$, define $Q(z) = (R - z)^{-1}$. From the resolvent formula, we obtain the following identity:

$$\begin{aligned} &Q((\lambda - \lambda_0 \pm \mathrm{i}\mu)^{-1}) - Q\left( (\lambda - \lambda_0)^{-1} \mp \mathrm{i} \frac{\mu}{(\lambda - \lambda_0)^2 + \mu^2} \right) \\ &= \frac{-\mu^2}{[(\lambda - \lambda_0)^2 + \mu^2](\lambda - \lambda_0)} Q((\lambda - \lambda_0 \pm \mathrm{i}\mu)^{-1}) Q\left( (\lambda - \lambda_0)^{-1} \mp \mathrm{i} \frac{\mu}{(\lambda - \lambda_0)^2 + \mu^2} \right). \end{aligned} \tag{A.20}$$

Remember that $\|Q(z)\| \leq |\Im(z)|^{-1}$; hence,

$$\left\| Q((\lambda - \lambda_0 \pm \mathrm{i}\mu)^{-1}) - Q\left( (\lambda - \lambda_0)^{-1} \mp \mathrm{i} \frac{\mu}{(\lambda - \lambda_0)^2 + \mu^2} \right) \right\| \leq \frac{(\lambda - \lambda_0)^2 + \mu^2}{|\lambda - \lambda_0|}, \tag{A.21}$$

and, subsequently,

$$\sup_{\lambda \in I, \mu \in (0,1)} |\lambda - \lambda_0 \pm \mathrm{i}\mu|^{-1} \|\langle A\rangle^{-\nu}(R - (\lambda - \lambda_0 \pm \mathrm{i}\mu)^{-1})^{-1}\langle A\rangle^{-\nu}\|$$

$$\leq \sup_{\lambda\in I,\mu>0} \frac{1}{|\lambda-\lambda_0|}\left(\frac{(\lambda-\lambda_0)^2+1}{|\lambda-\lambda_0|} + \|\langle A\rangle^{-\nu}(R-(\lambda-\lambda_0)^{-1}\pm \mathrm{i}\mu)^{-1}\langle A\rangle^{-\nu}\|\right). \tag{A.22}$$

It remains to demonstrate that the r.h.s. of (A.17) is finite. If $K \subset J$ is a compact subset, then $R$ obeys a Mourre estimate on every open subset contained in $\tilde{K} = \{(\lambda-\lambda_0)^{-1} \mid \lambda \in K\}$ (see [ABG96, Proposition 7.2.5]); hence, by Theorem A.5,

$$\sup_{\lambda\in\tilde{I},\mu>0} \|\langle A\rangle^{-\nu}(R-\lambda\pm \mathrm{i}\mu)^{-1}\langle A\rangle^{-\nu}\| < \infty. \tag{A.23}$$

Also, $\|\langle A\rangle^{\nu}R\langle A\rangle^{-\nu}\| < \infty$ for $\nu \in [0,1]$ because $R \in C^1(A)$. In fact,

$$(A+\mathrm{i})R(A+\mathrm{i})^{-1} = R + [A,R](A+\mathrm{i})^{-1} \tag{A.24}$$

is a sum of bounded operators. It follows that $\langle A\rangle R\langle A\rangle^{-1}$ is also bounded, and from Lemma A.8, we conclude $\|\langle A\rangle^{\nu}R\langle A\rangle^{-\nu}\| \leq \|R\|^{1-\nu}\|\langle A\rangle R\langle A\rangle^{-1}\|^{\nu} < \infty$. □

**Lemma A.8** (Interpolation, [Am09, Proposition 6.17]). *Let $X$ be a bounded operator and $S_1$, $S_2$ positive invertible self-adjoint operators. Assume that $S_1$ or $S_2$ is bounded. If the closure of $S_1XS_2$ is bounded, then, for $\nu \in [0,1]$, the closure of $S_1^{\nu}XS_2^{\nu}$ is also bounded and*

$$\|S_1^{\nu}XS_2^{\nu}\| \leq \|X\|^{1-\nu}\|S_1XS_2\|^{\nu}. \tag{A.25}$$

*A.3 Locally smooth operators.* We introduce locally smooth operators for a family $H = (H_1,\ldots,H_n)$ of strongly commuting self-adjoint operators. Most of the results of this section are straightforward generalisations of those for a single self-adjoint operator (see e.g. [ABG96, Section 7.1]). The only exception is Kato's Theorem (Theorem A.12), whose proof requires a new idea. We denote by $E$ the spectral measure of the family $H$ and by $\sigma(H) \subset \mathbb{R}^n$ the joint spectrum. Note that the intersection $D(H) = D(H_1) \cap \cdots \cap D(H_n)$ is dense in $\mathcal{H}$. We consider $D(H)$ as a Banach space equipped with the graph topology.

Set $\mathcal{N} = \{1,\ldots,n\}$ and let $a = \{a_1,\ldots,a_k\} \subset \mathcal{N}$ be a subset (w.l.o.g. $a_1 < \cdots < a_k$) with $|a| = k$ elements. For $x \in \mathbb{R}^n$, we denote the vector $(x_{a_1},\ldots,x_{a_k}) \in \mathbb{R}^k$ by $x_a$, and the vector $(x_{b_1},\ldots,x_{b_{n-k}})$ by $x^a$, where $\{b_1,\ldots,b_{n-k}\} = \mathcal{N}\backslash a$ and $b_1 < \cdots < b_{n-k}$. We identify $x$ with $x_a \oplus x^a$. For a subset $K \subset \mathbb{R}^n$, we define the following sets (see Fig. 2):

$$K(\mathcal{N}) = K, \tag{A.26}$$

$$K(a) = \{x \in \mathbb{R}^n \mid \exists y \in \mathbb{R}^{n-|a|} : x_a \oplus y \in K\},\ \emptyset \neq a \subsetneq \mathcal{N}, \tag{A.27}$$

$$K(\emptyset) = \mathbb{R}^n \backslash \bigcup_{\emptyset\neq a\subset\mathcal{N}} K(a). \tag{A.28}$$

Observe that the sets $K(a)$, $a \subset \mathcal{N}$, cover $\mathbb{R}^n$, $K$ is contained in $K(a)$ if $a \neq \emptyset$, and

$$\mathbb{R}^n\backslash K = K(\emptyset) \cup \bigcup_{\emptyset\neq a\subsetneq\mathcal{N}} (K(a)\backslash K). \tag{A.29}$$

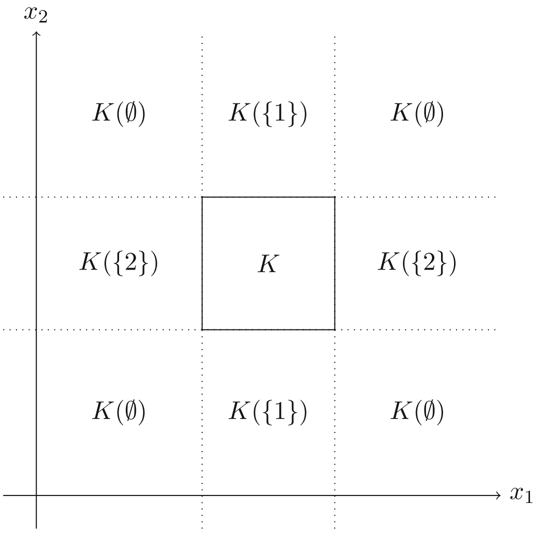


**Fig. 2.** Partition of the set $\mathbb{R}^2$ into the sets $K(a)$, $a \subset \{1, 2\}$

For products of resolvents of $H_1, \ldots, H_n$, we use the following notation ($\lambda, \mu \in \mathbb{R}^n$):

$$R_a(\lambda + \mathrm{i}\mu) = \prod_{j \in a} (H_j - \lambda_j - \mathrm{i}\mu_j)^{-1}, \tag{A.30}$$

$$\begin{aligned} \Im R_a(\lambda + \mathrm{i}\mu) &= \prod_{j \in a} \Im (H_j - \lambda_j - \mathrm{i}\mu_j)^{-1} \\ &= \mu_{a_1} \ldots \mu_{a_k} R_a(\lambda \pm \mathrm{i}\mu)^* R_a(\lambda \pm \mathrm{i}\mu). \end{aligned} \tag{A.31}$$

We abbreviate $R_{\mathcal{N}}(\lambda + \mathrm{i}\mu)$ by $R(\lambda + \mathrm{i}\mu)$. The following definition is a natural generalisation of locally $H$-smooth operators for a family of commuting self-adjoint operators.

**Definition A.9.** Let $\mathcal{G}$ be a Hilbert space. A continuous operator $T : D(H) \to \mathcal{G}$ is locally $H$-smooth on an open set $J \subset \mathbb{R}^n$ if, for every $\emptyset \neq a \subset \mathcal{N}$ and every compact subset $K \subset J$, a constant $C_{K(a)}$ exists such that, for all $f \in \mathcal{H}$,

$$\int_{\mathbb{R}^{|a|}} \|T\, \mathrm{e}^{\mathrm{i}x_a \cdot H_a} E(K(a)) f\|_{\mathcal{G}}^2 \,\mathrm{d}x_a \le C_{K(a)} \|f\|_{\mathcal{H}}^2. \tag{A.32}$$

In the case of a single self-adjoint operator $H$ (i.e. $n = 1$), a continuous operator $T : D(H) \to \mathcal{G}$ is locally $H$-smooth on $J \subset \mathbb{R}$ if, for every compact subset $K \subset J$,

$$\int_{\mathbb{R}} \|T\, \mathrm{e}^{\mathrm{i}xH} E(K) f\|_{\mathcal{G}}^2 \,\mathrm{d}x \le C_K \|f\|_{\mathcal{H}}^2. \tag{A.33}$$

This definition coincides with the one provided in [ABG96, p.274]. It is possible to define locally $H$-smooth operators by demanding (A.32) only for $a = \mathcal{N}$, and some of the subsequent results can be generalised if this weaker definition is used instead. However, we prefer the above definition because Theorem A.12 offers an equivalent characterisation of locally $H$-smooth operators in terms of resolvent estimates.

We state two identities that will be useful below. Define $\mathbb{R}_+ = (0, \infty)$. For every $\mu \in \mathbb{R}^n_+$,

$$R_a(\lambda + \mathrm{i}\mu) = \mathrm{i}^{|a|} \int_{\mathbb{R}_+^{|a|}} \mathrm{e}^{\mathrm{i}\lambda_a \cdot x_a}\, \mathrm{e}^{-\mathrm{i}x_a \cdot H_a - \mu_a \cdot x_a}\, \mathrm{d}x_a. \tag{A.34}$$

The second identity is a consequence of the resolvent identity (A.34) (see [ABG96, (7.1.2), (7.1.11)]):

$$\int_{\mathbb{R}^{|a|}} \|T\, \mathrm{e}^{\mathrm{i}x_a \cdot H_a} f\|^2\, \mathrm{d}x_a = \frac{2^{|a|}}{\pi^{|a|}} \sup_{\mu \in (0,1)^n} \int_{\mathbb{R}^{|a|}} \|T \Im R_a(\lambda + \mathrm{i}\mu) f\|^2\, \mathrm{d}\lambda_a. \tag{A.35}$$

**Proposition A.10.** *If $T$ is locally $H$-smooth, the optimal constant for the bound (A.32) is*

$$C^0_{K(a)} = 2^{|a|} \sup_{\lambda \in \mathbb{R}^n, \mu \in (0,1)^n} \mu_{a_1} \ldots \mu_{a_k} \|T E(K(a)) R_a(\lambda + \mathrm{i}\mu)\|^2. \tag{A.36}$$

*Moreover, a continuous operator $T : D(H) \to \mathcal{G}$ is locally $H$-smooth on $J$ if $C^0_{K(a)} < \infty$ for every $\emptyset \neq a \subset \mathcal{N}$ and every compact subset $K \subset J$.*

*Proof.* The proof is similar to Step (i) of the proof of [ABG96, Proposition 7.1.1]. For $\mu \in (0, 1)^n$, we obtain the following estimate from the resolvent identity (A.34) and the Cauchy–Schwarz inequality:

$$\begin{aligned} \|T E(K(a)) R_a(\lambda + \mathrm{i}\mu) f\|^2 &\leq \left( \int_{\mathbb{R}_+^{|a|}} \mathrm{e}^{-\mu_a \cdot x_a} \|T\, \mathrm{e}^{-\mathrm{i}x_a \cdot H_a} E(K(a)) f\|\, \mathrm{d}x_a \right)^2 \\ &\leq \frac{1}{2^{|a|}\mu_{a_1} \ldots \mu_{a_k}} \int_{\mathbb{R}^{|a|}} \|T\, \mathrm{e}^{-\mathrm{i}x_a \cdot H_a} E(K(a)) f\|^2\, \mathrm{d}x_a. \end{aligned} \tag{A.37}$$

Applying the assumption that $T$ is locally $H$-smooth on $J$, yields $C^0_{K(a)} \leq C_{K(a)}$ (i.e. $C^0_{K(a)}$ is the optimal constant for (A.32)).

Next, we establish that $T$ is locally $H$-smooth on $J$ if $C^0_{K(a)} < \infty$ for every $\emptyset \neq a \subset \mathcal{N}$ and every compact subset $K \subset J$. From (A.35) and (A.31), it follows that

$$\begin{aligned} \int_{\mathbb{R}^{|a|}} \|T\, \mathrm{e}^{\mathrm{i}x_a \cdot H_a} E(K(a)) f\|^2\, \mathrm{d}x_a &= \frac{2^{|a|}}{\pi^{|a|}} \sup_{\mu \in (0,1)^n} \int_{\mathbb{R}^{|a|}} \|T \Im R_a(\lambda + \mathrm{i}\mu) E(K(a)) f\|^2\, \mathrm{d}\lambda_a \\ &\leq \frac{2^{|a|}}{\pi^{|a|}} \sup_{\lambda \in \mathbb{R}^n, \mu \in (0,1)^n} \mu^2_{a_1} \ldots \mu^2_{a_k} \|T E(K(a)) R_a(\lambda + \mathrm{i}\mu)\|^2 \int_{\mathbb{R}^{|a|}} \|R_a(\lambda - \mathrm{i}\mu) f\|^2\, \mathrm{d}\lambda_a. \end{aligned} \tag{A.38}$$

To conclude, we utilise the following identity:

$$\mu_{a_1} \ldots \mu_{a_k} \int_{\mathbb{R}^{|a|}} \|R_a(\lambda - \mathrm{i}\mu) f\|^2\, \mathrm{d}\lambda_a = \pi^{|a|} \|f\|^2. \tag{A.39}$$

Thus, $T$ is locally $H$-smooth on $J$ if $C^0_{K(a)} < \infty$. □

Proposition A.10 demonstrates the close connection between the notion of locally $H$-smooth operators and the boundary values of the resolvents $R_a(\lambda + \mathrm{i}\mu)$ as $\mu \downarrow 0$. This connection is further clarified in Kato's theorem, which we present and prove in a generalised form below. To prepare the theorem, we repeat the above proposition in the case $n = 1$.

**Proposition A.11.** *If $n = 1$, a continuous operator $T : D(H) \to \mathcal{G}$ is locally $H$-smooth if and only if $C_K^0 < \infty$ for every compact subset $K \subset J$. Moreover,*

$$C_K^0 \leq 8 \sup_{\lambda \in K, \mu \in (0,1)} \|T \Im R(\lambda + \mathrm{i}\mu) T^*\|. \tag{A.40}$$

The proof of (A.40) is Step (ii) in the proof of [ABG96, Proposition 7.1.1].

**Theorem A.12.** *A continuous operator $T : D(H) \to \mathcal{H}$ is locally $H$-smooth on $J$ if and only if, for every $\emptyset \neq a \subset \mathcal{N}$ and every compact subset $K \subset J$,*

$$\sup_{\lambda \in K, \mu \in (0,1)^n} \|T \Im R_a(\lambda + \mathrm{i}\mu) T^*\| < \infty. \tag{A.41}$$

*Proof.* The strategy of the proof is to demonstrate that (A.41) is equivalent to $C^0_{K(a)} < \infty$. If this is proved, the theorem follows from Proposition A.10. We observe that, for both directions of the proof, it suffices to consider compact hyperrectangles in $J$ instead of arbitrary compact subsets $K \subset J$. This follows from the fact that every compact set in $J$ can be covered by finitely many compact hyperrectangles in $J$.

Let $T$ be locally $H$-smooth on $J$. We show (A.41) for $a = \mathcal{N}$. The general case is similar. Let $K = I_1 \times \cdots \times I_n \subset J$ be a compact hyperrectangle, where $I_1, \ldots, I_n \subset \mathbb{R}$ are compact intervals. Let $\tilde{K} = \tilde{I}_1 \times \cdots \times \tilde{I}_n \subset J$ be another compact hyperrectangle such that, for every $j \in \mathcal{N}$, $I_j \subset \tilde{I}_j$ and $\operatorname{dist}(I_j, \mathbb{R} \backslash \tilde{I}_j) > 0$. Let $\lambda \in K$. From (A.29) applied to $\tilde{K}$, it follows that

$$E(\mathbb{R}^n \backslash \tilde{K}) \leq E(\tilde{K}(\emptyset)) + \sum_{\emptyset \neq a \subsetneq \mathcal{N}} E(\tilde{K}(a) \backslash \tilde{K}), \tag{A.42}$$

and, accordingly,

$$\begin{aligned} \|TE(\mathbb{R}^n \backslash \tilde{K}) R(\lambda + \mathrm{i}\mu)\|^2 &\leq \|TE(\tilde{K}(\emptyset)) R(\lambda + \mathrm{i}\mu)\|^2 + \sum_{\emptyset \neq a \subsetneq \mathcal{N}} \|TE(\tilde{K}(a) \backslash \tilde{K}) R(\lambda + \mathrm{i}\mu)\|^2 \\ &\leq \|TE(\tilde{K}(\emptyset)) R(\lambda + \mathrm{i}\mu)\|^2 + \sum_{\emptyset \neq a \subsetneq \mathcal{N}} \|TE(\tilde{K}(a) \backslash \tilde{K}) R_a(\lambda + \mathrm{i}\mu)\|^2 \|E(\tilde{K}(a) \backslash \tilde{K}) R_{\mathcal{N} \backslash a}(\lambda + \mathrm{i}\mu)\|^2; \end{aligned} \tag{A.43}$$

hence, by Proposition A.10,

$$\begin{aligned} \sup_{\lambda \in K, \mu \in (0,1)^n} \mu_1 \ldots \mu_n \|TE(\mathbb{R}^n \backslash \tilde{K}) R(\lambda + \mathrm{i}\mu)\|^2 &\leq \sup_{\lambda \in K} \|TE(\tilde{K}(\emptyset)) R(\lambda)\|^2 \\ &+ \sum_{\emptyset \neq a \subsetneq \mathcal{N}} \frac{1}{2^{|a|}} C^0_{\tilde{K}(a)} \sup_{\lambda \in K} \|E(\tilde{K}(a) \backslash \tilde{K}) R_{\mathcal{N} \backslash a}(\lambda)\|^2 < \infty, \end{aligned} \tag{A.44}$$

where we used that $C^0_{\tilde{K}(a)} < \infty$ due to the assumption that $T$ is locally $H$-smooth on $J$. Thus, we obtain (A.41):

$$\sup_{\lambda\in K,\mu\in(0,1)^n} \|T\Im R(\lambda+\mathrm{i}\mu)T^*\| \le \sup_{\lambda\in K,\mu\in(0,1)^n} \mu_1\dots\mu_n\|TE(\tilde{K})R(\lambda+\mathrm{i}\mu)\|^2 + \sup_{\lambda\in K,\mu\in(0,1)^n} \mu_1\dots\mu_n\|TE(\mathbb{R}^n\backslash\tilde{K})R(\lambda+\mathrm{i}\mu)\|^2 < \infty. \tag{A.45}$$

Next, we prove that $T$ is locally $H$-smooth on $J$ if (A.41) holds for every $\emptyset \ne a \subset \mathcal{N}$ and every compact subset $K \subset J$. We demonstrate that $C^0_K < \infty$ for every compact hyperrectangle $K = I_1 \times \cdots \times I_n$. The case $C^0_{K(a)} < \infty$ is similar. We consider the contributions in the supremum defining $C^0_K$ from the points $\lambda \notin K$. If $\lambda \notin K$, an $a \subsetneq \mathcal{N}$ exists such that $\lambda \in K(a)$. For every $j \in \mathcal{N}\backslash a$, we choose $\kappa_j \in I_j$ such that $\mathrm{dist}(\lambda_j, I_j) = |\lambda_j - \kappa_j|$, and we define $\tilde{\lambda} \in K$ to be the element that satisfies $\tilde{\lambda}_j = \lambda_j$ if $j \in a$ and $\tilde{\lambda}_j = \kappa_j$ if $j \in \mathcal{N}\backslash a$. We have

$$R(\lambda+\mathrm{i}\mu) = \prod_{j\in\mathcal{N}\backslash a} (1+(\lambda_j-\kappa_j)(H_j-\lambda_j-\mathrm{i}\mu_j)^{-1})R(\tilde{\lambda}+\mathrm{i}\mu); \tag{A.46}$$

thus,

$$\begin{aligned}\|TE(K)R(\lambda+\mathrm{i}\mu)\| &\le \|TE(K)R(\tilde{\lambda}+\mathrm{i}\mu)\| \\ &\quad\times \|E(K)\prod_{j\in\mathcal{N}\backslash a}(1+(\lambda_j-\kappa_j)(H_j-\lambda_j-\mathrm{i}\mu_j)^{-1})\| \\ &\le 2^{n-|a|}\|TE(K)R(\tilde{\lambda}+\mathrm{i}\mu)\|.\end{aligned} \tag{A.47}$$

It follows that

$$C^0_K \le 8^n \sup_{\lambda\in K,\mu\in(0,1)^n} \|T\Im R(\lambda+\mathrm{i}\mu)T^*\| < \infty. \tag{A.48}$$

This estimate can be compared with (A.40). However, we have obtained the factor $8^n$ only in the case that $K \subset J$ is a compact hyperrectangle, and (A.48) might not generalise to arbitrary compact sets $K$. □

In the remainder of this section, we discuss an example of a locally smooth operator, which is important for the main part of the paper. Let $\mathcal{H}_{\mathrm{ac}}(H) \subset \mathcal{H}$ be the jointly absolutely continuous subspace of $H$.

**Definition A.13.** For $f \in \mathcal{H}_{\mathrm{ac}}(H)$, let $\rho_f$ be the Radon–Nikodym derivative (w.r.t. the Lebesgue measure on $\mathbb{R}^n$) of the spectral measure $\langle f, E(\cdot)f\rangle$. For $a \subset \mathcal{N}$, we denote by $\mathcal{M}(H)_a$ the set of all vectors $f \in \mathcal{H}_{\mathrm{ac}}(H)$ for which

$$|||f|||_a = \sup_{x_a\in\mathbb{R}^{|a|}} \left(\int_{\mathbb{R}^{n-|a|}} \rho_f(x_a \oplus x^a)\,\mathrm{d}x^a\right)^{\frac{1}{2}} < \infty. \tag{A.49}$$

We abbreviate $\mathcal{M}(H)_{\mathcal{N}}$ by $\mathcal{M}(H)$ and $|||f|||_{\mathcal{N}}$ by $|||f|||$.

Observe that $\mathcal{M}(H)_\emptyset = \mathcal{H}_{\mathrm{ac}}(H)$ and that $\mathcal{M}(H)$ is the set of all vectors $f \in \mathcal{H}_{\mathrm{ac}}(H)$ for which the Radon–Nikodym derivative $\rho_f$ is a bounded function. In general, for $\emptyset \neq a \subsetneq \mathcal{N}$, $\mathcal{M}(H)$ is not a subset of $\mathcal{M}(H)_a$. If, for every $j \in \mathcal{N} \backslash a$, an $h_j \in \mathbb{R}_+$ exists such that $f = E(\{|x_j| \leq h_j\})f$, then

$$\||f\||_a^2 = \sup_{x_a \in \mathbb{R}^{|a|}} \int_{\mathbb{R}^{n-|a|}} \rho_f(x_a \oplus x^a)\, \mathrm{d}x^a \leq \prod_{j \in \mathcal{N}\backslash a} 2h_j \||f\||^2. \tag{A.50}$$

This follows from the fact that in this case $\rho_f(x) = \prod_{j \in \mathcal{N}\backslash a} \chi(|x_j| \leq h_j)\rho_f(x)$, where $\chi$ denotes the characteristic function.

**Lemma A.14.** *For every $a \subset \mathcal{N}$, $\mathcal{M}(H)_a$ is dense in $\mathcal{H}_{\mathrm{ac}}(H)$ (w.r.t. the norm in $\mathcal{H}$).*

*Proof.* It suffices to prove that $\mathcal{M}(H)$ is dense in $\mathcal{H}_{\mathrm{ac}}(H)$ because $f \in \mathcal{M}(H)$ is approximated by $f_R = E(\{|x| \leq R\})f$ as $R \to \infty$ and $f_R \in \mathcal{M}(H)_a$ due to (A.50). We may assume that a cyclic vector $f \in \mathcal{H}_{\mathrm{ac}}(H)$ for $H$ exists. Otherwise, we decompose $\mathcal{H}_{\mathrm{ac}}(H)$ into subspaces such that each subspace has a cyclic vector. By spectral theory, the Hilbert space $\mathcal{H}_{\mathrm{ac}}(H)$ is unitarily equivalent to $L^2(\sigma(H), \rho_f\, \mathrm{d}x)$ in such a way that $H$ is unitarily equivalent to the multiplication operator by $x$. For $g \in L^2(\sigma(H), \rho_f\, \mathrm{d}x)$, we have $\rho_g = |g|^2\rho_f$. Because every $g \in L^2(\sigma(H), \rho_f\, \mathrm{d}x)$ is approximated by $g\chi(|g|^2\rho_f \leq R)$ as $R \to \infty$, the space $\mathcal{M}(H)$ is dense in $\mathcal{H}_{\mathrm{ac}}(H)$. □

The following proposition demonstrates that the operator $T_f : \mathcal{H} \to \mathbb{C}, g \mapsto \langle f, g\rangle$, is (locally) $H$-smooth on $\mathbb{R}^n$ if $f \in \bigcap_{a \subset \mathcal{N}} \mathcal{M}(H)_a$. The proposition is a generalisation of [RS79, Lemma XI.3.1].

**Proposition A.15.** *For $\emptyset \neq a \subset \mathcal{N}$, $f \in \mathcal{M}(H)_a$ and $g \in \mathcal{H}$,*

$$\int_{\mathbb{R}^{|a|}} |\langle f, \mathrm{e}^{\mathrm{i}x_a \cdot H_a} g\rangle|^2\, \mathrm{d}x_a \leq (2\pi)^{|a|} \||f\||_a^2\, \|g\|^2\,. \tag{A.51}$$

*If, for every $j \in \mathcal{N}\backslash a$, an $h_j \in \mathbb{R}_+$ exists such that $f = E(\{|x_j| \leq h_j\})f$, then*

$$\int_{\mathbb{R}^{|a|}} |\langle f, \mathrm{e}^{\mathrm{i}x_a \cdot H_a} g\rangle|^2\, \mathrm{d}x_a \leq (2\pi)^{|a|} \prod_{j \in \mathcal{N}\backslash a} 2h_j \||f\||^2\, \|g\|^2\,. \tag{A.52}$$

*Proof.* We demonstrate that $T_f$ is locally $H$-smooth on $\mathbb{R}^n$. Let $K \subset \mathbb{R}^n$ be a compact subset. By Proposition A.10, it suffices to show that, for every $\emptyset \neq a \subset \mathcal{N}$,

$$\begin{aligned} &\sup_{\lambda \in \mathbb{R}^n, \mu \in (0,1)^n} \mu_1 \ldots \mu_n \|T_f E(K(a)) R_a(\lambda + \mathrm{i}\mu)\|^2 \\ &\leq \sup_{\lambda \in \mathbb{R}^n, \mu \in (0,1)^n} \|T_f \Im R_a(\lambda + \mathrm{i}\mu) T_f^*\| \leq \pi^{|a|} \||f\||_a^2. \end{aligned} \tag{A.53}$$

The first inequality is obvious. To obtain the second inequality, we observe the following:

$$\begin{aligned} \sup_{\mu \in (0,1)^n} \|T_f \Im R_a(\lambda + \mathrm{i}\mu) T_f^*\| &= \sup_{\mu \in (0,1)^n} |\langle f, \Im R_a(\lambda + \mathrm{i}\mu) f\rangle| \\ &= \sup_{\mu \in (0,1)^n} \int_{\mathbb{R}^n} \prod_{j \in a} \Im \frac{1}{\sigma_j - \lambda_j - \mathrm{i}\mu_j} \rho_f(\sigma)\, \mathrm{d}\sigma \\ &\leq \pi^{|a|} \int_{\mathbb{R}^{n-|a|}} \rho_f(\lambda_a \oplus \sigma^a)\, \mathrm{d}\sigma^a. \end{aligned} \tag{A.54}$$

We conclude that, for every compact subset $K \subset \mathbb{R}^n$,

$$\int_{\mathbb{R}^{|a|}} |T_f \, \mathrm{e}^{\mathrm{i}x_a \cdot H_a} E(K(a)) g|^2 \, \mathrm{d}x_a \leq (2\pi)^{|a|} |||f|||_a^2 \|g\|^2. \tag{A.55}$$

The first statement of the lemma follows from Fatou's lemma because the bound on the r.h.s. is independent of $K$. The second statement follows from (A.50). □

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# Mourre theory and spectral analysis of energy-momentum operators in relativistic quantum field theory

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**Abstract**
A central task of theoretical physics is to analyse spectral properties of quantum mechanical observables. In this endeavour, Mourre's conjugate operator method emerged as an effective tool in the spectral theory of Schrödinger operators. This paper introduces a novel class of examples from relativistic quantum field theory that are amenable to Mourre's method. By assuming Lorentz covariance and the spectrum condition, we derive a limiting absorption principle for the energy-momentum operators and provide new proofs of the absolute continuity of the energy-momentum spectra. Moreover, under the assumption of dilation covariance, we show that the spectrum of the relativistic mass operator is purely absolutely continuous in $(0, \infty)$.

**Keywords** Spectral theory · Mourre's conjugate operator method · Absence of singular continuous spectrum · Representations of the Poincaré group · Dilation-covariant representations

**Mathematics Subject Classification** 81Q10

## 1 Introduction

Understanding spectral properties of quantum mechanical observables is of fundamental importance in theoretical physics. By analysing the spectrum, insights into the system's stability and long-time evolution can be obtained. Specifically, the spectrum of a self-adjoint operator decomposes into a pure point, absolutely continuous and singular continuous part, where the pure-point part corresponds to bound states and the absolutely continuous part to scattering states. Typically, the singular continuous spectrum, which evades a simple physical interpretation, is empty. One of the main

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objectives of this paper is to prove the absence of singular continuous spectrum for relativistic energy-momentum operators.

Various mathematical techniques have been developed to investigate the spectrum of a self-adjoint operator. One powerful method in this endeavour is Mourre's conjugate operator method, which is based on a strictly positive commutator estimate. The idea of analysing spectral properties of self-adjoint operators through commutator identities and estimates can be traced back to the pioneering works of Putnam [19], Kato [12], and Lavine [14]. However, it was Mourre [17] who advanced this approach by introducing local commutator estimates.

Mourre's method led to substantial progress in the spectral and scattering theory of Schrödinger operators. In his seminal paper, Mourre [17] demonstrated the absence of singular continuous spectrum for 2- and 3-body Schrödinger operators. The applicability of the method to $N$-body systems was extended by Perry, Sigal, and Simon [18] and by Froese and Herbst [10]. Mourre's method also played a decisive role in establishing asymptotic completeness of $N$-body Schrödinger operators (see [5] for a textbook exposition). On an abstract level, the mathematical theory underlying the conjugate operator method was notably improved by Amrein, Boutet de Monvel, and Georgescu [1].

While originally developed for non-relativistic quantum mechanics, Mourre's method has also been extended to other areas. Worthy of note, it has been applied in non-relativistic quantum electrodynamics (QED). Among many works, we mention here that a Mourre estimate was proved by Dereziński and Gérard [6] for confined Pauli–Fierz Hamiltonians, by Fröhlich, Griesemer, and Schlein [8] for a Hamiltonian describing Compton scattering, by Fröhlich, Griesemer, and Sigal [9] for the standard model of non-relativistic QED, by Chen, Faupin, Fröhlich, and Sigal [4] for dressed electrons, and by Møller and Rasmussen [16] for the translation-invariant massive Nelson model.

The application of Mourre's conjugate operator method in relativistic quantum field theory is more difficult due to the abstract nature of the Hamiltonian. Typically, the renormalised Hamiltonian is derived through a limiting procedure or is defined axiomatically as the generator of time translations. Although a Mourre estimate was established for the spatially cut-off $P(\varphi)_2$ Hamiltonian by Dereziński and Gérard [7] and Gérard and Panati [11], it remained an open problem whether a Mourre estimate can be proved for the Hamiltonian in the infinite-volume limit. Recently, the author [13] applied Mourre's method within the axiomatic framework of Haag–Kastler quantum field theory to prove the existence of asymptotic observables. In this context, Mourre's method was implemented through scattering theory by comparing the abstract dynamics generated by the Hamiltonian to a more concrete free dynamics.

In this paper, we apply Mourre's conjugate operator method in the relativistic setting directly to the energy-momentum operators, which are defined axiomatically as the generators of spacetime translations. By assuming Lorentz covariance and the spectrum condition, we prove Mourre estimates for the energy-momentum operators $P = (P_0, \mathbf{P})$, using the generators of Lorentz boosts $\mathbf{K}$ to construct conjugate operators. Our Mourre estimates yield the following limiting absorption principle:

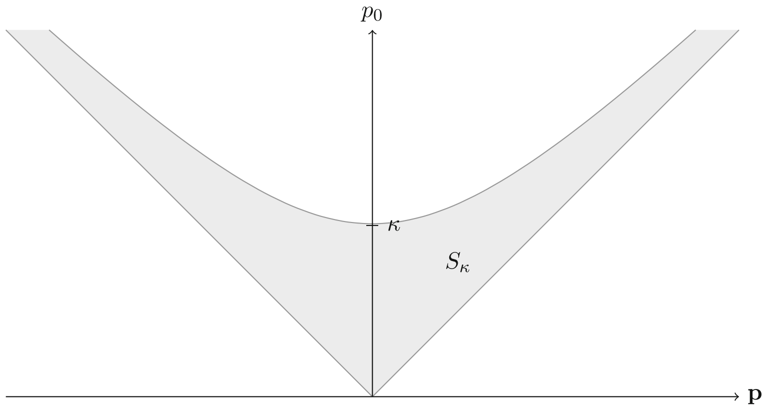


**Fig. 1** The set $S_\kappa \subset V_+$ is invariant under Lorentz boosts

**Theorem 1.1** *Let $U : \mathcal{P} \to \mathfrak{B}(\mathcal{H})$ be a strongly continuous unitary representation of the Poincaré group $\mathcal{P} = \mathcal{L} \ltimes \mathbb{R}^d$ on a Hilbert space $\mathcal{H}$, $P = (P_0, \mathbf{P})$ the generators of the translation subgroup $U|_{\mathbb{R}^d}$, $E$ the joint spectral measure of $P$, and $\mathbf{K}$ the generators of Lorentz boosts. Assume that the energy-momentum operators $P$ obey the spectrum condition (see Assumption 4.1). For $\kappa > 0$, define the following Lorentz-invariant sets (see Fig. 1):*

$$S_\kappa = \{\Lambda_1(t_1)\dots\Lambda_s(t_s)(p_0, \mathbf{0}) \mid t_1, \dots, t_s \in \mathbb{R},\, p_0 \in [0, \kappa]\}^-, \tag{1.1}$$

*where $\Lambda_j(t_j)$ are the Lorentz boosts in the spatial direction $j \in \{1, \dots, s = d-1\}$ and $\{\dots\}^-$ denotes the closure in $\mathbb{R}^d$. For all compact subsets $I_0 \subset (\kappa, \infty)$ and $I_j \subset \mathbb{R}\backslash\{0\}$, for every $\nu > 1/2$,*

$$\sup_{\lambda \in I_0, \mu > 0} \|E(S_\kappa)\langle K_j\rangle^{-\nu}(P_0 - \lambda \mp \mathrm{i}\mu)^{-1}\langle K_j\rangle^{-\nu}E(S_\kappa)\| < \infty, \tag{1.2}$$

$$\sup_{\lambda \in I_j, \mu > 0} \|E(S_\kappa)\langle K_j\rangle^{-\nu}(P_j - \lambda \mp \mathrm{i}\mu)^{-1}\langle K_j\rangle^{-\nu}E(S_\kappa)\| < \infty. \tag{1.3}$$

We illustrate our line of argument for the momentum operators $\mathbf{P}$. Formally, the momentum operator $P_j$ and the generator of Lorentz boosts $K_j$ in the spatial direction $j$ satisfy the following commutation relation:

$$[P_j, \mathrm{i}K_j] = P_0. \tag{1.4}$$

By virtue of the spectrum condition, the energy $P_0$ is strictly positive on spectral subspaces of $P_j$ that are separated from 0. To make sense of the commutator and to apply the results of Mourre theory, it is necessary to demonstrate that $P_j$ lies in the regularity classes $C^k(K_j)$ (see Definition 3.1). However, this is not generally the case: because the inclusion $D(P_0) \subset D(P_j)$ of domains, which follows from the spectrum condition, can be proper, the form defined by the commutator $[P_j, \mathrm{i}K_j]$ on $D(P_j) \cap D(K_j)$

does not necessarily have a continuous extension to $D(P_j)$. To address this problem, we introduce the Lorentz-invariant sets $S_\kappa$, which cover the closed forward light cone $V_+$. Within the sets $S_\kappa$, the energy stays bounded relative to the momentum, ensuring that the operators $P_{0,\kappa} = E(S_\kappa)P_0$ and $P_{j,\kappa} = E(S_\kappa)P_j$ are bounded relative to each other. Moreover, the commutation relation (1.4) is preserved with $P_0$ and $P_j$ replaced by $P_{0,\kappa}$ and $P_{j,\kappa}$, respectively, because $E(S_\kappa)$ commutes with $K_j$. Additionally, we have $D(P_{0,\kappa}) = D(P_{j,\kappa})$ and $P_{j,\kappa} \in C^\infty(K_j)$ (see Proposition 4.2). The limiting absorption principle (1.3) can then be derived from the standard results of Mourre theory.

As a corollary of Theorem 1.1, we provide a new proof of the well-known theorem asserting that the energy-momentum spectra are purely absolutely continuous if translation-invariant vectors are removed [15]. More precisely, we prove the following proposition, where $\mathcal{H}_{\mathrm{ac}}(A)$ denotes the absolutely continuous spectral subspace of a commuting family of self-adjoint operators $A$ and $x \cdot y = x_0 y_0 - \mathbf{x} \cdot \mathbf{y}$, $x, y \in \mathbb{R}^d$, denotes the Minkowski scalar product.

**Proposition 1.2** *Let $U$ be as in Theorem 1.1 and $e \in \mathbb{R}^d \setminus \{0\}$ a spacetime vector such that $e \cdot e \neq 0$ if $d = 2$. Assume that the energy-momentum operators $P = (P_0, \mathbf{P})$ obey the spectrum condition. If $Q_0$ denotes the projection onto the subspace of translation-invariant vectors, then*

$$\mathcal{H} = Q_0\mathcal{H} \oplus \mathcal{H}_{\mathrm{ac}}(e \cdot P) = Q_0\mathcal{H} \oplus \mathcal{H}_{\mathrm{ac}}(\mathbf{P}). \tag{1.5}$$

The first identity in (1.5) was originally proved by Maison [15, Satz 2] for $d = 4$ through the application of Wigner's theorem. The second identity (absolute continuity of the joint momentum spectrum) has been proved under differing assumptions in the realm of quantum field theory. Buchholz and Fredenhagen [3, Proposition 2.2] utilised the locality principle to establish this identity, and Bachmann, Dybalski, and Naaijkens [2, Lemma 4.16] presented a simplified proof under the additional assumption of the existence of a vacuum vector.

An immediate consequence of Proposition 1.2 and the Riemann–Lebesgue lemma is the following clustering property. If $f, g \in \mathcal{H}$ are two arbitrary vectors, then

$$\lim_{t \to \pm\infty} \langle f, U(te)g\rangle = \langle f, Q_0 g\rangle. \tag{1.6}$$

In quantum field theory, the clustering property is well-known for space-like directions (i.e. $e \cdot e < 0$) due to the locality principle. It is somewhat unexpected that this property is also valid for light-like (i.e. $e \cdot e = 0$) and time-like directions (i.e. $e \cdot e > 0$) under the assumption of Lorentz covariance.

In two spacetime dimensions (i.e. $d = 2$), the clustering property may not hold for light-like directions. This stems from the potential presence of massless excitations, as elaborated upon in the remark subsequent to Proposition 4.9.

**Proposition 1.3** *Let $U$ be as in Theorem 1.1 and let $E_{P_0 \pm P_1}$ denote the spectral measure of $P_0 \pm P_1$. If $d = 2$, then*

$$\mathcal{H} = E_{P_0 \pm P_1}(\{0\})\mathcal{H} \oplus \mathcal{H}_{\mathrm{ac}}(P_0 \pm P_1). \tag{1.7}$$

Another operator of interest in quantum field theory is the mass operator $M = \sqrt{P_0^2 - |\mathbf{P}|^2}$. We prove the absence of singular continuous spectrum for the mass operator if $U$ satisfies dilation covariance. A unitary representation $U : \mathbb{R}^d \to \mathfrak{B}(\mathcal{H})$ of the translation group $\mathbb{R}^d$ is dilation-covariant if a self-adjoint operator $D$ exists such that, for all $t \in \mathbb{R}$, $x \in \mathbb{R}^d$,

$$e^{itD} U(x) e^{-itD} = U(e^{-t} x), \tag{1.8}$$

that is, the adjoint action of $e^{itD}$ scales the translation $x$ by the factor $e^{-t}$.

**Proposition 1.4** *Let $U : \mathbb{R}^d \to \mathfrak{B}(\mathcal{H})$ be a strongly continuous unitary representation of the translation group $\mathbb{R}^d$. Assume that the generators $P = (P_0, \mathbf{P})$ of $U$ obey the spectrum condition. If $U$ is dilation-covariant, then, for every compact subset $I \subset (0, \infty)$ and every $\nu > 1/2$,*

$$\sup_{\lambda \in I, \mu > 0} \| \langle D \rangle^{-\nu} (M - \lambda \mp i\mu)^{-1} \langle D \rangle^{-\nu} \| < \infty. \tag{1.9}$$

*The spectrum of $M$ is purely absolutely continuous in $(0, \infty)$.*

Given that the spectrum of the mass operator $M \geq 0$ is contained in the interval $[0, \infty)$, the proposition implies that the pure point spectrum of $M$ must be empty or $\{0\}$, while the singular continuous spectrum is empty. Under the assumptions of Proposition 1.2, the mass operator may have other eigenvalues than 0. The assumption of dilation covariance excludes additional mass shells within the energy-momentum spectrum.

The structure of the paper is as follows. In Sect. 2, we define locally smooth operators for a commuting family of self-adjoint operators, and we relate the existence of a locally smooth operator to the regularity of the joint spectrum. In Sect. 3, we review essential results of Mourre's conjugate operator method. We explain how to obtain locally smooth operators based on a strictly positive commutator estimate. In Sect. 4, we present the proofs of our results.

## 2 Locally smooth operators

Let $U : \mathbb{R}^n \to \mathfrak{B}(\mathcal{H})$ be a strongly continuous unitary representation of $\mathbb{R}^n$ on a Hilbert space $\mathcal{H}$ with self-adjoint generators $H = (H_1, \ldots, H_n)$. We denote by $D(H)$ the intersection of the domains of $H_1, \ldots, H_n$, by $E$ the spectral measure of the family $H$, and by $\sigma(H) \subset \mathbb{R}^n$ its joint spectrum. We consider $D(H)$ as a Banach space equipped with the graph norm.

**Definition 2.1** A continuous operator $T : D(H) \to \mathcal{H}$ is **locally $H$-smooth** on an open set $J \subset \mathbb{R}^n$ if, for every compact subset $K \subset J$, a constant $C_K$ exists such that, for all $f \in \mathcal{H}$,

$$\int_{\mathbb{R}^n} \| T U(x) E(K) f \|^2 \, dx \leq C_K \| f \|^2. \tag{2.1}$$

In [13, Definition A.9], we defined locally $H$-smooth operators by demanding that all subfamilies of $H$ are locally $H$-smooth in the sense of the above definition. The advantage of this stronger definition is that it allows locally $H$-smooth operators to be equivalently characterised by resolvent estimates which resemble a limiting absorption principle (see [13, Theorem A.12] and Proposition 2.4 below for the case $n = 1$). In this paper, we prefer the weaker formulation of Definition 2.1 because it is sufficient for analysing the regularity of the joint spectrum $\sigma(H)$.

The following proposition is a straightforward generalisation of [21, Theorem XIII.23].

**Proposition 2.2** *If $T$ is locally $H$-smooth on $J$, then $E(J)\overline{\operatorname{ran}(T^*)} \subset E(J)\mathcal{H}_{\mathrm{ac}}(H)$. If, additionally, $\ker(T) = \{0\}$, then the joint spectrum of $H$ is purely absolutely continuous in $J$.*

***Proof*** Let $f \in E(J)\overline{\operatorname{ran}(T^*)}$. Due to the inner regularity of the spectral measure $E$, we can approximate $f$ by a sequence $\{f_j\}_{j\in\mathbb{N}}$ such that, for every $j \in \mathbb{N}$, $f_j \in E(K_j)\operatorname{ran}(T^*)$ for a compact subset $K_j \subset J$. If, for every $j \in \mathbb{N}$, $f_j \in E(J)\mathcal{H}_{\mathrm{ac}}(H)$, then also $f \in E(J)\mathcal{H}_{\mathrm{ac}}(H)$ because $E(J)\mathcal{H}_{\mathrm{ac}}(H)$ is a closed subspace of $\mathcal{H}$. We write $f_j = E(K_j)T^*g_j$ with $g_j \in D(T^*)$. Let $\mu_{f_j}$ be the spectral measure associated with $f_j$, and let $F_j$ be its inverse Fourier transform:

$$F_j(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}p\cdot x}\,\mathrm{d}\mu_{f_j}(p) = \frac{1}{(2\pi)^n}\langle E(K_j)T^*g_j, U(x)f_j\rangle. \tag{2.2}$$

The function $F_j$ is square-integrable because $|F_j(x)| \le (2\pi)^{-n}\|g_j\|\,\|TU(x)E(K_j)f_j\|$ and $T$ is locally $H$-smooth on $J$ by assumption. Thus, $\mathrm{d}\mu_{f_j}(p) = \hat{F}_j(p)\,\mathrm{d}p$ is an absolutely continuous measure.

To prove the second statement, we apply the decomposition $\mathcal{H} = \ker(T) \oplus \overline{\operatorname{ran}(T^*)}$. If $\ker(T) = \{0\}$, then $E(J)\mathcal{H} = E(J)\overline{\operatorname{ran}(T^*)} \subset E(J)\mathcal{H}_{\mathrm{ac}}(H)$. We conclude $E(J)\mathcal{H} = E(J)\mathcal{H}_{\mathrm{ac}}(H)$ because $\mathcal{H}_{\mathrm{ac}}(H) \subset \mathcal{H}$, that is, the joint spectrum of $H$ is purely absolutely continuous in $J$. □

It is possible to determine the constant $C_K$ for the bound (2.1), which is useful to verify the $H$-smoothness of an operator $T$. The following proposition is proved as in [13, Proposition A.10].

**Proposition 2.3** *A continuous operator $T : D(H) \to \mathcal{H}$ is locally $H$-smooth on $J$ if and only if, for all compact subsets $K \subset J$,*

$$C_K^0 = 2^n \sup_{\lambda\in\mathbb{R}^n,\mu\in(0,1)^n} \mu_1\ldots\mu_n\|TE(K)R(\lambda+\mathrm{i}\mu)\|^2 < \infty, \tag{2.3}$$

*where*

$$R(\lambda+\mathrm{i}\mu) = \prod_{j=1}^{n}(H_j - \lambda_j - \mathrm{i}\mu_j)^{-1}. \tag{2.4}$$

If $n = 1$, we can formulate a necessary and sufficient condition for $C^0_K < \infty$ in terms of a limiting absorption principle (see Theorem 3.5).

**Proposition 2.4** *Let $n = 1$. For every compact subset $K \subset \mathbb{R}$, $C^0_K < \infty$ if and only if*

$$\sup_{\lambda \in K, \mu \in (0,1)} \| T \Im R(\lambda + \mathrm{i}\mu) T^* \| < \infty. \tag{2.5}$$

The proof of Proposition 2.4 is contained in the proof of [1, Proposition 7.1.1]; see [13, Theorem A.12] for a generalisation to $n \geq 2$. In the next section, we apply Propositions 2.3 and 2.4 to construct locally smooth operators (see Corollary 3.6).

## 3 Mourre's conjugate operator method

Mourre's conjugate operator method is a mathematical tool to analyse the spectrum of a self-adjoint operator based on a positive commutator estimate. In this section, we outline the framework and state the most important results of the method, in particular the limiting absorption principle. At the end of this section, we apply Mourre's method to construct locally smooth operators.

We introduce the following regularity classes, which are relevant for defining commutators.

**Definition 3.1** Let $A$ be a self-adjoint operator on $\mathcal{H}$ and let $k \in \mathbb{N} \cup \{\infty\}$. We denote by $C^k(A)$ the space of all self-adjoint operators $H$ such that $t \mapsto \mathrm{e}^{\mathrm{i}tA}(H + \mathrm{i})^{-1} \mathrm{e}^{-\mathrm{i}tA}$ is a $C^k$-map in the strong operator topology. We denote by $C^k_{\mathrm{u}}(A)$ the subspace of operators $H$ for which the same map is $C^k$ in norm.

We collect some properties of the class $C^1(A)$ from [1, Theorem 6.2.10]. If $H \in C^1(A)$, then $D(A) \cap D(H)$ is dense in $D(H)$ (equipped with the graph topology). Moreover, the sesquilinear form defined by the commutator $HA - AH$ on $D(A) \cap D(H)$ extends to a continuous sesquilinear form on $D(H)$. We denote the extended sesquilinear form by $[H, A]$. It is possible to characterise the class $C^1(A)$ in terms of the commutator $[H, A]$.

**Proposition 3.2** [1, Theorem 6.2.10 (a)] *A self-adjoint operator $H$ belongs to the class $C^1(A)$ if and only if there exists $z \in \rho(H)$ such that $\{f \in D(A) \mid (H - z)^{-1} f \in D(A)$ and $(H - \overline{z})^{-1} f \in D(A)\}$ is a core for $A$ and, for all $f \in D(A) \cap D(H)$,*

$$|\langle Hf, Af \rangle - \langle Af, Hf \rangle| \leq c(\|Hf\|^2 + \|f\|^2). \tag{3.1}$$

Having formalised the commutator of two self-adjoint unbounded operators, we are able to define strictly positive commutator estimates.

**Definition 3.3** A self-adjoint operator $H$ obeys a **Mourre estimate** on an open and bounded subset $J \subset \mathbb{R}$ if a self-adjoint operator $A$ (**conjugate operator**) exists such that $H \in C^1(A)$ and, for an $a > 0$,

$$E(J)[H, \mathrm{i}A]E(J) \geq aE(J), \tag{3.2}$$

where $E$ is the spectral measure of $H$.

***Example*** Let $H$ be $A$-homogeneous, that is, for every $t, x \in \mathbb{R}$,

$$\mathrm{e}^{\mathrm{i}tA}\,\mathrm{e}^{\mathrm{i}xH}\,\mathrm{e}^{-\mathrm{i}tA} = \mathrm{e}^{\mathrm{i}\,\mathrm{e}^{-t}xH}. \tag{3.3}$$

It is not difficult to show that (3.3) implies $H \in C^\infty(A)$ and $[H, \mathrm{i}A] = H$. Moreover, for every open and bounded interval $J = (a, b) \subset (0, \infty)$, $0 < a < b < \infty$,

$$E(J)[H, \mathrm{i}A]E(J) = HE(J) \geq aE(J), \tag{3.4}$$

that is, $H$ obeys a Mourre estimate on $J$ with conjugate operator $A$. Similarly, for every open and bounded interval $I = (a, b) \subset (-\infty, 0)$, $-\infty < a < b < 0$,

$$E(I)[H, -\mathrm{i}A]E(I) = -HE(I) \geq -bE(I), \tag{3.5}$$

that is, $H$ obeys a Mourre estimate on $I$ with conjugate operator $-A$.

A consequence of the Mourre estimate on $J$ is that $H$ has no eigenvalues in $J$. This fact emerges as a direct corollary of the virial theorem [1, Proposition 7.2.10].

**Theorem 3.4** (Virial theorem) *Let $H \in C^1(A)$. If $f$ is an eigenvector of $H$, then $\langle f, [H, A]f\rangle = 0$.*

Next, we state the limiting absorption principle. To formulate the limiting absorption principle under optimal conditions, we introduce new regularity classes. Let $p \in [1, \infty]$ and $k \in \mathbb{N}$. For $0 < s < k$, we denote by $\mathcal{C}^{s,p}(A)$ the real interpolation space [1, (5.2.6)]

$$\mathcal{C}^{s,p}(A) = (C^k_{\mathrm{u}}(A), C^0_{\mathrm{u}}(A))_{\theta,p}, \ \theta = 1 - s/k. \tag{3.6}$$

Moreover, we write $H \in C^{k+0}(A)$ [1, p. 204] if $H \in C^k(A)$ and if the operator-valued function $S(t) = \mathrm{e}^{\mathrm{i}tA}\mathrm{ad}^k_A((H + \mathrm{i})^{-1})\,\mathrm{e}^{-\mathrm{i}tA}$ is Dini continuous in norm, that is,

$$\int_{|t|<1} \|S(t) - S(0)\| \frac{\mathrm{d}t}{t} < \infty. \tag{3.7}$$

We mention the following chain of inclusions [1, (5.2.14)]:

$$C^{k+1}(A) \subset C^{k+0}(A) \subset \mathcal{C}^{k,1}(A) \subset C^k_{\mathrm{u}}(A), \ k \in \mathbb{N}. \tag{3.8}$$

Note that $C^{k+1}(A) \subset C^{k+0}(A)$ as a consequence of the mean value theorem and the uniform boundedness principle.

We say that $H$ has a spectral gap if $\sigma(H) \neq \mathbb{R}$ and we write $\langle A\rangle$ for $\sqrt{A^2 + 1}$.

**Theorem 3.5** (Limiting absorption principle) *Let $H \in \mathcal{C}^{1,1}(A)$ if $H$ has a spectral gap and $H \in C^{1+0}(A)$ if $\sigma(H) = \mathbb{R}$. If $H$ obeys a Mourre estimate on $J$, then, for every compact subset $K \subset J$ and every $\nu > 1/2$,*

$$\sup_{\lambda \in K, \mu > 0} \|\langle A\rangle^{-\nu}(H - \lambda \mp \mathrm{i}\mu)^{-1}\langle A\rangle^{-\nu}\| < \infty. \tag{3.9}$$

***Remark*** The assumption $H \in \mathcal{C}^{1,1}(A)$ is optimal on the Besov scale in the sense that counterexamples of the limiting absorption principle for operators possessing less regularity exist [1, Section 7.B]. A proof of the limiting absorption principle under the assumption $H \in \mathcal{C}^{1,1}(A)$ has been accomplished in the case that $H$ has a spectral gap [1, Theorem 7.4.1]. Sahbani [22] proved the limiting absorption principle under the more restrictive assumption $H \in C^{1+0}(A)$. It is an open problem to prove the limiting absorption principle for $H \in \mathcal{C}^{1,1}(A)$ if $\sigma(H) = \mathbb{R}$.

According to Proposition 2.2, $H$ has no singular spectrum in $J$ if an injective locally $H$-smooth operator on $J$ exists. If each $H_i$ obeys a Mourre estimate with conjugate operator $A_i$ such that, for $i \neq j$, $A_i$ commutes strongly with $H_j$, then, for every $\nu > 1/2$, $\langle A_1\rangle^{-\nu} \dots \langle A_n\rangle^{-\nu}$ is an injective operator that is locally $H$-smooth. This is a corollary of the limiting absorption principle and the results from Sect. 2.

**Corollary 3.6** *Consider a commuting family of self-adjoint operators $H = (H_1, \dots, H_n)$ and a corresponding family $A = (A_1, \dots, A_n)$ of self-adjoint operators such that, for every $i \in \{1, \dots, n\}$, $H_i \in \mathcal{C}^{1,1}(A_i)$ if $H_i$ has a spectral gap and $H_i \in C^{1+0}(A_i)$ if $\sigma(H_i) = \mathbb{R}$. If $H_i$ obeys a Mourre estimate on $J_i$ with conjugate operator $A_i$ and, for $i \neq j$, $A_i$ commutes strongly with $H_j$, then the joint spectrum of $H$ is purely absolutely continuous in the product region $J = J_1 \times \dots \times J_n$.*

***Proof*** We demonstrate that the injective operator $T = \langle A_1\rangle^{-\nu} \dots \langle A_n\rangle^{-\nu}$ is locally $H$-smooth on $J$ for every $\nu > 1/2$. If this claim is proved, the statement follows from Proposition 2.2. We show that the constant $C_K^0$ from Proposition 2.3 is finite for every compact subset $K \subset J$. It suffices to consider compact hyperrectangles $K = K_1 \times \dots \times K_n$, where $K_i \subset J_i$ are compact intervals, because every compact subset $K \subset J$ can be covered by finitely many compact hyperrectangles in $J$. If $E_i$ denotes the spectral measure of $H_i$, then

$$E(K) = E_1(K_1) \dots E_n(K_n). \tag{3.10}$$

By assumption, $A_i$ commutes strongly with $H_j$ if $i \neq j$; hence,

$$\begin{aligned} C_K^0 &= 2^n \sup_{\lambda \in \mathbb{R}^n, \mu \in (0,1)^n} \mu_1 \dots \mu_n \|\langle A_1\rangle^{-\nu} \dots \langle A_n\rangle^{-\nu} E(K) R(\lambda + \mathrm{i}\mu)\|^2 \\ &\leq \prod_{i=1}^n 2 \sup_{\lambda_i \in \mathbb{R}, \mu_i \in (0,1)} \mu_i \|\langle A_i\rangle^{-\nu} E_i(K_i) R_i(\lambda_i + \mathrm{i}\mu_i)\|^2. \end{aligned} \tag{3.11}$$

Each factor on the r.h.s. is finite due to Proposition 2.4 and the limiting absorption principle (Theorem 3.5). □

## 4 Spectral analysis of relativistic energy-momentum operators

In this section, we establish Mourre estimates for the energy-momentum operators $P = (P_0, \mathbf{P})$ and verify the limiting absorption principle of Theorem 1.1 (see Theorem 4.3 and Theorem 4.6 below). Moreover, we analyse the spectra of the energy-momentum operators: in Proposition 4.4 for the momentum operators $\mathbf{P}$, in Proposition 4.7 for the energy operator $P_0$, and in Proposition 4.9 for the operators $e \cdot P$, where $e$ is a light-like vector. The relativistic mass operator $M$ and its spectrum is studied in Sect. 4.2.

### 4.1 Representations of the Poincaré group

The Poincaré group is the semi-direct product $\mathcal{P} = \mathcal{L} \ltimes \mathbb{R}^d$ of the Lorentz group $\mathcal{L} = O(d-1, 1)$ and the translation group $\mathbb{R}^d$. Its multiplication law is defined as follows:

$$(\lambda_1, a_1) \cdot (\lambda_2, a_2) = (\lambda_1\lambda_2, a_1 + \lambda_1 a_2), \ \lambda_1, \lambda_2 \in \mathcal{L}, a_1, a_2 \in \mathbb{R}^d. \tag{4.1}$$

Let $U : \mathcal{P} \to \mathfrak{B}(\mathcal{H})$ be a strongly continuous unitary representation of the Poincaré group on a Hilbert space $\mathcal{H}$. The generators of the translation subgroup $U|_{\mathbb{R}^d}$ are the energy-momentum operators $P = (P_0, \mathbf{P})$, such that, for $x \in \mathbb{R}^d$, $U(1, x) = \mathrm{e}^{\mathrm{i}x\cdot P}$, where $x \cdot P = x_0 P_0 - \mathbf{x} \cdot \mathbf{P}$. Let $E$ be the joint spectral measure of $P$. A vector $f \in \mathcal{H}$ is translation-invariant if $U(1, x) f = f$ for all $x \in \mathbb{R}^d$. We denote by $Q_0 = E(\{0\})$ the projection onto the subspace of translation-invariant vectors.

Let $\Lambda_1(t), \dots, \Lambda_s(t)$ be the Lorentz boosts in the space directions $1, \dots, s = d-1$, for example,

$$\Lambda_1(t) = \begin{pmatrix} \cosh(t) & \sinh(t) & 0 & \cdots & 0 \\ \sinh(t) & \cosh(t) & 0 & \cdots & 0 \\ 0 & 0 & 1 & & 0 \\ \vdots & \vdots & & \ddots & \\ 0 & 0 & 0 & & 1 \end{pmatrix}, \tag{4.2}$$

and let $K_1, \dots, K_s$ be the self-adjoint generators of the Lorentz boosts, that is, $U(\Lambda_j(t), 0) = \mathrm{e}^{\mathrm{i}tK_j}$, $j \in \{1, \dots, s\}$. We mention the following identities, which are consequences of the multiplication law ($t \in \mathbb{R}$, $x \in \mathbb{R}^d$):

$$\mathrm{e}^{\mathrm{i}tK_j}\, \mathrm{e}^{-\mathrm{i}x_0 P_0}\, \mathrm{e}^{-\mathrm{i}tK_j} = \mathrm{e}^{-\mathrm{i}x_0(\cosh(t)P_0 - \sinh(t)P_j)}, \tag{4.3}$$

$$\mathrm{e}^{\mathrm{i}tK_j}\, \mathrm{e}^{-\mathrm{i}x_j P_j}\, \mathrm{e}^{-\mathrm{i}tK_j} = \mathrm{e}^{-\mathrm{i}x_j(\cosh(t)P_j - \sinh(t)P_0)}. \tag{4.4}$$

Formally, by differentiating in $x = 0$ and $t = 0$, these identities are equivalent to the commutation relations $[P_0, \mathrm{i}K_j] = P_j$ and $[P_j, \mathrm{i}K_j] = P_0$.

Our analysis below crucially depends on the spectrum condition. The spectrum condition states that the energy-momentum spectrum is contained within the forward light cone, which is the largest Lorentz-invariant set where the energy is nonnegative.

This condition is a quantum field theory axiom in the frameworks of Wightman and Haag–Kastler. It will be assumed for the remainder of this section.

**Assumption 4.1** (*Spectrum condition*) The joint spectrum of the energy-momentum operators $P$ is a subset of the closed forward light cone $V_+ = \{p = (p_0, \mathbf{p}) \in \mathbb{R}^d \mid p_0 \geq |\mathbf{p}|\}$ (i.e. $\sigma(P) \subset V_+$).

From the spectrum condition, it follows that the momentum operators are relatively bounded relative to the energy operator $P_0$, that is, $|\mathbf{P}| \leq P_0$. This implies the inclusion $D(P_0) \subset D(P_j)$ of domains, which can be proper. As explained in the introduction, this is problematic for defining the commutators $[P_0, \mathrm{i}K_j]$ and $[P_j, \mathrm{i}K_j]$ in a way suitable for Mourre theory. Specifically, $P_0$ and $P_j$ are not necessarily elements of the regularity classes $C^k(K_j)$.

The Lorentz-invariant sets $S_\kappa$, defined in (1.1), cover the light cone $V_+$ (i.e. $\bigcup_{\kappa>0} S_\kappa = V_+$). These sets are constructed so that the energy within $S_\kappa$ remains bounded relative to the momentum. Consequently, the operators $P_{0,\kappa} = E(S_\kappa)P_0$ and $\mathbf{P}_\kappa = E(S_\kappa)\mathbf{P}$ are bounded relative to each other:

$$|\mathbf{P}_\kappa| \leq P_{0,\kappa} \leq C_\kappa(1 + |\mathbf{P}_\kappa|). \tag{4.5}$$

Moreover, the subspaces $E(S_\kappa)\mathcal{H}$ cover the Hilbert space $\mathcal{H}$, and it holds that

$$\mathcal{H}_{\mathrm{ac}}(\mathbf{P}) = \bigcup_{\kappa>0} \mathcal{H}_{\mathrm{ac}}(\mathbf{P}_\kappa), \ \ \mathcal{H}_{\mathrm{ac}}(P_0) = \bigcup_{\kappa>0} \mathcal{H}_{\mathrm{ac}}(P_{0,\kappa}). \tag{4.6}$$

In the following three subsections, we prove Mourre estimates and absence of singular continuous spectrum for the momentum operators $\mathbf{P}$, the energy operator $P_0$, and the light-cone operators $e \cdot P$, where $e$ is a light-like vector.

### 4.1.1 Momentum operators

The following proposition, which proves that $P_{0,\kappa}$ and $P_{j,\kappa}$ are elements of the regularity class $C^\infty(K_j)$, is essential for establishing the Mourre estimate and applying the results of Sect. 3.

**Proposition 4.2** *For every $\kappa > 0$ and $j \in \{1, \dots, s\}$, $P_{0,\kappa} \in C^\infty(K_j)$, $P_{j,\kappa} \in C^\infty(K_j)$, and*

$$[P_{0,\kappa}, \mathrm{i}K_j] = P_{j,\kappa}, \tag{4.7}$$

$$[P_{j,\kappa}, \mathrm{i}K_j] = P_{0,\kappa}. \tag{4.8}$$

***Proof*** The spectral projection $E(S_\kappa)$ commutes with $K_j$ because $S_\kappa$ is a Lorentz-invariant set. Thus, we can replace $P_0$ and $P_j$ in (4.4) with $P_{0,\kappa}$ and $P_{j,\kappa}$:

$$\mathrm{e}^{\mathrm{i}tK_j}\,\mathrm{e}^{-\mathrm{i}x_j P_{j,\kappa}}\,\mathrm{e}^{-\mathrm{i}tK_j} = \mathrm{e}^{-\mathrm{i}x_j(\cosh(t)P_{j,\kappa} - \sinh(t)P_{0,\kappa})}. \tag{4.9}$$

From (4.9), we obtain the following identity:

$$e^{itK_j}(P_{j,\kappa}+i)^{-1}e^{-itK_j} = (\cosh(t)P_{j,\kappa} - \sinh(t)P_{0,\kappa} + i)^{-1}. \tag{4.10}$$

The r.h.s. is a smooth function in $t$ in the strong operator topology because $P_{0,\kappa}$ and $P_{j,\kappa}$ are bounded relative to each other; hence, $P_{j,\kappa} \in C^\infty(K_j)$. Moreover, by differentiating (4.9) in $t = 0$ and $x_j = 0$, we obtain, for every $f, g \in D(P_{j,\kappa}) \cap D(K_j)$,

$$\langle P_{j,\kappa} f, K_j g\rangle - \langle K_j f, P_{j,\kappa} g\rangle = -i\langle f, P_{0,\kappa} g\rangle. \tag{4.11}$$

Because $P_{j,\kappa} \in C^1(K_j)$, the sesquilinear form (4.11) has a unique extension to $D(P_{j,\kappa}) = D(P_{0,\kappa})$, yielding the commutator identity $[P_{j,\kappa}, iK_j] = P_{0,\kappa}$. The proof of $P_{0,\kappa} \in C^\infty(K_j)$ and $[P_{0,\kappa}, iK_j] = P_{j,\kappa}$ is analogous. □

**Theorem 4.3** *Under the assumptions of Theorem 1.1, the following limiting absorption principle holds for every $\kappa > 0$, every compact subset $I_j \subset \mathbb{R}\backslash\{0\}$, and every $\nu > 1/2$:*

$$\sup_{\lambda \in I_j, \mu > 0} \|E(S_\kappa)\langle K_j\rangle^{-\nu}(P_j - \lambda \mp i\mu)^{-1}\langle K_j\rangle^{-\nu}E(S_\kappa)\| < \infty. \tag{4.12}$$

***Proof*** Let $a > 0$ and let $J$ be an open and bounded subset of $(-\infty, -a] \cup [a, \infty)$. From Proposition 4.2 and the spectrum condition ($P_{0,\kappa} \geq |P_{j,\kappa}|$), we obtain the following Mourre estimate:

$$E_{j,\kappa}(J)[P_{j,\kappa}, iK_j]E_{j,\kappa}(J) = P_{0,\kappa}E_{j,\kappa}(J) \geq aE_{j,\kappa}(J), \tag{4.13}$$

where $E_{j,\kappa}$ denotes the spectral measure of $P_{j,\kappa}$. Moreover, $P_{j,\kappa} \in C^{1+0}(K_j)$ because $C^\infty(K_j) \subset C^{1+0}(K_j)$ by (3.8). Thus, the limiting absorption principle (4.12) follows from Theorem 3.5 and the fact that $E(S_\kappa)$ commutes with $K_j$. □

**Proposition 4.4** *Under the assumptions of Proposition 1.2, $\mathcal{H} = Q_0\mathcal{H} \oplus \mathcal{H}_{ac}(\mathbf{P})$, and, for every space-like vector $e \in \mathbb{R}^d$, $\mathcal{H} = Q_0\mathcal{H} \oplus \mathcal{H}_{ac}(e \cdot P)$.*

***Proof*** In the proof of Theorem 4.3, we demonstrated that $P_{j,\kappa}$ obeys a Mourre estimate on every open and bounded subset of $(-\infty, -a] \cup [a, \infty)$, $a > 0$, with conjugate operator $K_j$. Moreover, $K_i$ commutes strongly with $P_{j,\kappa}$ if $i \neq j$. Because $a > 0$ can be arbitrary small, it follows from Corollary 3.6 that $E(\{p \in S_\kappa \mid \forall j \in \{1, \ldots, s\}\colon p_j \neq 0\})\mathcal{H} \subset \mathcal{H}_{ac}(\mathbf{P}_\kappa)$. By taking the union over $\kappa > 0$, we obtain $E(\{p \in V_+ \mid \forall j \in \{1, \ldots, s\}\colon p_j \neq 0\})\mathcal{H} \subset \mathcal{H}_{ac}(\mathbf{P})$, and, according to Lemma 4.5, $E(\{p \in V_+ \mid \forall j \in \{1, \ldots, s\}\colon p_j \neq 0\})\mathcal{H} = E(V_+\backslash\{0\})\mathcal{H}$. Thus, the first statement of the proposition follows from the decomposition $\mathcal{H} = Q_0\mathcal{H} \oplus E(V_+\backslash\{0\})\mathcal{H}$. The second statement can be proved by a similar argument or can be derived from the first statement. □

**Lemma 4.5** *Let $f \in \mathcal{H}$. If $P_\mu f = 0$ for one $\mu \in \{0, \ldots, s\}$, then $P_\mu f = 0$ for all $\mu \in \{0, \ldots, s\}$, that is, if a vector is translation-invariant in one spacetime direction, then it is translation-invariant in all spacetime directions.*

***Proof*** If $P_0 f = 0$, then $P_j f = 0$ for all $j \in \{1, \dots, s\}$ due to the spectrum condition. If $P_j f = 0$ for one $j \in \{1, \dots, s\}$, then also $P_{j,\kappa} f = 0$ for all $\kappa > 0$. From the commutation relation $[P_{j,\kappa}, \mathrm{i}K_j] = P_{0,\kappa}$ and the virial theorem (Theorem 3.4), it follows that

$$\|\sqrt{P_{0,\kappa}} f\|^2 = \langle f, [P_{j,\kappa}, \mathrm{i}K_j] f\rangle = 0; \tag{4.14}$$

hence, $P_{0,\kappa} f = 0$ for all $\kappa > 0$, that is, $P_0 f = 0$. □

### 4.1.2 Energy operator

Constructing a conjugate operator for the energy operator $P_0$ is more difficult. We cannot choose the generator $K_j$ of a Lorentz boost because $[P_{0,\kappa}, K_j] = P_{j,\kappa}$, and $P_{j,\kappa}$ has no definite sign on any spectral subspace of $P_{0,\kappa}$. The conjugate operator $\overline{A_\kappa}$, which we construct below, is adapted from [1, Lemma 7.6.4].

**Theorem 4.6** *Under the assumptions of Theorem 1.1, the following limiting absorption principle holds for every $\kappa > 0$, every compact subset $I_0 \subset (\kappa, \infty)$, and every $\nu > 1/2$:*

$$\sup_{\lambda \in I_0, \mu > 0} \|E(S_\kappa)\langle K_j\rangle^{-\nu}(P_0 - \lambda \mp \mathrm{i}\mu)^{-1}\langle K_j\rangle^{-\nu}E(S_\kappa)\| < \infty. \tag{4.15}$$

***Proof*** For convenience, we choose $j = 1$. Let $\kappa > 0$, $\theta \in C_c^\infty((\kappa, \infty))$ a real-valued function, and set $F(P_\kappa) = \theta(P_{0,\kappa})/P_{1,\kappa}$. The operator $F(P_\kappa)$ is well-defined because if $p \in S_\kappa$ with $p_0 \in \mathrm{supp}(\theta)$, then $p_1$ is separated from 0. We define the following operator on the domain $D(A_\kappa) = D(K_1)$:

$$A_\kappa = \frac{1}{2}(F(P_\kappa)K_1 + K_1 F(P_\kappa)). \tag{4.16}$$

By Lemma 4.8, the operator $A_\kappa$ is essentially self-adjoint, $P_{0,\kappa} \in C^\infty(\overline{A_\kappa})$, where $\overline{A_\kappa}$ is the self-adjoint closure of $A_\kappa$, and, for $0 < a < b$,

$$E_{0,\kappa}(\kappa + (a, b))[P_{0,\kappa}, \mathrm{i}\overline{A_\kappa}]E_{0,\kappa}(\kappa + (a, b)) = \theta(P_{0,\kappa})E_{0,\kappa}(\kappa + (a, b)), \tag{4.17}$$

where $E_{0,\kappa}$ is the spectral measure of $P_{0,\kappa}$. If we select $\theta \in C_c^\infty((\kappa, \infty))$ such that $\theta = 1$ on $\kappa + (a, b)$, then $P_{0,\kappa}$ obeys a Mourre estimate on $\kappa + (a, b)$. From Theorem 3.5, it follows that

$$\sup_{\lambda \in I_0, \mu > 0} \|E(S_\kappa)\langle \overline{A_\kappa}\rangle^{-\nu}(P_0 - \lambda \mp \mathrm{i}\mu)^{-1}\langle \overline{A_\kappa}\rangle^{-\nu}E(S_\kappa)\| < \infty. \tag{4.18}$$

To replace $\langle \overline{A_\kappa}\rangle^{-\nu}$ with $\langle K_1\rangle^{-\nu}$, we observe that $\langle \overline{A_\kappa}\rangle^{-\nu}\langle K_1\rangle^{\nu}$ is a bounded operator which commutes with $E(S_\kappa)$. □

**Proposition 4.7** *Under the assumptions of Proposition 1.2, $\mathcal{H} = Q_0\mathcal{H} \oplus \mathcal{H}_{\mathrm{ac}}(e \cdot P)$ for every time-like vector e.*

***Proof*** It suffices to prove the proposition for $e = (1, 0, \dots, 0)$ because $e \cdot P$ is unitarily equivalent to $P_0$ if $e$ is time-like. In the proof of Theorem 4.6, we demonstrated that $P_{0,\kappa}$ obeys a Mourre estimate on $\kappa + (a, b)$ for every $0 < a < b$. From Corollary 3.6, it follows that $E(\{p \in S_\kappa \mid p_0 \in \kappa + (a, b)\})\mathcal{H} \subset \mathcal{H}_{\mathrm{ac}}(P_{0,\kappa})$. We take the union over $\kappa$, $a$, and $b$:

$$E(V_+\backslash\{\mathbf{p} = 0\})\mathcal{H} = \bigcup_{\kappa>0,b>a>0} E(\{p \in S_\kappa \mid p_0 \in \kappa + (a, b)\})\mathcal{H} \subset \mathcal{H}_{\mathrm{ac}}(P_0). \tag{4.19}$$

According to Lemma 4.5, the l.h.s. is equal to $E(V_+\backslash\{0\})\mathcal{H}$. □

**Lemma 4.8** *Let $\kappa > 0$, $\theta \in C_c^\infty((\kappa, \infty))$ a real-valued function, and set $F(P_\kappa) = \theta(P_{0,\kappa})/P_{1,\kappa}$. The symmetric operator*

$$A_\kappa = \frac{1}{2}(F(P_\kappa)K_1 + K_1F(P_\kappa)) \tag{4.20}$$

*is essentially self-adjoint on $D(K_1)$. If we denote its self-adjoint closure by $\overline{A_\kappa}$, then $P_{0,\kappa} \in C^\infty(\overline{A_\kappa})$ and*

$$[P_{0,\kappa}, \mathrm{i}\overline{A_\kappa}] = \theta(P_{0,\kappa}). \tag{4.21}$$

***Proof*** (i) If $G \in C_c^\infty(\mathbb{R}^d, \mathbb{R})$ and $j \in \{1, \dots, s\}$, then $G(P_\kappa) \in C^\infty(K_j)$ and the following commutator identity holds:

$$\begin{aligned}[G(P_\kappa), \mathrm{i}K_j] &= -\partial_t\big|_0 \mathrm{e}^{\mathrm{i}tK_j} G(P_\kappa)\, \mathrm{e}^{-\mathrm{i}tK_j} \\ &= -\partial_t\big|_0 G(\cosh(t)P_{0,\kappa} - \sinh(t)P_{j,\kappa}, P_{1,\kappa}, \dots, \cosh(t)P_{j,\kappa} \\ &\quad - \sinh(t)P_{0,\kappa}, \dots, P_{s,\kappa}) \\ &= \partial_0 G(P_\kappa)P_{j,\kappa} + \partial_j G(P_\kappa)P_{0,\kappa}. \end{aligned} \tag{4.22}$$

Applying this identity to $F(P_\kappa)$, we obtain the following commutator:

$$[F(P_\kappa), \mathrm{i}K_1] = \theta'(P_{0,\kappa}) - \theta(P_{0,\kappa})\frac{P_{0,\kappa}}{P_{1,\kappa}^2}. \tag{4.23}$$

The operator on the r.h.s. is bounded. It follows that $F(P_\kappa)$ leaves $D(K_1)$ invariant. In particular, $A_\kappa$ is well-defined on $D(K_1)$.
(ii) We apply Nelson's commutator theorem [20, Theorem X.36] to establish the essential self-adjointness of $A_\kappa$ on $D(K_1)$. Setting $N = K_1^2 + 1$, we define the integer scale $\mathcal{H}_k$, $k \in \mathbb{Z}$, corresponding to $N$ as the completion of $D(N^{k/2})$ with respect to

the norm $\|f\|_k = \|N^{k/2} f\|$. Clearly, $A_\kappa$ is a symmetric bounded operator from $\mathcal{H}_n$ to $\mathcal{H}_{-n}$ for every $n \geq 1$. Moreover, as an operator identity from $\mathcal{H}_{n+2}$ to $\mathcal{H}_{-n-2}$,

$$NA_\kappa - A_\kappa N = 2K_1[K_1, F(P_\kappa)]K_1 + \frac{1}{2}[K_1, [K_1, [K_1, F(P_\kappa)]]]. \tag{4.24}$$

The triple commutator in the second summand is a bounded operator on $\mathcal{H}$ according to (4.22). It follows that, for every $f \in \mathcal{H}_{n+2} \subset \mathcal{H}_n$, the commutator $NA_\kappa f - A_\kappa N f$ is an element of $\mathcal{H}_{-n}$, and

$$\|NA_\kappa f - A_\kappa N f\|_{-n} \leq c\|f\|_n; \tag{4.25}$$

hence, the commutator $NA_\kappa - A_\kappa N$ extends to a bounded operator from $\mathcal{H}_n$ to $\mathcal{H}_{-n}$. We conclude that $A_\kappa$ is essentially self-adjoint on any core of $N$, particularly on $D(K_1^2)$. The closure of $A_\kappa$ restricted to $D(K_1^2)$ coincides with the closure of $A_\kappa$ defined on $D(K_1)$. Thus, $A_\kappa$ is essentially self-adjoint on $D(K_1)$.

(iii) If $\chi \in C_c^\infty(\mathbb{R})$ is a function such that $\chi = 1$ on $\operatorname{supp}(\theta)$, then $\overline{A_\kappa} = \chi(P_{0,\kappa})\overline{A_\kappa}$. In fact, if $f \in D(K_1)$, then, by (4.23),

$$\begin{aligned} \overline{A_\kappa} f = A_\kappa f = F(P_\kappa)K_1 f + \frac{1}{2}\left(\theta'(P_{0,\kappa}) - \theta(P_{0,\kappa})\frac{P_{0,\kappa}}{P_{1,\kappa}^2}\right) f \\ = \chi(P_{0,\kappa})A_\kappa f = \chi(P_{0,\kappa})\overline{A_\kappa} f. \end{aligned} \tag{4.26}$$

This identity extends to $f \in D(\overline{A_\kappa})$ by approximating $f$ with elements from $D(K_1)$ in the graph topology of $\overline{A_\kappa}$.

(iv) We prove $P_{0,\kappa} \in C^1(\overline{A_\kappa})$. Utilising $[P_{0,\kappa}, \mathrm{i}K_1] = P_{1,\kappa}$ (see Proposition 4.2), it is easy to verify that, for every $f \in D(K_1) \cap D(P_{0,\kappa})$,

$$|\langle P_{0,\kappa} f, \overline{A_\kappa} f\rangle - \langle \overline{A_\kappa} f, P_{0,\kappa} f\rangle| = |\langle f, \theta(P_{0,\kappa}) f\rangle| \leq \|\theta(P_{0,\kappa})\|\|f\|^2. \tag{4.27}$$

By approximating $f \in D(\overline{A_\kappa})$ in the graph topology of $\overline{A_\kappa}$ with elements from $D(K_1)$, it follows from the previous step that (4.27) is valid for $f \in D(\overline{A_\kappa})$. Moreover, $\{f \in D(\overline{A_\kappa}) \mid (P_{0,\kappa} \pm \mathrm{i})^{-1} f \in D(\overline{A_\kappa})\}$ contains the core $D(K_1)$. In fact, if $f \in D(K_1)$, then

$$\begin{aligned} \lim_{t\to 0} \frac{1}{\mathrm{i}t}(\mathrm{e}^{\mathrm{i}tK_1} - 1)(P_{0,\kappa} \pm \mathrm{i})^{-1} f = -\mathrm{i}P_{1,\kappa}(P_{0,\kappa} \pm \mathrm{i})^{-2} f \\ + (P_{0,\kappa} \pm \mathrm{i})^{-1} K_1 f \in \mathcal{H}. \end{aligned} \tag{4.28}$$

We conclude $P_{0,\kappa} \in C^1(\overline{A_\kappa})$ by Proposition 3.2.

(v) From the previous step, it follows that the commutator $[P_{0,\kappa}, \mathrm{i}\overline{A_\kappa}] = \theta(P_{0,\kappa})$ is a bounded operator. By similar arguments as in step (iv), we compute the higher-order commutators:

$$\mathrm{ad}^k_{-\mathrm{i}\overline{A_\kappa}}(P_{0,\kappa}) = \theta_{k-1}(P_{0,\kappa}),\ \theta_k(x) = [\theta(x)\partial_x]^k \theta(x). \tag{4.29}$$

All commutators are bounded operators; thus, $P_{0,\kappa} \in C^\infty(\overline{A_\kappa})$. □

### 4.1.3 Light-cone operators

We prove Proposition 1.2 for nonzero light-like vectors $e$ and Proposition 1.3. It suffices to consider $e = (1, \mp 1, 0, \ldots, 0)$ because $e \cdot P$ is unitarily equivalent to $P_0 + P_1$ or $P_0 - P_1$ if $e$ is light-like.

**Proposition 4.9** *Under the assumptions of Proposition 1.2, $\mathcal{H} = Q_0\mathcal{H} \oplus \mathcal{H}_{\mathrm{ac}}(P_0 \pm P_1)$ if $d \geq 3$ and $\mathcal{H} = E_{P_0 \pm P_1}(\{0\})\mathcal{H} \oplus \mathcal{H}_{\mathrm{ac}}(P_0 \pm P_1)$ if $d = 2$.*

***Proof*** From (4.3) and (4.4), we obtain the following identity ($t, x \in \mathbb{R}$):

$$\mathrm{e}^{\mathrm{i}tK_1}\, \mathrm{e}^{\mathrm{i}x(P_0 \pm P_1)}\, \mathrm{e}^{-\mathrm{i}tK_1} = \mathrm{e}^{\mathrm{i}\,\mathrm{e}^{\mp t} x (P_0 \pm P_1)}. \tag{4.30}$$

In the terminology of the example subsequent to Definition 3.3, the operator $P_0 \pm P_1$ is $\pm K_1$-homogeneous. It follows that $P_0 \pm P_1$ obeys a Mourre estimate on every open and bounded interval that is separated from 0. Thus, by Corollary 3.6, $E(V_+ \backslash \{p_0 \pm p_1 \neq 0\})\mathcal{H} \subset \mathcal{H}_{\mathrm{ac}}(P_0 \pm P_1)$. If $d \geq 3$, then $E(V_+ \backslash \{p_0 \pm p_1 \neq 0\})\mathcal{H} = E(V_+ \backslash \{0\})\mathcal{H}$. In fact, for $f \in \mathcal{H}$, $P_0 f = \pm P_1 f$ implies $P_2 f = 0$ by the spectrum condition. And, by Lemma 4.5, $P_2 f = 0$ implies $P_0 f = P_1 f = 0$. □

***Remark*** Assume that an eigenstate $f \in \mathcal{H}/Q_0\mathcal{H}$ with eigenvalue 0 of the mass operator $M = \sqrt{P_0^2 - |\mathbf{P}|^2}$ exists. If $d = 2$, then $P_0 f = |\mathbf{P}| f = |P_1| f$ for such a massless excitation. If $f_\pm$ denotes the positive/negative momentum component of $f$, then $(P_0 \mp P_1) f_\pm = 0$. This illustrates that, in the case $d = 2$, the subspace $E_{P_0 \pm P_1}(\{0\})\mathcal{H}$ can be larger than $Q_0\mathcal{H}$.

### 4.2 Dilation-covariant representations

Let $U : \mathbb{R}^d \to \mathfrak{B}(\mathcal{H})$ be a strongly continuous unitary representation of the translation group, whose generators obey the spectrum condition (Assumption 4.1). In this subsection, we assume that $U$ is dilation-covariant, that is, a self-adjoint operator $D$ exists such that, for every $\mu \in \{0, \ldots, s\}$ and $t, x \in \mathbb{R}$,

$$\mathrm{e}^{\mathrm{i}tD}\, \mathrm{e}^{\mathrm{i}xP_\mu}\, \mathrm{e}^{-\mathrm{i}tD} = \mathrm{e}^{\mathrm{i}\,\mathrm{e}^{-t} x P_\mu}. \tag{4.31}$$

We analyse the spectrum of the mass operator $M = \sqrt{P_0^2 - |\mathbf{P}|^2}$, which is a well-defined self-adjoint operator due to the spectrum condition.

***Proof of Proposition 1.4*** In the terminology of the example subsequent to Definition 3.3, $M$ is $D$-homogeneous due to (4.31), implying $M \in C^\infty(D)$ and

$$[M, \mathrm{i}D] = M. \tag{4.32}$$

Thus, $M$ obeys a Mourre estimate with conjugate operator $D$ on every open and bounded subset of $[a, \infty)$, $a > 0$. The limiting absorption principle (1.9) for the mass operator follows from Theorem 3.5. That the spectrum of $M$ is purely absolutely continuous in $(0, \infty)$ is a consequence of Corollary 3.6. □

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