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Stochastic quantization and Osterwalder-Schrader axioms for quantum field theory models

A doctoral dissertation in natural sciences
in the area of mathematics

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Kwantowanie stochastyczne i aksjomaty Osterwaldera-Schradera dla modeli kwantowej teorii pola

Rozprawa doktorska w naukach ścisłych i przyrodniczych w dyscyplinie matematyka
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Contents

Summary

The purpose of this thesis is to construct the $P(\Phi)_2$ Euclidean quantum field theory, extending our discussion from [37]. We first construct the finite volume measure of the $P(\Phi)_2$ model on $\mathscr{D}'(\mathbb{S}_R)$, where $\mathbb{S}_R$ is a 2-sphere of the radius $R \in \mathbb{N}_+$. To this end, we rely on the Nelson hypercontractivity estimate. Then, we utilize the parabolic stochastic quantization and the energy method with the help of the pull-back map $\jmath_R^* : \mathscr{D}'(\mathbb{S}_R) \to \mathscr{S}'(\mathbb{R}^2)$, induced by the stereographic projection $\jmath_R$, to obtain the infinite volume measure on $\mathscr{S}'(\mathbb{R}^2)$. To this end, we verify the global existence in time of solutions for the corresponding stochastic partial differential equations with fixed cutoffs, a topic that was only briefly addressed in [37].

The rest of the thesis is devoted to verifying the Osterwalder-Schrader axioms: regularity, reflection positivity and invariance under translations, rotations, and reflections. Such a combination of axioms, which is crucial for obtaining a non-trivial local relativistic QFT on Minkowski space-time, was not obtained in earlier constructions based on the stochastic quantization, such as [5, 6, 55].

Podsumowanie

Celem tej pracy jest skonstruowanie euklidesowej kwantowej teorii pola $P(\Phi)_2$, rozszerzając naszą dyskusję z [37]. Najpierw konstruujemy miarę na $\mathscr{D}'(\mathbb{S}_R)$ dla modelu $P(\Phi)_2$ w skończonej objętości, gdzie $\mathbb{S}_R$ jest 2-sferą o promieniu $R \in \mathbb{N}_+$. W tym celu opieramy się na oszacowaniu Nelsona dotyczącym półgrup hiperkontraktywnych. Następnie wykorzystujemy paraboliczne kwantowanie stochastyczne i metodę energetyczną aby za pomocą pull-backu $\jmath_R^* : \mathscr{D}'(\mathbb{S}_R) \to \mathscr{S}'(\mathbb{R}^2)$, zadanego przez rzut stereograficzny $\jmath_R$, uzyskać miarę w nieskończonej objętości, mianowicie na $\mathscr{S}'(\mathbb{R}^2)$. W tym celu sprawdzamy istnienie globalnych w czasie rozwiązań dla odpowiednich stochastycznych równań różniczkowych cząstkowych z ustalonymi obcięciami, który to temat został jedynie krótko poruszony w [37].

Pozostała część pracy poświęcona jest weryfikacji aksjomatów Osterwaldera-Schradera: regularności, dodatniości względem odbicia oraz niezmienniczości względem przesunięć, obrotów i odbić. Takiej kombinacji aksjomatów, kluczowej dla uzyskania nietrywialnej lokalnej relatywistycznej kwantowej teorii pola na czasoprzestrzeni Minkowskiego, nie uzyskano we wcześniejszych konstrukcjach opartych na kwantowaniu stochastycznym, takich jak [5, 6, 55].

Introduction and the main result

Over the past 60 years several attempts have been made to reconcile the principles of quantum mechanics with special relativity. The resulting theories are known as quantum field theories (QFT's). In order to make QFT's mathematically precise, using axiomatic setting, several approaches were proposed. The first satisfactory approach was the Wightman formalism on Minkowski spacetime, which assigns to each subset of space-time an operator-valued distribution as a quantum observable [95]. Later, the Haag-Kastler setting was invented, which assigns C^*-algebras of observables to open bounded regions of space-time [62]. Another significant endeavour is the Osterwalder-Schrader framework, which provides conditions under which Wightman quantum field theory can be constructed from a Euclidean QFT [76]. It is also the setting of this work.

To set the scene, we discuss in heuristic terms the Euclidean field theory using the probabilistic approach of [66]. Let (M, g) be a $d-$dimensional Riemannian manifold with the metric tensor g, which might enjoy some symmetries, Σ be another Riemannian manifold equipped with a volume measure σ, and let $\Phi(M)$ denote the set consisting of all the mappings $\phi : M \to \Sigma$. We aim to define a measure μ on $\Phi(M)$ representing the desired classical field measure such that it respects the symmetries of the underlying manifold M. To this end, consider the functional $\mathcal{L} : \Phi(M) \times M \to \mathbb{R}_+$, which is called the Lagrangian density and formally assigns to each point $\mathrm{m} \in M$ and each $\phi \in \Phi(M)$ a real number, i.e., $[\mathcal{L}(\phi)](\mathrm{m}) = \mathcal{L}(\phi, \mathrm{m})$. Integrating the Lagrangian density functional over the manifold M, one defines the action $\mathcal{A} : \Phi(M) \to \mathbb{R}$ by $\mathcal{A}(\phi) := \int_M [\mathcal{L}(\phi)](\mathrm{m}) \, \mathrm{vol}(\mathrm{dm})$, where $\mathrm{vol}(\mathrm{dm})$ is the Riemannian volume measure of M determined using the metric g. Using the action $\mathcal{A}$, one defines the measure

$$\mu(\mathrm{d}\phi) := \frac{1}{\mathcal{Z}} \exp\left(-\mathcal{A}(\phi)\right) \nu(\mathrm{d}\phi), \tag{1.0.1}$$

on $\Phi(M)$, where ν is a reference measure on $\Phi(M)$ informally given by $\nu(\mathrm{d}\phi) := \bigotimes_{\mathrm{m} \in M} \sigma(\mathrm{d}\phi(\mathrm{m}))$. The partition function $\mathcal{Z} := \int_{\Phi(M)} \mu(\mathrm{d}\phi) = \int_{\Phi(M)} \mathrm{e}^{-\mathcal{A}(\phi)} \, \nu(\mathrm{d}\phi)$

denotes the total mass of the measure μ. Note that most of the time such a measure is not well-defined and calculating $\mathcal{Z}$ for a given theory might not always be feasible. But if the measure (1.0.1) can be controlled, its moments give the Schwinger functions of the QFT in question. To define the moments, we need a multiplication in Σ. As we are interested in scalar fields, we set $\Sigma = \mathbb{R}$ in the following.

Parabolic stochastic quantization for a Euclidean scalar field was introduced by Parisi and Wu [77]. It is an approach to control mathematically the measure (1.0.1) using methods from the theory of stochastic partial differential equations (SPDE). In our context of fields defined on a Riemannian manifold M it works as follows: we start with the field $\phi(\mathrm{m})$, $\mathrm{m} \in M$, introduced above. Then, we define its counterpart, $\varPhi(t,\mathrm{m})$, $(t,\mathrm{m}) \in [0,T] \times M$, by introducing a fictitious time variable $t \in [0,T]$ such that $\varPhi(0,\mathrm{m}) = \phi(\mathrm{m})$. The random field $\varPhi(t,\mathrm{m})$ is coupled to an irregular Gaussian space-time white noise $\xi(t,\mathrm{m})$ whose typical realizations are distributions of negative regularity [25, Thm. 2.7]. Next, we shall demand that the time evolution of $\varPhi(t,\mathrm{m})$ obeys a stochastic differential equation of Langevin type[1]

$$\frac{\partial \varPhi(t,\mathrm{m})}{\partial t} = -\frac{1}{2}\frac{\delta \mathcal{A}}{\delta \varPhi}\left(\varPhi(t,\bullet)\right)(\mathrm{m}) + \xi(t,\mathrm{m}), \tag{1.0.2}$$

where $\frac{\delta \mathcal{A}}{\delta \varPhi}$ denotes the functional derivative of the action. Even if the action $\mathcal{A}$ describes the free field, the solution Φ inherits the distributional character of the white noise. If, subsequently, interaction is switched on, the functional derivative above becomes non-linear. Thus, evaluation on the distribution $\varPhi$ is ill-defined, which is a manifestation of the ultraviolet (UV) problem. It requires a modification of the equation, which is called renormalization. The simplest case, which suffices for polynomial interactions in $d = 2$, is the Wick ordering of the action. Higher dimensions and more complicated interactions require more advanced methods such as the theory of regularity structures [58], paracontrolled calculus [54], and renormalization group techniques [36, 65]. In these particular frameworks it is possible to classify the equations (1.0.2) into sub-critical, critical and super-critical, depending on their UV behaviour. This corresponds to the physicist's terminology of super-renormalizable, (just-)renormalizable and non-renormalizable actions, dictated by their behaviour under the Wilsonian block-averaging [53]. In the renormalizable cases these actions move away from the massless free field action (i.e., the Gaussian fixed point) and give rise

[1]There is a choice involved at this stage. Instead of the Langevin dynamics, one can choose the Fokker-Planck dynamics, the elliptic stochastic quantization, canonical stochastic quantization or the variational method [34, 56].

to an interacting effective action at macroscopic scales. This mechanism is particularly robust in the super-renormalizable case, in which the coupling constants have strictly negative dimensions (in the units of length). In the non-renormalizable case, the action moves towards the Gaussian fixed point which leads to trivialization.

If Φ evolves according to equation (1.0.2), its law evolves according to the corresponding Fokker-Planck equation and is expected to approach the measure (1.0.1) in the limit $t \to \infty$. For similar reasons, this measure should be the law of a stationary solution of (1.0.2). Physically, we can think of equation (1.0.2) as describing the return to equilibrium of a statistical physics system, whose (Landau) free energy is given by the Euclidean action $\mathcal{A}$, cf. [53, Eq. (8.41)].

Once the measure μ is constructed, one moves on to the study of its properties. In the case of the Euclidean spacetime $M = \mathbb{R}^d$, a convenient set of conditions, which guarantee the existence of an underlying Minkowskian QFT, are the Osterwalder-Schrader axioms. They concern an abstract family of distributions $S_n \in \mathscr{S}'(\mathbb{R}^d)$, $n \in \mathbb{N}_0$, which are interpreted as the moments of the measure μ:

$$S_m(f_1 \otimes \ldots \otimes f_m) = \int_{\mathscr{S}'(\mathbb{R}^d)} \phi(f_1) \ldots \phi(f_m) \, \mu(\mathrm{d}\phi), \tag{1.0.3}$$

$f_1, \ldots, f_m \in \mathscr{S}(\mathbb{R}^d)$, i.e., the Schwinger functions. The axioms encode their physically expected properties and add sufficient technical input for the subsequent reconstruction of a Wightman QFT on Minkowski spacetime.

Definition 1.0.1. *(The Osterwalder-Schrader axioms) [76].*

OS0 *(Regularity). It holds that* $S_0 = 1$. *There is a Schwartz norm* $\|\bullet\|_{\mathrm{s}}$ *on* $\mathscr{S}'(\mathbb{R}^d)$ *and* $K, \beta > 0$ *such that for all* $m \in \mathbb{N}$ *and* $f_1, \ldots, f_m \in \mathscr{S}(\mathbb{R}^d)$

$$|S_m(f_1 \otimes \ldots \otimes f_m)| \leq K^m (m!)^\beta \prod_{i=1}^m \|f_i\|_{\mathrm{s}}.$$

OS1 *(Euclidean invariance). For each* $m \in \mathbb{N}$ *and all* $f_1, \ldots, f_m \in \mathscr{S}(\mathbb{R}^d)$,

$$S_m(f_{1,(a,Q)} \otimes \ldots \otimes f_{m,(a,Q)}) = S_m(f_1 \otimes \ldots \otimes f_m),$$

where $f_{(a,Q)}(x) := f(Q^{-1}(x-a))$, $Q \in SO(d)$, *and* $a \in \mathbb{R}^d$.

OS2 *(Reflection positivity). Let* $\mathbb{R}^{dm}_+ := \{(x^{(1)}, \ldots, x^{(m)}) \in \mathbb{R}^{dm} : x^{(j)}_0 > 0, j = 1, \ldots, m\}$ *and*

$$\mathscr{S}(\mathbb{R}^{3m}_+; \mathbb{C}) := \{ f \in \mathscr{S}(\mathbb{R}^{3m}; \mathbb{C}) : \mathrm{supp}(f) \subset \mathbb{R}^{dm}_+ \}.$$

For all sequences $f^{(m)} \in \mathscr{S}(\mathbb{R}^{dm}_{+};\mathbb{C})$, $m \in \mathbb{N}_0$, *with finitely many non-zero elements*

$$\sum_{\ell,m\in\mathbb{N}_0} S_{\ell+m}(\overline{\Theta f^{(\ell)}} \otimes f^{(m)}) \geq 0,$$

where $(\Theta f^{(\ell)})(x_0^{(1)}, \vec{x}^{(1)}, \ldots, x_0^{(\ell)}, \vec{x}^{(\ell)}) := f(-x_0^{(1)}, \vec{x}^{(1)}, \ldots, -x_0^{(\ell)}, \vec{x}^{(\ell)})$.

OS3 *(Symmetry). For all* $m \in \mathbb{N}$, $f_1, \ldots, f_m \in \mathscr{S}(\mathbb{R}^d)$ *and* π *a permutation of* m *elements*

$$S_m(f_{\pi(1)} \otimes \ldots \otimes f_{\pi(m)}) = S_m(f_1 \otimes \ldots \otimes f_m).$$

OS4 *(Clustering). For all sequences* $f^{(m)}, g^{(m)} \in \mathscr{S}(\mathbb{R}^{dm}_{+};\mathbb{C})$, $m \in \mathbb{N}_0$, *with finitely many non-zero elements*

$$\lim_{\lambda\to\infty} \sum_{\ell,m\in\mathbb{N}_0} \left\{ S_{\ell+m}(\overline{\Theta f^{(\ell)}} \otimes g^{(m)}_{(\lambda a,I)}) - S_\ell(\overline{\Theta f^{(\ell)}}) S_m(g^{(m)}) \right\} = 0,$$

where $a = (0, \vec{a})$.

There are several, possibly inequivalent, versions of the Osterwalder-Schrader axioms in the literature [40, 50, 76]. We chose the above variant because the regularity condition OS0 is easy to check by methods of stochastic quantization (unlike the variant from the textbook of Glimm and Jaffe [50]). The reflection positivity axiom OS2 is somewhat stronger than in the work of Osterwalder and Schrader [76] and in this form it was formulated by Eckmann and Epstein in [40] to ensure the existence of time-ordered products on the Minkowski side. Clearly, the symmetry axiom OS3 is trivially satisfied if the distributions S_m are defined by (1.0.3) as moments of a measure.

The axioms are satisfied, in particular, by free field theories, in which case the measure μ, defining the Schwinger functions in (1.0.3) is Gaussian. Therefore, non-Gaussianity of the measure is generally accepted as a useful criterion for non-triviality of a QFT in question. Due to the necessity of renormalization, mentioned above, non-triviality of a QFT cannot be inferred from the mere presence of interaction terms in the action $\mathcal{A}$. For example, the ϕ^4-interaction term in the action gives rise to a trivial theory in $d \geq 4$ [1, 2, 44].

If the Schwinger functions, defined by (1.0.3), satisfy the Osterwalder-Schrader axioms of Def. 1.0.1, then their analytic continuation from imaginary to real time $x_{0,j} \to -\mathrm{i}x_{0,j}$, $j = 1, \ldots, m$, is possible and gives rise to the Wightman functions of the underlying QFT on Minkowski spacetime [76]. The fact that the Osterwalder-Schrader axioms imply the existence of a corresponding Wightman QFT is a central result of constructive QFT and an important motivation to study Euclidean QFT. Let us recall, for completeness, the Wightman setting as it links our discussion with quantum theory.

Let $M = \mathbb{R}^d$ be equipped with the Minkowski metric η which is invariant under the restricted Poincaré group $\mathcal{P}_+^\uparrow$. Consider a quadruple $\langle H, U, \hat{\phi}, \mathrm{Dom}\rangle$, where H is a separable Hilbert space consisting of physical states, $\mathrm{Dom} \subset H$ is a dense subspace, $\mathcal{P}_+^\uparrow \ni g \mapsto U(g) \in \mathcal{U}(H)$ is a group homomorphism, where $\mathcal{U}(H)$ is the group of unitary operators on H [2] and $\hat{\phi}$ is an operator-valued tempered distribution. Furthermore, a unit vector $\Omega \in H$ is called a vacuum state if $U(g)\,\Omega \subset \Omega$, $g \in \mathcal{P}_+^\uparrow$, and it is called unique if Ω is the only such vector in H up to multiplication by a phase.

Definition 1.0.2. *(The Wightman axioms) [83, IX.8] A relativistic real-valued QFT on $\mathscr{S}'(\mathbb{R}^d)$ is a quadruple $\langle H, U, \hat{\phi}, \mathrm{Dom}\rangle$ satisfying the Wightman axioms:*

(A) *H is a separable Hilbert space with a unique vacuum state $\Omega \in \mathrm{Dom} \subset H$ and U is a strongly continuous unitary representation on H of the restricted Poincaré group $\mathcal{P}_+^\uparrow$ such that $U(a,\Lambda)\mathrm{Dom} \subset \mathrm{Dom}$ for all $(a,\Lambda) \in \mathcal{P}_+^\uparrow$.*

(B) *The projection-valued measure E on $\mathbb{R}^d$ corresponding to $U(a) = \mathrm{e}^{ia^\mu P_\mu}$ has support in the closed forward light cone, i.e., $\overline{V}_+ = \{(p_0, p_1) \in \mathbb{R}^d \,|\, p_0 \geq |p_1|\}$.*

(C) *For $f \in \mathscr{S}(\mathbb{R}^d)$ it holds $\mathrm{Dom} \subset \mathrm{Dom}(\hat{\phi}(f))$, $\hat{\phi}(f)\mathrm{Dom} \subset \mathrm{Dom}$ and for any fixed $\varphi \in \mathrm{Dom}$, the map $f \mapsto \hat{\phi}(f)\varphi$ is linear. Moreover, for any $\varphi_1, \varphi_2 \in \mathrm{Dom}$ the mapping $\mathscr{S}(\mathbb{R}^d) \ni f \mapsto \langle \varphi_1,\, \hat{\phi}(f)\,\varphi_2\rangle \in \mathbb{R}$ is a tempered distribution.*

(D) *(Locality) $[\hat{\phi}(f_1), \hat{\phi}(f_2)] = 0$ if $\mathrm{supp} f_1$ and $\mathrm{supp} f_2$ spacelike separated. (In the sense of weak commutativity on Dom).*

(E) *(Covariance) $U(a,\Lambda)\,\hat{\phi}(f)\,U(a,\Lambda)^* = \hat{\phi}(f_{(a,\Lambda)})$ for all $(a,\Lambda) \in \mathcal{P}_+^\uparrow$ and all $f \in \mathscr{S}(\mathbb{R}^d)$ with $f_{(a,\Lambda)} = f(\Lambda^{-1}(x-a))$.*

(F) *The dense subset $\mathrm{Dom} \subset H$ is constructed out of the vacuum via $\mathrm{Dom} = \mathrm{Span}\{\hat{\phi}(f_1), \cdots \hat{\phi}(f_m)\,\Omega \,|\, f_1, \cdots, f_m \in \mathscr{S}(\mathbb{R}^d), m \in \mathbb{N}_0\}$.*

The spectral condition in Def. 1.0.2, Item (B), determines the physical masses of the quantum fields via the point spectrum of the mass operator $M := (P_\mu P^\mu)^{1/2}$. For non-zero isolated masses Haag-Ruelle scattering theory gives a canonical construction of the scattering matrix [62]. If it is different from the identity, we say that the QFT is interacting. This condition for non-triviality is stronger than the non-Gaussianity of the functional measure μ mentioned above. It was verified for some polynomial interactions for $d = 2, 3$, cf. [50]. In this thesis we will only consider non-Gaussianity.

[2] The Wigner theorem gives an operator $U(g)$ for each $g \in \mathcal{P}_+^\uparrow$, which is unique up to phase.

Review of the literature

There are a large number of works devoted to constructions of the Φ^4_d quantum field theory in different dimensions. Let us briefly describe the most relevant results in $d = 2, 3$ settings.

- The finite volume measure of the Φ^4_2 model was first constructed by Nelson [74]. Due to the fact that finite volume Φ^4_2 measure is absolutely continuous with respect to the Gaussian reference measure, the construction of this model was relatively easy to carry out.
- Glimm and Jaffe constructed the infinite volume measure of the $P(\Phi)_2$ model on Minkowski space-time and verified all the Wightman axioms [50]. Glimm, Jaffe, and Spencer constructed the infinite volume measure of the $P(\Phi)_2$ model and verified all of the Osterwalder-Schrader axioms including exponential decay of correlation functions for small coupling constant [48].
- Glimm showed the existence of the Φ^4_3 Hamiltonian in the Minkowski setting [51]. Glimm and Jaffe proved the positivity of the Φ^4_3 Hamiltonian [52]. The first complete constructions of the model are given in [45, 46, 70].
- Hairer studied the dynamical Φ^4_3 model using the theory of regularity structures and verified the local in time well-posedness of this model [58]. Mourrat and Weber investigated its long-time behaviour [71]. Gubinelli and Hofmanová constructed global in space and time solutions of the Φ^4_3 model [54]. Later, Jagannath and Perkowski gave a simpler argument for the global well-posedness of the Φ^4_3 model [63].
- Shen, Zhua, and Zhu obtained the perturbative expansion for the k-point correlation functions of the Φ^4_2 model on the plane using the stochastic quantization. The same approach is claimed to give the perturbative expansion for the $P(\Phi)_2$ model [94].
- Albeverio and Kusuoka constructed the measure of the Φ^4_3 model of Euclidean quantum field theory using the stochastic quantization and verified rotational invariance, non-Gaussianity and reflection positivity [5, 6]. Note that based on the observation made in [13], reflection positivity in [6] might not hold true.
- Gubinelli and Hofmanová constructed the measure of the Φ^4_3 model of Euclidean quantum field theory via stochastic quantization and verified non-Gaussianity, reflection positivity, translation invariance and exponential integrability. However, rotational invariance and clustering were not addressed. Their construc-

tion also includes the dynamical fractional Φ^4_3 measure, which is proved to be reflection positive [55].

- Duch, Dybalski and Jahandideh constructed the Euclidean $P(\Phi)_2$ model using the stochastic quantization and verified the rotational and translational invariance, integrability, non-Gaussianity, and reflection positivity [37]. This thesis is based on [37].
- Duch, Gubinelli, Rinaldi constructed the measure of the fractional Φ^4_3 model of Euclidean quantum field theory via stochastic quantization and verified non-Gaussianity, reflection positivity, translation invariance and exponential integrability [38].
- Bailleul, Dang, Ferdinand, and Tô constructed the Φ^4_3 measure on an arbitrary 3-dimensional compact Riemannian manifold without boundary using stochastic quantization [12, 14, 15]. To this end, they used the technique introduced by Jagannath–Perkowski in [63] to develop a new Cole-Hopf transform. In particular, the used counter-terms in their construction do not depend on the metric of the underlying Riemannian manifold.
- Bauerschmidt, Dagallier, Weber proved uniqueness of the invariant measure of the Φ^4_2 SPDE up to the critical temperature using the Holley–Stroock approach, which was originally established to verify the uniqueness of invariant measures of Glauber dynamics of lattice spin systems [16].
- Duch, Hairer, Yi, and Zhao characterised the infinite-volume Φ^4_3 measure at high temperature (small coupling) as the unique invariant measure of the associated stochastic equation, and proved that it satisfies all Osterwalder-Schrader axioms, including invariance under translations, rotations, and reflections, as well as exponential decay of correlations [39].

$P(\Phi)_2$ model and the main result

Let us adapt the above setting for a class of polynomial interactions known as the $P(\Phi)_2$ models, which can be seen as the simplest examples in the realm of the interacting QFT's. Fix $n \in 2\mathbb{N}_+$, $n \geq 4$, and a real polynomial $P(\tau) = \sum_{m=0}^{n} a_m \tau^m$, $\tau \in \mathbb{R}$, with $a_0, \ldots, a_{n-1} \in \mathbb{R}$ and $a_n = 1/n$. Observe that such a polynomial is bounded from below. Recall the setting introduced in the beginning of this chapter: Let $M = \mathbb{S}_R$, which is the two-sphere of radius $R \in \mathbb{N}_+$ with the metric tensor g_R induced from the metric on $\mathbb{R}^3$ and with the symmetry group $O(3)$, $\Sigma = \mathbb{R}$, $\Phi(M)$ is replaced with $\mathscr{D}'(\mathbb{S}_R)$ and $\mathrm{vol(dm)} = \rho_R(\mathrm{dx})$, which is invariant under the action of the group $SO(3)$.

Consider the probability measure

$$\mu_R(\mathrm{d}\phi) := \frac{1}{\mathcal{Z}_R} \exp\left(-\int_{\mathbb{S}_R} \lambda :P(\phi(\mathrm{x})): \rho_R(\mathrm{dx})\right) \nu_R(\mathrm{d}\phi), \tag{1.0.4}$$

where $\nu_R := \mathcal{N}(0, G_R)$ indicates the free field Gaussian measure with covariance $G_R := (1-\Delta_R)^{-1}$, $(-\Delta_R)$ is the Laplace-Beltrami operator, $\lambda \in (0,\infty)$ is the coupling constant, $:\bullet:$ refers to the Wick renormalization with respect to the measure ν_R and $\mathcal{Z}_R \in (0,\infty)$ is the partition function. Observe that we used the Gaussian measure ν_R as the reference measure on $\mathscr{D}'(\mathbb{S}_R)$ due to the fact that the Lebesgue measure does not exist in infinite dimensional settings. Note that $(\mathscr{D}'(\mathbb{S}_R), \mathrm{Borel}(\mathscr{D}'(\mathbb{S}_R)), \nu_R)$ forms a probability space. The existence of the measure (1.0.4) is shown by first introducing an ultraviolet (UV) regularized measure $\mu_{R,N}$ and then controlling the limit $N \to \infty$ using the Nelson argument, cf. Ch. 3. Such simple treatment of the UV problem is special to polynomial interactions in $d = 2$ and reflects the fact that the perturbed measure (1.0.4) is absolutely continuous w.r.t. ν_R.

On account of Eq. (1.0.2) one writes the corresponding parabolic SPDE associated to the measure (1.0.4)

$$(\partial_t + \frac{1}{2}Q_R)\,\Phi_R(t,\mathrm{x}) = -\frac{1}{2}\lambda :P'(\Phi_R(t,\mathrm{x})): + \xi(t,\mathrm{x}), \tag{1.0.5}$$

where $(t,\mathrm{x}) \in \mathbb{R}_+ \times \mathbb{S}_R$, $P'(\tau) := \partial_\tau P(\tau)$ and $Q_R := (1-\Delta_R)$. For $R \in \mathbb{N}_+$ we let $\jmath_R : \mathbb{R}^2 \to \mathbb{S}_R \backslash \{(0,0,R)\}$ be the stereographic projection and $\jmath_R^* : \mathscr{D}'(\mathbb{S}_R) \to \mathscr{S}'(\mathbb{R}^2)$ the corresponding pull-back of distributions. We denote by $\jmath_R^* \sharp \mu_R$ the measure on $\mathscr{S}'(\mathbb{R}^2)$ obtained by the push-forward of μ_R with $\jmath_R^*$. We shall make use of the SPDE (1.0.5) and the intertwining property of the map $\jmath_R^*$ in order to prove the existence of limit points of the measures $(\jmath_R^* \sharp \mu_R)_{R \in \mathbb{R}_+}$ on $\mathscr{S}'(\mathbb{R}^2)$ and to verify that such limiting measures satisfy the Osterwalder-Schrader axioms. The following theorem is our main result.

Theorem 1.0.3. *The sequence of measures $(\jmath_R^* \sharp \mu_R)_{R \in \mathbb{N}_+}$ on $\mathscr{S}'(\mathbb{R}^2)$ has a weakly convergent subsequence. Every accumulation point μ of $(\jmath_R^* \sharp \mu_R)_{R \in \mathbb{N}_+}$ is invariant under the Euclidean symmetries of the plane and reflection positive. Moreover, there exists a ball $B \subset \mathscr{S}(\mathbb{R}^2)$ with respect to some Schwartz semi-norm centered at the origin such that for all $f \in B$ it holds*

$$\int \exp(\phi(f)^n)\,\mu(\mathrm{d}\phi) \leq 2\,. \tag{1.0.6}$$

Remark 1.0.4. *By the above theorem μ satisfies all the Osterwalder-Schrader axioms of Def. 1.0.1 possibly with the exception of the cluster property OS4. It is known that the $P(\Phi)_2$ measure on the plane is unique provided $\lambda \in (0, \infty)$ is sufficiently small [48]. In general uniqueness does not hold and the model exhibits phase transitions [49, Ch. 16]. The cluster property is only expected to hold in pure phases. Our construction of the $P(\Phi)_2$ measure does not need any smallness assumption on λ. However, it does not give any information about the uniqueness of the infinite volume limit or the decay of correlation functions. In what follows we set $\lambda = 1$.*

Proof. The existence of a weakly convergent subsequence of $(\jmath_R^* \sharp \mu_R)_{R \in \mathbb{N}_+}$ follows from tightness and Prokhorov's theorem. The proof of tightness is presented in Sec. 5.2 and uses parabolic stochastic quantization combined with a PDE energy estimate. The invariance of μ under the Euclidean symmetries is established in Sec. 6.4 and is based on the fact that for all $R \in \mathbb{N}_+$ the measure μ_R is invariant under the action of the orthogonal group $O(3)$. The proof that μ is reflection positive is given in Sec. 6.3 and is based on the fact that for all $R \in \mathbb{N}_+$ the measure μ_R is reflection positive. The bound (1.0.6) is proved in Prop. 6.1.1 with the help of the Hairer-Steele argument [61]. □

Plan of the thesis

In Ch. 2 we provide the required background to comprehend the ideas, which will be utilized in the sequel. Sec. 2.3 is an extended version of [37, App. A]. Lemmas 2.4.10, 2.5.14 correspond to [37, Lemmas B.1, B.4], respectively.

In Ch. 3 we introduce the regularized $P(\Phi)_2$ measure $\mu_{R,N}$ on the sphere $\mathbb{S}_R$ with a certain UV cutoff $N \in \mathbb{N}_+$ in the frequency space. We shall write $\mu_{R,N}$ as a perturbation of the regularized Gaussian measure $\nu_{R,N}$ by the $P(\Phi)_2$ interaction. In Sec. 3.4 we verify that the measure $\mu_{R,N}$ converges weakly, as $N \to \infty$, to the probability measure μ_R formally given by Eq. (1.0.4). We use the Nelson hypercontractivity estimate for this purpose. We also investigate a certain auxiliary measure $\mu^g_{R,N}$, $g \in \mathscr{S}(\mathbb{R}^2)$, which coincides with $\mu_{R,N}$ when $g = 0$ and will be used in Ch. 6. Ch. 3 extends [37, Sec. 2].

In Ch. 4 we study the stochastic quantization equations of the measures $\mu^g_{R,N}$ and $\nu_{R,N}$ using the Da Prato-Debussche trick. We show the local and global in time existence of solutions with fixed $R, N \in \mathbb{N}_+$ in Sec. 4.3.1 and in Sec. 4.3.2, respectively. We utilize the Banach fixed point theorem and an appropriate Sobolev embedding to verify the local existence. The global existence will be shown with the help of the energy technique. The fact that the measures $\mu^g_{R,N}$ and $\nu_{R,N}$ are invariant for the associated semi-groups is the subject of Sec. 4.4. This chapter is largely independent of [37] as these topics were only briefly addressed in Sec. 3 of this latter publication.

In Ch. 5, we apply the energy technique to prove a certain a priori bound, which is uniform in $R, N \in \mathbb{N}_+$. This a priori bound will be used in order to verify that the sequence of measures $(\jmath_R^* \sharp \mu_R^g)_{R \in \mathbb{N}_+}$ on $\mathscr{S}'(\mathbb{R}^2)$ is weakly converging along a subsequence. This chapter expands on [37, Secs. 5,6].

In Ch. 6, we investigate the Osterwalder-Schrader axioms. Sec. 6.2 is designated for verifying regularity. Sec. 6.3 is devoted to the proof of reflection positivity. In Sec. 6.4 we use the intertwining property of the map $\jmath_R^*$ to show that an infinite volume $P(\Phi)_2$ measure is invariant under translations, rotations and reflections of the plane. The proof relies on the invariance of the finite volume measure μ_R on $\mathscr{D}'(\mathbb{S}_R)$ under the action of the orthogonal group $O(3)$, combined with the fact that the symmetry groups of $\mathbb{S}_R$ and $\mathbb{R}^2$ have the same dimension. This chapter extends [37, Secs. 7,8,9].

Appendix A provides some well-known facts on Bessel potential. In Appendix B, which extends [37, Apps. B,C], we prove some bounds for the Wick ordered fields, which are essential for proving the local and global in time existence of the stochastic random fields in Ch. 4 as well as to prove tightness in Ch. 5. In Appendices C, D, and E we provide the complete proofs of Lemma 4.3.9, Lemma 5.1.3, and Remark 6.4.15. These proofs were largely left for the reader in [37].

Preliminaries

In this section, we collect various technical results on a 2-dimensional sphere of radius $R \in [1, \infty)$, i.e., $\mathbb{S}_R$. We gather some necessary facts on spherical harmonics in Sec. 2.2. In Sec. 2.3, we define function spaces on $\mathbb{R}^2$ and $\mathbb{S}_R$ and collect some useful Sobolev embeddings. We summarize all required functional analysis techniques in Sec. 2.4. Sec. 2.5 contains a review of all the essential techniques in probability. Sec. 2.7 is meant to provide the necessary background on the stochastic differential equations.

2.1 2-dimensional sphere

Let $R \in [1, \infty)$. We define a two-dimensional sphere of radius R, i.e., $(\mathbb{S}_R, g_R)$ as a regular surface of the Euclidean space $\mathbb{R}^3$ via

$$\mathbb{S}_R := \{\mathrm{x} = (\mathrm{x}_1, \mathrm{x}_2, \mathrm{x}_3) \in \mathbb{R}^3 \,|\, \mathrm{x}_1^2 + \mathrm{x}_2^2 + \mathrm{x}_3^2 = R^2\},$$

where g_R stands for the metric tensor on $\mathbb{S}_R$ induced from the metric on $\mathbb{R}^3$. We let $N = (0, 0, R)$ be the north pole of $\mathbb{S}_R$. Note that a two-dimensional sphere $\mathbb{S}_R$ represents a Riemannian manifold with the Ricci scalar curvature $2/R^2$ [24, Eq. (3.158)] [1]. We let $\rho_R(\mathrm{dy}) = \sqrt{\det g_R}\,\mathrm{dy}$ denote the Riemannian volume measure [57, Thm 3.11] or [4, Def. 1.74] such that $\int_{\mathbb{S}_R} \rho_R(\mathrm{dy}) = 4\pi R^2$. Moreover, $\|\bullet\|_{L_p(\mathbb{S}_R)}$ denotes the L_p-norm on $\mathbb{S}_R$ with respect to the canonical measure $\rho_R(\mathrm{dx})$ for all $p \in [1, \infty)$.

The symmetry group of a 2-sphere, including reflections, is known as the orthogonal group $O(3)$ [19, Prop. 18.5.2]. The symmetry group consisting of only rotations is the special orthogonal group $SO(3)$. $O(3)$ and $SO(3)$ act transitively[2] as groups of isometries of $\mathbb{S}_R$ and they give rise to groups of unitary operators on $L_2(\mathbb{S}_R)$. Note that for all $R \in \mathbb{N}_+$ the measure $\rho_R(\mathrm{dx})$ is invariant under the action of the group

[1] The sectional curvature of an d-sphere of radius R is $1/R^2$ and its Ricci scalar curvature is $d(d-1)/R^2$.

[2] Namely, for all $\mathrm{x}, \mathrm{y} \in \mathbb{S}_R$ there exists $Q \in SO(3)$ such that $Q \cdot \mathrm{x} = \mathrm{y}$ [18, Def. 1.4.3].

$SO(3)$. The transitive action of $SO(3)$ makes $\mathbb{S}_R$ a homogeneous space[3], i.e., for all $R \in \mathbb{N}_+$ the two sphere $\mathbb{S}_R$ can be identified as a quotient space $SO(3)/SO(2)$ [4].

The tangent space of $\mathbb{S}_R$ at $\mathrm{x} \in \mathbb{S}_R$, i.e., $\mathrm{T_x}\mathbb{S}_R$ is the orthogonal complement of x, i.e., $\mathrm{T_x}\mathbb{S}_R = \mathrm{x}^\perp = \{v \in \mathbb{R}^3 \,|\, \langle \mathrm{x}, v\rangle = 0\}$, where $\langle \mathrm{x}, v\rangle = \sum_{i=0}^{2} \mathrm{x}_i \cdot v_i$ denotes the inner product on $\mathbb{R}^3$. Note that if v, w is an orthonormal basis of $\mathrm{T_x}\,\mathbb{S}_R$, then x, v, w is an orthonormal basis of $\mathbb{R}^3$.

Remark 2.1.1. *From the fact that for all $n \geq 1$ the determinant on $O(n)$ gives rise to a short exact sequence $\mathbb{1} \to SO(n) \xrightarrow{\imath} O(n) \xrightarrow{\det} \mathbb{Z}_2 \to \mathbb{1}$, one infers that $O(n) \cong SO(n) \rtimes_\varphi \mathbb{Z}_2$, where the map $\varphi : \mathbb{Z}_2 \to \mathrm{Aut}(SO(n))$ defined by $\varphi(1)(Q) = Q$ and $\varphi(-1)(Q) = \mathcal{R}\, Q\, \mathcal{R}^{-1}$ for $Q \in SO(n)$ and $\mathcal{R} \in O(n)$ with $\det \mathcal{R} = -1$.*

Remark 2.1.2. *Let $N = (0, 0, R)$ be the north pole of $\mathbb{S}_R$. Observe that $N^\perp = \mathrm{T}_N\mathbb{S}_R = \{(x_1, x_2, 0) \in \mathbb{R}^3\}$. This implies that the isotropy group of N is isomorphic to $SO(2)$. More precisely, the subgroup of $SO(3)$ fixing N is a copy of $SO(2)$, which stabilizes the x_1x_2-plane. In Sec. 6.4 this fact will be used to establish a one-to-one correspondence between the action of the orthogonal group $O(3)$ on $\mathbb{S}_R$ and the action of the Euclidean group $T(2) \rtimes O(2)$ on $\mathbb{R}^2$, see Remark 3.1.2.*

Remark 2.1.3. *Let $p : SO(3) \to SO(3)/SO(2)$ denote the canonical projection from the special orthogonal group $SO(3)$ onto the homogeneous space $SO(3)/SO(2)$, which is smooth. The differential of p at the identity $\mathrm{e} \in SO(3)$ is the linear map $\mathrm{T_e}\, p : so(3) \to \mathrm{T}_{[\mathrm{e}]}(SO(3)/SO(2))$, where $[\mathrm{e}] = p(\mathrm{e})$. Note that both p and $\mathrm{T_e}\, p$ are surjective. This implies that $\mathrm{T}_{[\mathrm{e}]}(SO(3)/SO(2))$ is isomorphic to $so(3)/so(2)$. One shows that $\mathrm{T}_{[g]}(SO(3)/SO(2))$ is also isomorphic to $so(3)/so(2)$, where $[g] \in SO(3)/SO(2)$ is arbitrary. This is due the fact that $SO(3)/SO(2)$ is a homogeneous space.*

Definition 2.1.4. *For all $\mathrm{x}, \mathrm{y} \in \mathbb{S}_R$, we set the geodesic distance on $\mathbb{S}_R$ by*

$$\mathrm{dist}(\mathrm{x}, \mathrm{y}) := d_R(\mathrm{x}, \mathrm{y}) = R\,\theta(\mathrm{x}, \mathrm{y})\,, \quad \theta(\mathrm{x}, \mathrm{y}) = \arccos \frac{(\mathrm{x} \cdot \mathrm{y})}{R^2} \in [0, \pi].$$

[3] The set X is called a homogeneous space under G (for the action ϕ) if (G, X, ϕ) is transitive [18, Def. 1.5.4].

[4] To see this, fix a vector in $\mathbb{R}^3$ such that its orbit under the $SO(3)$-action gives rise to the canonical embedding $\mathbb{S}_1 \hookrightarrow \mathbb{R}^3$. Moreover, the subgroup $SO(2)$, embedded inside $SO(3)$, stabilizes the axis formed by this vector. Hence, by the Orbit-Stabilizer Theorem we have the identification $\mathbb{S}_1 \equiv SO(3)/SO(2)$.

In fact, the geodesic distances on a unit sphere are simply given by arc lengths, angles between two arbitrary vectors, from the origin in the ambient Euclidean space [67, p. 82]. In particular, one has $0 \le d_R(\mathrm{x}, \mathrm{y}) \le R\pi$. *Moreover, it holds*

$$\|\mathrm{x} - \mathrm{y}\| = \sqrt{2R^2 - 2R^2 \cos(\frac{\mathrm{x} \cdot \mathrm{y}}{R^2})} = 2R \sin \frac{d_R(\mathrm{x}, \mathrm{y})}{2R} ,$$

where $\|\bullet\|$ *denotes the Euclidean norm on* $\mathbb{R}^3$. *This implies that* $\frac{2}{\pi} d_R(\mathrm{x}, \mathrm{y}) \le \|\mathrm{x}-\mathrm{y}\| \le d_R(\mathrm{x}, \mathrm{y})$, *which implies that the two distances are comparable.*

Laplace-Beltrami operator

Let (M, g) be a $d-$dimensional Riemannian manifold with local coordinates $\mathrm{x} = (\mathrm{x}_1, \mathrm{x}_2, \cdots, \mathrm{x}_d)$. The Laplace–Beltrami operator is given by [57, Eq. (3.40)]

$$\Delta_g = \frac{1}{\sqrt{|g|}} \sum_{i,j=1}^{d} \frac{\partial}{\partial \mathrm{x}_j} (\sqrt{|g|}\, g^{ij} \frac{\partial}{\partial \mathrm{x}_i}) ,$$

where $\mathrm{d}s^2 = \sum_{i,j=1}^{d} g_{ij}\, \mathrm{dx}^i\, \mathrm{dx}^j$ and $|g| = |\det(g)|$. In particular, in the Euclidean space $\mathbb{R}^3$ with the standard metric $g_{ij}(x) = \delta_{ij}$ for all $x \in \mathbb{R}^3$, one has $\Delta_{\mathbb{R}^3} = \sum_{i=1}^{3} \partial^2/\partial x_i^2$, which in the spherical coordinates is of the following form

$$\Delta_{\mathbb{R}^3} = \sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2} \equiv \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_{\mathbb{S}_1}, \tag{2.1.1}$$

where $\Delta_{\mathbb{S}_1}$ is the Laplace-Beltrami operator on the unit sphere with the usual round metric. It is often written $-L^2$ in physics texts. More precisely, in spherical coordinates (θ, φ), one has

$$\Delta_{\mathbb{S}_1} = \frac{1}{\sin\theta} \frac{\partial}{\partial \theta}(\sin\theta \frac{\partial}{\partial \theta}) + \frac{1}{\sin\theta^2} \frac{\partial^2}{\partial \varphi^2} .$$

We denote the Laplacian on $\mathbb{S}_R$ by $(-\Delta_R)$ such that $\Delta_R = \Delta_{\mathbb{S}_1}/R^2$.

Example 2.1.5. *Let* (M, g) *be a* $d-$*dimensional manifold. Consider the conformal deformation* $\tilde{g}$ *associated to the metric* g, *i.e.,* $\tilde{g} = \mathrm{e}^{2\nu} g$, *where* $\nu \in C^\infty(M)$. *One has*

$$\tilde{g}_{i,j} = \mathrm{e}^{2\nu}\, g_{ij}, \quad \tilde{g}^{ij} = \mathrm{e}^{-2\nu}\, g^{ij}, \quad \sqrt{\det \tilde{g}} = \mathrm{e}^{d\nu} \sqrt{\det g} .$$

Hence,

$$\Delta_{\tilde g} = \frac{1}{e^{d\nu}\sqrt{\det g}} \sum_{i,j=1}^{d} \frac{\partial}{\partial \mathrm{x}_j}\Big(e^{(d-2)\nu}\sqrt{\det g}\, g^{ij}\frac{\partial}{\partial \mathrm{x}_j}\Big)$$
$$= e^{-2\nu}\Delta_g + (d-2)e^{-2\nu}\, g^{ij}\,\frac{\partial \nu}{\partial \mathrm{x}_i}\,\frac{\partial}{\partial \mathrm{x}_j}\,.$$

In particular, if $(M,\tilde g)$ is a 2-dimensional Riemannian manifold with a metric given by $(\tilde g)_{jk}(x) = e^{2\nu(x)}\,\delta_{jk}$ it holds

$$\Delta_{\tilde g} f = e^{-2\nu}\sum_{j,k=1}^{2}\frac{\partial}{\partial \mathrm{x}_j}(e^{-2\nu}\,\delta^{jk}\,e^{2\nu}\frac{\partial}{\partial \mathrm{x}_k} f) = e^{-2\nu}\Delta f\,,$$

where $\Delta V = \frac{\partial^2 V}{\partial x_1^2} + \frac{\partial^2 V}{\partial x_2^2}$ is the flat Laplacian [97, App. C, Cor. 3.5]. We shall make use of this fact once we introduce the stereographic projection.

Remark 2.1.6. *For all $R \in \mathbb{N}_+$ we have $\Delta_R = \vec{\nabla}_R \cdot \vec{\nabla}_R$, where $\vec{\nabla}_R$ is the gradient on the sphere. For any $f, g \in C^2(\mathbb{S}_R)$, by the Green–Beltrami identity one has [9, Prop. 3.3 and Cor. 3.4] or [57, Thm 3.16]*

$$\int_{\mathbb{S}_R} g(\mathrm{x})\,(\Delta_R f)(\mathrm{x})\,\rho_R(d\mathrm{x}) = -\int_{\mathbb{S}_R} (\vec{\nabla}_R g)(\mathrm{x})\cdot(\vec{\nabla}_R f)(\mathrm{x})\,\rho_R(d\mathrm{x}) = \int_{\mathbb{S}_R} f(\mathrm{x})(\Delta_R g)(\mathrm{x})\,\rho_R(d\mathrm{x}).$$

Let $\langle \bullet, \bullet\rangle_{L_2(\mathbb{S}_R)}$ denote the inner product on $L_2(\mathbb{S}_R)$. It holds,

$$\langle f, \Delta_R\, g\rangle_{L_2(\mathbb{S}_R)} = \langle \Delta_R\, f, g\rangle_{L_2(\mathbb{S}_R)}, \quad \langle f, \Delta_R\, f\rangle_{L_2(\mathbb{S}_R)} \le 0\,.$$

In general, on a complete Riemannian manifold M, the Laplace-Beltrami operator is essentially self-adjoint [96, Thm. 2.4] or [57, Theorem. 11.5], i.e., it has a unique self-adjoint extension.

Stereographic projection

Definition 2.1.7. *Let $R \in [1,\infty)$ and $N = (0,0,R)$. Define the map $\jmath_R : \mathbb{R}^2 \to \mathbb{S}_R \setminus N \subset \mathbb{R}^3$ by*

$$\jmath_R(x_1,x_2) := \mathrm{x} \equiv (\mathrm{x}_1,\mathrm{x}_2,\mathrm{x}_3) = \frac{R(4Rx_1, 4Rx_2, x_1^2 + x_2^2 - 4R^2)}{x_1^2 + x_2^2 + 4R^2}\,.$$

Note that $-R \le \mathrm{x}_3 < R$. We call the point $(x_1,x_2) = x \in \mathbb{R}^2$ the stereographic coordinate of the point $\mathrm{x} \in \mathbb{S}_R \setminus N$. Note that $\jmath_R$ is always locally invertible. Let

$\jmath_R^{-1} : (\mathbb{S}_R \setminus N) \subset \mathbb{R}^3 \to \mathbb{R}^2$. *It holds*

$$\jmath_R^{-1}(\mathrm{x}_1, \mathrm{x}_2, \mathrm{x}_3) := x \equiv (x_1, x_2) = \frac{2R(\mathrm{x}_1, \mathrm{x}_2)}{R - \mathrm{x}_3}\,.$$

Moreover, for $x, y \in \mathbb{R}^2$ *such that* $\jmath_R(x) = \mathrm{x}$ *and* $\jmath_R(y) = \mathrm{y}$ *one has*

$$\sum_{i=1}^{3}(\mathrm{x}_i - \mathrm{y}_i)^2 = \|\mathrm{x} - \mathrm{y}\|^2 = \frac{16R^4}{(4R^2 + |x|^2)(4R^2 + |y|^2)}\,\|x - y\|^2\,.$$

Observe that for all $R \in \mathbb{N}_+$ *the mappings* $\jmath_R$ *and* $\jmath_R^{-1}$ *are rational functions, hence, are smooth. Thus,* $\jmath_R$ *is a local diffeomorphism.*

Recall the Riemannian volume measure was given by $\rho_R(\mathrm{dy}) = \sqrt{\det g_R}\,\mathrm{dy}$. One verifies that the metric tensor g_R on $\mathbb{S}_R$ in the stereographic coordinates is of the form

$$\jmath_R^*\,(g_R)_{i,j}(x) := w_R(x)\,\delta_{i,j} \tag{2.1.2}$$

with $w_R(x) := \sqrt{\det \jmath_R^* g_R(x)} = 16R^4/(4R^2 + x_1^2 + x_2^2)^2 \in C^\infty(\mathbb{R}^2)$ [90, p.12]. This implies that $\mathbb{S}_R$ is locally conformally flat[5]. Note that for all $R \in \mathbb{N}_+$ and all $x \in \mathbb{R}^2$ it holds $w_R(x) < 1$.

Remark 2.1.8. *Let* $f \in C(\mathbb{S}_R)$*, then* $\jmath_R^* f = f \circ \jmath_R \in C(\mathbb{R}^2)$*. Note that for* $f \in C_{\mathrm{c}}^\infty(\mathbb{R}^2)$ *the function* $f \circ \jmath_R^{-1} \in C_{\mathrm{c}}(\mathbb{S}_R \setminus N)$ *has unique continuous extension to* $\mathbb{S}_R$*. From Remark 2.5.15 Items (A),(B) for all* $f \in C^\infty(\mathbb{S}_R \setminus N)$ *it holds*

$$\int_{\mathbb{R}^2} \jmath_R^* f(x)\, w_R(x)\,\mathrm{d}x = \int_{\mathbb{R}^2} f \circ \jmath_R(x)\, w_R(x)\,\mathrm{d}x = \int_{\mathbb{S}_R} f(\mathrm{x})\,\rho_R(\mathrm{dx})\,.$$

Moreover, by Remark 2.5.15 Item (C) for $\phi \in \mathscr{D}'(\mathbb{S}_R)$*,* $f \in C_{\mathrm{c}}^\infty(\mathbb{R}^2)$ *one has*

$$\begin{aligned}\langle \jmath_R^*\phi, f\rangle_{L_2(\mathbb{R}^2)} &= \int_{\mathbb{R}^2} \jmath_R^*\phi(x)\, f(x)\,\mathrm{d}x \\ &= \int_{\mathbb{S}_R} \phi(\mathrm{y})\, f(\jmath_R^{-1}(\mathrm{y}))\, w_R^{-1}(\jmath_R^{-1}(\mathrm{y}))\,\rho_R(\mathrm{dy}) = \langle \phi, (w_R^{-1} f) \circ \jmath_R^{-1}\rangle_{L_2(\mathbb{S}_R)}\,.\end{aligned}$$

Remark 2.1.9. *Let* $\hat{\Delta} := \jmath_R^*\Delta_R$ *denote the Laplacian associated to the metric* $\jmath_R^* g_R$ *on* $\mathbb{R}^2$*. Using Example 2.1.5 one gets* $\hat{\Delta} = w_R^{-1}\Delta$*, where* Δ *denotes the Laplacian on* $\mathbb{R}^2$*. Alternatively, one can use [78, Sec. 1] with* $g_1 = \jmath_R^* g_R, g = \delta_{i,j}, f = \frac{1}{2}\log(w_R(x))$ *and* $n = 2$*. It holds* $\jmath_R^*\,\Delta_R = w_R^{-1}\Delta\,\jmath_R^*$*.*

[5] A Riemannian manifold (M, g) is said to be locally conformally flat if it is locally conformal to Euclidean space. Recall that a conformal structure on M is an equivalence class of metrics with $\hat{g} = \mathrm{e}^{2\omega}\, g$ for some function $\omega \in C^\infty(M)$.

2.2 Spherical harmonics

Let $\mathbb{Y}_l(\mathbb{R}^d)$ be the space of the homogeneous harmonics of degree l in $\mathbb{R}^d$, and $\mathbb{Y}_l^d := \mathbb{Y}_l(\mathbb{R}^d)|_{\mathbb{S}_1^{d-1}}$ be the spherical harmonic space of order l in a unit $d-1$-dim sphere. We set $\{Y_{l,m} : 1 \le m \le N_{l,d}\}$ to be an orthonormal basis of $\mathbb{Y}_l^d$, i.e.,

$$\int_{\mathbb{S}_1^{d-1}} Y_{l,j}(\eta)\, \overline{Y_{l',k}(\eta)}\, \mathrm{d}\eta = \delta_{ll'}\, \delta_{jk}, \quad 1 \le j,k \le N_{l,d}\,,$$

where $\mathrm{d}\eta$ is the Riemannian volume measure on $\mathbb{S}_1^{d-1}$ and $N_{l,d} = \dim \mathbb{Y}_l^d$ such that for l sufficiently large $\dim \mathbb{Y}_l^d = O(l^{d-2})$. Given $Y_l \in \mathbb{Y}_l^d$ and $H_l \in \mathbb{Y}_l(\mathbb{R}^d)$, it holds $H_l(r\mathrm{x}) = r^l Y_l(\mathrm{x})$ for all $\mathrm{x} \in \mathbb{S}_1^{d-1}$ and $r > 0$ [9, Sec. 2.1.3]. For $d = 3$, using Eq. (2.1.1) one gets

$$\Delta H_l(r\mathrm{x}) = \Delta(r^l Y_l)(\mathrm{x}) = 0 \quad \text{iff} \quad \Delta_{\mathbb{S}_1} Y_l(\mathrm{x}) = -l(l+1)\, Y_l(\mathrm{x})\,.$$

Recall that $\Delta_R = \Delta_{\mathbb{S}_1}/R^2$. The above expression implies that the Laplace-Beltrami operator $(-\Delta_R)$ has non-negative discrete spectrum and an isolated simple eigenvalue at zero, i.e., $0 = \lambda_0 < \lambda_1 \le \lambda_2 \le \cdots \to \infty$. Moreover, the non-zero functions in the space $\mathbb{Y}_l^3$ are the eigenfunctions of $(-\Delta_R)$ with eigenvalue $l(l+1)/R^2$ for all $l \in \mathbb{N}_0$ and with the multiplicity $N_{l,3} = \dim \mathbb{Y}_l^3 = (2l+1)$ [6] [30, Thm. 1.4.5]. Note that each $\mathbb{Y}_l^3$ is the eigenspace of $(-\Delta_R)$, which is preserved by the action of the group $O(3)$.

Remark 2.2.1. *Observe that the first non-zero eigenvalue is $\lambda_1 = 2/R^2$, which can be deemed as two (the dimension of $\mathbb{S}_R$) times the sectorial curvature $1/R^2$. This is an instance of a more general result known as the Lichnerowicz-Obata theorem [68] or [69, Thm. 5.1].*

Legendre polynomials

The Legendre polynomial of degree l in $d-$dimensions can be determined as the restriction of the Legendre harmonic on the unit sphere via [9, Eq. (2.19)]

$$P_{l,d}(t) = l!\,\Gamma(\frac{d-1}{2}) \sum_{k=0}^{\lfloor l/2 \rfloor} (-1)^k \frac{(1-t^2)^k\, t^{l-2k}}{4^k\, k!\,(l-2k)!\,\Gamma(k+(d-1)/2)}\,.$$

[6] In general, the eigenvalues of the (negative) Laplace-Beltrami operator on $\mathbb{S}_1^{d-1}$ are $l^2 + (d-2)l$ where $l \in \mathbb{N}_0$. The multiplicity of this eigenvalue is $\frac{(l+d/2-1)\prod_{j=1}^{d-3}(l+j)}{(d/2-1)\cdots(d-3)!}$ [89, Thm. 2.9].

In particular, for all $l \in \mathbb{N}_0, d \geq 2, t \in [-1,1]$ it holds [9, Eq. (2.116)]

$$|P_{l,d}(t)| \leq 1 = P_{l,d}(1)\,. \tag{2.2.1}$$

The Legendre polynomial $P_{l,d}$ satisfies the following differential equation [9, Eq. (2.82)] or [8, (15.75)]

$$(1-t^2)^{\frac{3-d}{2}} \frac{\mathrm{d}}{\mathrm{d}t}\left[(1-t^2)^{\frac{d-1}{2}} \frac{\mathrm{d}}{\mathrm{d}t} P_{l,d}(t)\right] + l(l+d-2)P_{l,d}(t) = 0\,.$$

Using the preceding expression with $t = \cos\theta$ and $d = 3$ one has

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\cos\theta\, P_l(\cos\theta)\right) - \frac{\mathrm{d}^2}{\mathrm{d}\theta^2}\left(\sin\theta\, P_l(\cos\theta)\right) = l(l+1)P_l(\cos\theta)\sin\theta, \tag{2.2.2}$$

where P_l denotes the $l-$th Legendre polynomial in $d = 3$.

Remark 2.2.2. *Let $j \in \mathbb{N}$. We denote the derivatives of $t \mapsto P_l(t)$ with respect to $t \in [-1,1]$ of order j by $t \mapsto P_l^j(t)$. It holds that [9, Eqs. (2.118–120)]*

$$\max_{t\in[-1,1]} |P_l^j(t)| = O(l^{2j})\,. \tag{2.2.3}$$

Utilizing the mean value theorem for any $t, s \in [-1,1]$ and some τ between t and s one obtains

$$|P_l(t) - P_l(s)| = |P_l^1(\tau)(t-s)| \leq c\, l^2\, |t-s|,$$

where we used Eq. (2.2.3) with $j = 1$. This implies that

$$|P_l(\mathrm{x}\cdot\mathrm{y}/R^2) - P_l(\mathrm{z}\cdot\mathrm{y}/R^2)| \leq c\, l^2\, |\mathrm{x}-\mathrm{z}|/R, \qquad \forall\ \mathrm{x},\mathrm{y},\mathrm{z} \in \mathbb{S}_R\,.$$

Remark 2.2.3. *For all $\mathrm{x},\mathrm{y} \in \mathbb{S}_R$ it holds that [8, Eq. (16.57)], [9, Thm. 2.9], [30, Eq. (1.6.7)]*

$$\sum_{m=-l}^{l} Y_{l,m}(\mathrm{x})\overline{Y}_{l,m}(\mathrm{y}) = \frac{(2l+1)}{4\pi R^2}\, P_l(\frac{\mathrm{x}\cdot\mathrm{y}}{R^2}) = (2l+1)\mathcal{P}_{R,l}(\mathrm{x},\mathrm{y}), \tag{2.2.4}$$

where $\mathcal{P}_{R,l} : L_2(\mathbb{S}_R) \to L_2(\mathbb{S}_R)$ is such that $(2l+1)\mathcal{P}_{R,l}$ is the orthogonal projection onto the eigenspace $\mathbb{Y}_l^3$ of the operator $(-\Delta_R)$. In particular, one obtains

$$\sum_{m=-l}^{l} Y_{l,m}(\mathrm{x})\overline{Y_{l,m}}(\mathrm{x}) = \frac{(2l+1)}{4\pi R^2}\, P_l(1) = \frac{(2l+1)}{4\pi R^2}\,.$$

This gives rise to $\max\{|Y_{l,m}(\mathrm{x})| : \mathrm{x} \in \mathbb{S}_R\,,\, -l \leq m \leq l\} \leq \sqrt{(2l+1)/(4\pi R^2)}$.

Remark 2.2.4. *Let $f \in L_2(\mathbb{S}_R)$. One has [9, Thm 2.38]*

$$f(\mathrm{x}) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} f_{lm}\, Y_{l,m}(\mathrm{x}) = \int_{\mathbb{S}_R} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} f(\mathrm{y}) \overline{Y_{l,m}}(\mathrm{y})\, Y_{l,m}(\mathrm{x})\, \rho_R(\mathrm{dy})\,.$$

Moreover, the projection of $f \in L_2(\mathbb{S}_R)$ into $\mathbb{Y}_l^3$ can be read off as

$$(2l+1)\,(\mathcal{P}_{R,l} f)(\mathrm{x}) := f_l(\mathrm{x}) = \frac{2l+1}{4\pi R^2} \int_{\mathbb{S}_R} f(\mathrm{y})\, P_l(\mathrm{x}\cdot\mathrm{y}/R^2)\, \rho_R(\mathrm{dy}),$$

where we used Eq. (2.2.4). As a result, we have the Parseval equality on $L_2(\mathbb{S}_R)$, i.e., $\|f\|^2_{L_2(\mathbb{S}_R)} = \sum_{l=0}^{\infty} \|f_l\|^2_{L_2(\mathbb{S}_R)}$, which is a consequence of the fact that $L_2(\mathbb{S}_R) = \bigoplus_{l=0}^{\infty} \mathbb{Y}_l^3$ [9, Eq. (2.143)].

Remark 2.2.5. *It holds [9, Eq. 2.79]*

$$\int_{-1}^{1} P_{l_1}(t)\, P_{l_2}(t)\, \mathrm{d}t = \frac{2}{2l_1+1}\, \delta_{l_1,l_2}\,.$$

2.3 Function spaces

Definition 2.3.1. *[101, Def. 6.1] We say that $w \in C^\infty(\mathbb{R}^d)$ is an admissible weight iff there exist $b \in [0,\infty)$ and $c \in (0,\infty)$ such that $0 < w(x) \leq c\, w(y)\, (1+|x-y|)^b$ for all $x,y \in \mathbb{R}^d$ and for every $a \in \mathbb{N}_0^d$ there exists $c_a \in (0,\infty)$ such that $|\partial^a w(x)| \leq c_a\, w(x)$ for all $x \in \mathbb{R}^d$.*

Definition 2.3.2. *(A) Let $w \in C^\infty(\mathbb{R}^d)$ be an admissible weight, $p \in [1,\infty]$ and $\alpha \in \mathbb{R}$, $n \in \mathbb{N}_0$. By definition $L_p(\mathbb{R}^d, w)$ is the Banach space with the norm $\|f\|_{L_p(\mathbb{R}^d,w)} := \|wf\|_{L_p(\mathbb{R}^d)}$ [101, Eq. 6.5].*

(B) The weighted Bessel potential space $L_p^\alpha(\mathbb{R}^d, w)$ is the Banach space with the norm $\|f\|_{L_p^\alpha(\mathbb{R}^d,w)} := \|(1-\Delta)^{\alpha/2} f\|_{L_p(\mathbb{R}^d,w)}$. We also set $L_p^\alpha(\mathbb{R}^d) = L_p^\alpha(\mathbb{R}^d, 1)$.

(C) The weighted Sobolev space $W_p^n(\mathbb{R}^d, w)$ is the Banach space with the norm

$$\|f\|_{W_p^n(\mathbb{R}^d,w)} = \textstyle\sum_{a\in\mathbb{N}^d, |a|\leq n} \|\partial^a f\|_{L_p(\mathbb{R}^d,w)}\,.$$

Definition 2.3.3. *Let $\|\bullet\|_{L_p(\mathbb{S}_R)}$ denote the L_p-norm on $\mathbb{S}_R$ defined with respect to the canonical measure $\rho_R(\mathrm{dx})$. Note that for all $0 < q \leq p \leq \infty$ and all $R \in \mathbb{N}_+$ there are $C \in (0,\infty)$ such that $\|\bullet\|_{L_q(\mathbb{S}_R)} \leq C\, \|\bullet\|_{L_p(\mathbb{S}_R)}$. For $R \in (0,\infty)$ we define the Bessel potential space $L_p^\alpha(\mathbb{S}_R)$ on $\mathbb{S}_R \subset \mathbb{R}^3$ as the Banach space with the norm $\|f\|_{L_p^\alpha(\mathbb{S}_R)} := \|(1-\Delta_R)^{\alpha/2} f\|_{L_p(\mathbb{S}_R)}$.*

Definition 2.3.4. *[99, Def. 2.3.1] Let $\Phi(\mathbb{R}^d)$ be a collection of functions $\{\varphi_j\}_{j\in\mathbb{N}_0} \subset \mathscr{S}(\mathbb{R}^d)$ such that for some $A, B, C > 0$ it holds*

$$\operatorname{supp}\varphi_0 \subset \{x \in \mathbb{R}^d \,:\, |x| < A\}, \quad \operatorname{supp}\varphi_j \subset \{x \in \mathbb{R}^d \,:\, B\,2^{j-1} \le |x| \le C\,2^{j+1}\}.$$

Moreover, for all $\alpha \in \mathbb{N}_0^n$ there exists $c_\alpha > 0$ such that for all $x \in \mathbb{R}^d$ one has

$$\sup_{x\in\mathbb{R}^d}\ \sup_{j=0,1,2\cdots} 2^{j|\alpha|}\,|\partial^\alpha \varphi_j(x)| \le c_\alpha, \quad \sum_{j=0}^{\infty} \varphi_j(x) = 1\,.$$

We call $\{\varphi_j\}_{j\in\mathbb{N}_0}$ a resolution of unity.

Remark 2.3.5. *Let $\{\varphi_j\}_{j\in\mathbb{N}_0} \in \Phi(\mathbb{R}^d)$. By construction for all $f \in \mathscr{S}'(\mathbb{R}^d)$ one has $f = \sum_{j=0}^{\infty} \mathcal{F}^{-1}\varphi_j\mathcal{F}f := \sum_{j=0}^{\infty} \Delta_j f$, which are called Littlewood-Paley dyadic blocks. Observe that $\Delta_j f$ for all $j \in \{0, 1, \cdots\}$ is an analytic function, i.e., it can be evaluated at a point.*

Definition 2.3.6. *[101, Def. 6.3] Let $s \in \mathbb{R}, 0 < q \le \infty, \{\varphi_j\}_{j\in\mathbb{N}_0} \in \Phi(\mathbb{R}^d)$ and $f \in \mathscr{S}'(\mathbb{R}^d)$. We define the weighted Triebel-Lizorkin norm for $0 < p < \infty$ and the weighted Besov norm for $0 < p \le \infty$ via*

$$\|f\|_{F^s_{p,q}(\mathbb{R}^d,w)} := \|(\sum_{j=0}^{\infty} 2^{jsq}\,|\mathcal{F}^{-1}\varphi_j\mathcal{F}f(\bullet)|^q)^{1/q}\|_{L_p(\mathbb{R}^d,w)} < \infty\,,$$

$$\|f\|_{B^s_{p,q}(\mathbb{R}^d,w)} := \Big(\sum_{j=0}^{\infty} 2^{jsq}\,\|\mathcal{F}^{-1}\varphi_j\mathcal{F}f\|^q_{L_p(\mathbb{R}^d,w)}\Big)^{1/q} < \infty\,.$$

Remark 2.3.7. *Let $\alpha \in \mathbb{R}$, $w \in C^\infty(\mathbb{R}^d)$ be an admissible weight and $f \in \mathscr{S}'(\mathbb{R}^d)$. Suppose $0 < p \le \infty$ ($p < \infty$ for the Triebel-Lizorkin spaces) and $0 < q \le \infty$.*

(A) The operator $f \mapsto wf$ is an isomorphic mapping from $B^s_{p,q}(\mathbb{R}^d, w)$ onto $B^s_{p,q}(\mathbb{R}^d)$ and from $F^s_{p,q}(\mathbb{R}^d, w)$ onto $F^s_{p,q}(\mathbb{R}^d)$ [101, Thm. 6.5]. This implies that the following norms are equivalent [7]

$$\|w\,f\|_{B^s_{p,q}(\mathbb{R}^d)} \sim \|f\|_{B^s_{p,q}(\mathbb{R}^d,w)}, \quad \|w\,f\|_{F^s_{p,q}(\mathbb{R}^d)} \sim \|f\|_{F^s_{p,q}(\mathbb{R}^d,w)}.$$

(B) Let $I_\alpha \,:\, f \mapsto \mathcal{F}^{-1}\big((1+|\zeta|^2)^{\alpha/2}\mathcal{F}f\big)$, which is called the lifting operator. It holds [101, Thm. 6.9]

$$I_\alpha B^s_{p,q}(\mathbb{R}^d, w) = B^{s-\alpha}_{p,q}(\mathbb{R}^d, w), \quad I_\alpha F^s_{p,q}(\mathbb{R}^d, w) = F^{s-\alpha}_{p,q}(\mathbb{R}^d, w)\,.$$

[7] If $c\|\bullet\|_B \le \|\bullet\|_A \le C\|\bullet\|_B$ for $c, C \in (0,\infty)$, then $\|\bullet\|_A \sim \|\bullet\|_B$.

(C) *For all* $1 < p < \infty$ *and* $m \in \mathbb{N}_0$ *it holds* $W_p^m(\mathbb{R}^d, w) = F_{p,2}^m(\mathbb{R}^d, w)$ *[101, Eq. (6.21)].*

Lemma 2.3.8. *Let* $w \in C^\infty(\mathbb{R}^d)$ *be an admissible weight,* $\alpha \in \mathbb{R}$ *and* $n \in \mathbb{N}_0$*.*

(A) *For all* $p \in (1, \infty)$*, the norms* $\|\bullet\|_{L_p^\alpha(\mathbb{R}^d,w)}$ *and* $\|w\bullet\|_{L_p^\alpha(\mathbb{R}^d)}$ *are equivalent.*

(B) *For all* $p \in (1, \infty)$*, the Bessel potential space* $L_p^\alpha(\mathbb{R}^d, w)$ *coincides with the Triebel-Lizorkin space* $F_{p,2}^\alpha(\mathbb{R}^d, w)$ *with equivalent norms.*

(C) *For all* $p \in (1, \infty)$*, the Sobolev space* $W_p^n(\mathbb{R}^d, w)$ *coincides with the Bessel potential space* $L_p^n(\mathbb{R}^d, w)$ *with equivalent norms.*

(D) *For all* $1 \le p \le \infty$ *it holds* $B_{p,1}^\alpha(\mathbb{R}^d, w) \hookrightarrow L_p^\alpha(\mathbb{R}^d, w) \hookrightarrow B_{p,\infty}^\alpha(\mathbb{R}^d, w)$ *continuously.*

Proof. From Remark 2.3.7 Item (A) with $q = 2$ one has $\|\bullet\|_{F_{p,2}^\alpha(\mathbb{R}^d,w)} \sim \|w\,\bullet\|_{F_{p,2}^\alpha(\mathbb{R}^d)}$. Moreover, for all $p \in (1, \infty)$, it holds $\|\bullet\|_{L_p^\alpha(\mathbb{R}^d)} \sim \|\bullet\|_{F_{p,2}^\alpha(\mathbb{R}^d)}$ [99, Sec. 2.5.6] or [88, Sec. 3.1.2]. This concludes Item (A) and Item (B).

Evoking Item (B) with $\alpha = n \in \mathbb{N}_+$ combined with Remark 2.3.7 Item (C), one concludes that both $\|\bullet\|_{L_p^n(\mathbb{R}^d,w)}$ norm and $\|\bullet\|_{W_p^n(\mathbb{R}^d,w)}$ norm are equivalent to $\|\bullet\|_{F_{p,2}^n(\mathbb{R}^d,w)}$ norm. This concludes Item (C).

To prove Item (D) one utilizes the fact that $B_{p,1}^0(\mathbb{R}^d) \hookrightarrow L_p(\mathbb{R}^d) \hookrightarrow B_{p,\infty}^0(\mathbb{R}^d)$ [99, Sec. 2.5.7, Prop. 1]. This implies that $B_{p,1}^0(\mathbb{R}^d, w) \hookrightarrow F_{p,2}^0(\mathbb{R}^d, w) \hookrightarrow B_{p,\infty}^0(\mathbb{R}^d, w)$, thanks to Item (B) and Remark 2.3.7 Item (A). Applying Remark 2.3.7 Item (B) with $I_{-\alpha}$ one gets $B_{p,1}^\alpha(\mathbb{R}^d, w) \hookrightarrow F_{p,2}^\alpha(\mathbb{R}^d, w) \hookrightarrow B_{p,\infty}^\alpha(\mathbb{R}^d, w)$. Now Item (B) concludes Item (D) and finishes the proof. □

Remark 2.3.9. *For* $\alpha \in \mathbb{R}$ *and* $p, q \in [1, \infty]$ *we have the following generalized Hölder inequality*

$$|\langle f, g\rangle_{L_2(\mathbb{R}^d,w^{1/2})}| \le C\,\|f\|_{L_p^\alpha(\mathbb{R}^d,w^{1/p})}\,\|g\|_{L_q^{-\alpha}(\mathbb{R}^d,w^{1/q})}, \qquad \frac{1}{p} + \frac{1}{q} = 1,$$

where $\langle \bullet, \bullet\rangle_{L_2(\mathbb{R}^d,w^{1/2})}$ *is the scalar product in* $L_2(\mathbb{R}^d, w^{1/2})$ *and the constant* $C \in (0, \infty)$ *depends only on the weight* w*.*

Proof. Suppose first that $1 < p < \infty$. By the Hölder inequality, we obtain

$$\begin{aligned}\langle f, g\rangle_{L_2(\mathbb{R}^d, w^{1/2})} &= \int \mathrm{d}x\, \bar{f}(x) g(x) w(x) = \int \mathrm{d}x\, (w^{1/p}\bar{f})(x)\, (w^{1/q} g)(x) \\ &= \int \mathrm{d}x\, (1-\Delta)^{\alpha/2} (w^{1/p}\bar{f})(x)\, (1-\Delta)^{-\alpha/2} (w^{1/q} g)(x) \\ &\leq \|w^{1/p} f\|_{L_p^\alpha(\mathbb{R}^d)} \|w^{1/q} g\|_{L_q^{-\alpha}(\mathbb{R}^d)}\,.\end{aligned}$$

Now, by Lemma 2.3.8, Item (B), and Remark 2.3.7, Item (A), we write

$$\|w^{1/p} f\|_{L_p^\alpha(\mathbb{R}^d)} \leq c \|w^{1/p} f\|_{F_{p,2}^\alpha(\mathbb{R}^d)} \leq c' \|f\|_{F_{p,2}^\alpha(\mathbb{R}^d, w^{1/p})} \leq c'' \|f\|_{L_p^\alpha(\mathbb{R}^d, w^{1/p})} \tag{2.3.1}$$

and analogously for $\|w^{1/q} g\|_{L_q^{-\alpha}(\mathbb{R}^d)}$. This concludes the proof for $1 < p < \infty$. For $p = \infty$ and $q = 1$ (or vice versa) $L_\infty^\alpha(\mathbb{R}^d)$ is understood as the Hölder-Zygmund class $B_{\infty,\infty}^\alpha(\mathbb{R}^d)$. As the restriction $p < \infty$ in Remark 2.3.7 does not apply to the Besov spaces, we conclude as in Eq. (2.3.1). □

Theorem 2.3.10. *Let w, v be admissible weights and*

$$-\infty < \alpha_2 \leq \alpha_1 < \infty, \qquad 1 \leq p_1 \leq p_2 \leq \infty.$$

(A) The embedding $L_{p_1}^{\alpha_1}(\mathbb{R}^d, w) \to L_{p_2}^{\alpha_2}(\mathbb{R}^d, v)$ is continuous if

$$p_2 < \infty, \qquad \alpha_1 - d/p_1 \geq \alpha_2 - d/p_2 \quad \textit{and} \quad \sup_{x \in \mathbb{R}^d} v(x)/w(x) < \infty.$$

(B) The embedding $L_{p_1}^{\alpha_1}(\mathbb{R}^d, w) \to L_\infty^{\alpha_2}(\mathbb{R}^d, v)$ is continuous if

$$\alpha_1 - d/p_1 > \alpha_2 \quad \textit{and} \quad \sup_{x \in \mathbb{R}^d} v(x)/w(x) < \infty.$$

(C) The embedding $L_{p_1}^{\alpha_1}(\mathbb{R}^d, w) \to L_{p_2}^{\alpha_2}(\mathbb{R}^d, v)$ is compact if

$$p_2 < \infty, \qquad \alpha_1 - d/p_1 > \alpha_2 - d/p_2 \quad \textit{and} \quad \lim_{|x| \to \infty} v(x)/w(x) = 0.$$

Proof. Parts (A) and (C) follow from [41, Sec. 4.2.3, Theorem] and the equivalence between $L_p^\alpha(\mathbb{R}^d, w)$ and $F_{p,2}^\alpha(\mathbb{R}^d, w)$ mentioned in Lemma 2.3.8 above. Part (B) is covered by [41, Sec. 4.2.3, Remark] and the embeddings stated in Lemma 2.3.8. □

Theorem 2.3.11. *Let w be an admissible weight, $\alpha \in [0, \infty)$ and $p, p_1, p_2 \in [1, \infty)$ be such that $1/p = 1/p_1 + 1/p_2$. Then there exists $C \in (0, \infty)$ such that for all $f \in L_{p_1}^\alpha(\mathbb{R}^d, w^{1/p_1})$ and $g \in L_{p_2}^\alpha(\mathbb{R}^d, w^{1/p_2})$*

$$\|fg\|_{L_p^\alpha(\mathbb{R}^d, w^{1/p})} \leq C\, \|f\|_{L_{p_1}^\alpha(\mathbb{R}^d, w^{1/p_1})}\, \|g\|_{L_{p_2}^\alpha(\mathbb{R}^d, w^{1/p_2})}.$$

Proof. The statement follows from the equivalence of the norms $\|\bullet\|_{L^\alpha_p(\mathbb{R}^d,w)}$ and $\|w\bullet\|_{L^\alpha_p(\mathbb{R}^d)}$, the fractional Leibniz rule [72, Ch. 2] and Theorem A.5 (A). Alternatively, one can use [21, Lemma 5]. □

Theorem 2.3.12. *Let w be an admissible weight, $p_1, p_2 \in [1,\infty)$, $\alpha_1, \alpha_2 \in \mathbb{R}$, $\theta \in (0,1)$ and*

$$\alpha = \theta\,\alpha_1 + (1-\theta)\,\alpha_2, \qquad\qquad \frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}.$$

There exists $C \in (0,\infty)$ such that for all $f \in L^{\alpha_1}_{p_1}(\mathbb{R}^d, w^{1/p_1}) \cap L^{\alpha_2}_{p_2}(\mathbb{R}^d, w^{1/p_2})$ it holds

$$\|f\|_{L^\alpha_p(\mathbb{R}^d,w^{1/p})} \le C\,\|f\|^{\theta}_{L^{\alpha_1}_{p_1}(\mathbb{R}^d,w^{1/p_1})}\,\|f\|^{1-\theta}_{L^{\alpha_2}_{p_2}(\mathbb{R}^d,w^{1/p_2})}.$$

Proof. The statement is a consequence of the Hölder inequality, cf. [21, Sec 3]. Applying first the Hölder inequality for series with $p' = \frac{1}{\theta}$ and $q' = \frac{1}{1-\theta}$ and then the Hölder inequality for integrals with $\frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}$ one has

$$\begin{aligned}
\|f\|_{L^\alpha_p(\mathbb{R}^d,w^{1/p})} &= \|f\|_{F^\alpha_{p,2}(\mathbb{R}^d,w^{1/p})} = \|(\sum_{j=0}^{\infty} 2^{2j\alpha}\,|\mathcal{F}^{-1}\varphi_j\mathcal{F}f(\bullet)|^2)^{1/2}\|_{L_p(\mathbb{R}^d,w^{1/p})} \\
&\le \|w^{1/p}\,(\sum_{j=0}^{\infty} 2^{2j\alpha_1}\,|\mathcal{F}^{-1}\varphi_j\mathcal{F}f(\bullet)|^2\,)^{\theta/2}\,(\sum_{j=0}^{\infty} 2^{2j\alpha_2}\,|\mathcal{F}^{-1}\varphi_j\mathcal{F}f(\bullet)|^2\,)^{(1-\theta)/2}\|_{L_p(\mathbb{R}^d)} \\
&\le \|w^{\frac{\theta}{p_1}}(\bullet)\,(\sum_{j=0}^{\infty} 2^{2j\alpha_1}\,|\mathcal{F}^{-1}\varphi_j\mathcal{F}f(\bullet)|^2\,)^{\theta/2}\|_{L_{p_1/\theta}(\mathbb{R}^d)} \\
&\quad\times\ \|w^{\frac{1-\theta}{p_2}}(\bullet)\,(\sum_{j=0}^{\infty} 2^{2j\alpha_2}\,|\mathcal{F}^{-1}\varphi_j\mathcal{F}f(\bullet)|)^2\,)^{(1-\theta)/2}\|_{L_{p_2/(1-\theta)}(\mathbb{R}^d)} \\
&\le C\,\|f\|^{\theta}_{F^{\alpha_1}_{p_1,2}(\mathbb{R}^d,w^{1/p_1})}\,\|f\|^{1-\theta}_{F^{\alpha_2}_{p_2,2}(\mathbb{R}^d,w^{1/p_2})}.
\end{aligned}$$

Using the fact that $\|\bullet\|_{L^\alpha_p(\mathbb{R}^d,w^{1/p})}$ is equivalent with $\|\bullet\|_{F^\alpha_{p,2}(\mathbb{R}^d,w^{1/p})}$, one concludes the proof. □

Lemma 2.3.13. *Let $w \in L_1(\mathbb{R}^2)$ be an admissible weight, $n \in \{3,4,\ldots\}$, $\delta \in (0,\infty)$ and $\kappa \in (0, 2/(n-1)(n-2))$. Then there exists $C \in (0,\infty)$ and $p \in [1,\infty)$ such that for all $m \in \{1,\ldots,n-1\}$ and $\Psi \in L^1_2(\mathbb{R}^2, w^{1/2}) \cap L_n(\mathbb{R}^2, w^{1/n})$, $Z \in L^{-\kappa}_p(\mathbb{R}^2, w^{1/p})$ it holds*

$$|\langle Z, \Psi^m\rangle_{L_2(\mathbb{R}^2,w^{1/2})}| \le C\,\|Z\|^p_{L^{-\kappa}_p(\mathbb{R}^2,w^{1/p})} + \delta\,\|\vec{\nabla}\Psi\|^2_{L_2(\mathbb{R}^2,w^{1/2})} + \delta\,\|\Psi\|^n_{L_n(\mathbb{R}^2,w^{1/n})} + \delta.$$

Proof. Let $1/r = (1-\kappa)/n + \kappa/2$, $1/q = m/r$ and $1/p' = 1 - 1/q$. By Hölder's inequality

$$|\langle Z, \Psi^m \rangle_{L_2(\mathbb{R}^2, w^{1/2})}| \leq C \, \|Z\|_{L^{-\kappa}_{p'}(\mathbb{R}^2, w^{1/p'})} \, \|\Psi^m\|_{L^{\kappa}_q(\mathbb{R}^2, w^{1/q})},$$

for some $C \in (0,\infty)$. Theorem 2.3.11 implies that

$$\|\Psi^m\|_{L^{\kappa}_q(\mathbb{R}^d, w^{1/q})} \leq C \, \|\Psi\|^m_{L^{\kappa}_r(\mathbb{R}^d, w^{1/r})}$$

and Theorem 2.3.12 implies that

$$\|\Psi\|_{L^{\kappa}_r(\mathbb{R}^d, w^{1/r})} \leq C \, \|\Psi\|^{\kappa}_{L^1_2(\mathbb{R}^d, w^{1/2})} \, \|\Psi\|^{1-\kappa}_{L_n(\mathbb{R}^d, w^{1/n})}$$

for some $C \in (0,\infty)$. Combining the above bounds we obtain

$$|\langle Z, \Psi^n \rangle_{L^2(\mathbb{R}^d, w^{1/2})}| \leq C \, \|Z\|_{L^{-\kappa}_{p'}(\mathbb{R}^d, w^{1/p'})} \, \|\Psi\|^{m\kappa}_{L^1_2(\mathbb{R}^d, w^{1/2})} \, \|\Psi\|^{m(1-\kappa)}_{L_n(\mathbb{R}^d, w^{1/n})}$$

for some $C \in (0,\infty)$. Hence, by the Young inequality with $1/p' + (m\kappa)/2 + (m(1-\kappa))/n = 1$ for every $\delta \in (0,\infty)$ there is $C \in (0,\infty)$ such that

$$\begin{aligned} &|\langle Z, \Psi^m \rangle_{L_2(\mathbb{R}^2, w^{1/2})}| \\ &\qquad \leq C \, \|Z\|^{p'}_{L^{-\kappa}_{p'}(\mathbb{R}^2, w^{1/p'})} + \delta \, \|\Psi\|^2_{L^1_2(\mathbb{R}^2, w^{1/2})} + \delta \, \|\Psi\|^n_{L_n(\mathbb{R}^2, w^{1/n})}. \end{aligned} \tag{2.3.2}$$

Observe that by the Hölder inequality and the assumption $w \in L_1(\mathbb{R}^2)$ for all $q, r \in [1,\infty)$ such that $q \leq r$ there exists $C \in (0,\infty)$ such that $\|\bullet\|_{L_q(\mathbb{R}^2, w^{1/q})} \leq C \, \|\bullet\|_{L_r(\mathbb{R}^2, w^{1/r})}$. Hence, the bound (2.3.2) implies the statement of the lemma with $1/p = (2 - \kappa(n-1)(n-2))/2n$. □

Lemma 2.3.14. *Let $p \in [2,\infty)$ and $\alpha = 1 - 2/p$. Then there exists $C \in (0,\infty)$ such that $\|f\|_{L_p(\mathbb{S}_R)} \leq C \, \|f\|_{L^{\alpha}_2(\mathbb{S}_R)}$ for all $f \in L^{\alpha}_2(\mathbb{S}_R)$ and all $R \in [1,\infty)$.*

Proof. See e.g. [20, Theorem 6] or [103, Theorem II.2.7(ii)]. □

Remark 2.3.15. *Let $p \in [2,\infty)$. Thanks to Lemma 2.3.14 there exist $C, C' \in (0,\infty)$ such that for all $\Psi \in L^{2-2/p}_2(\mathbb{S}_R)$ and all $R \in [1,\infty)$ one has*

$$\|\vec{\nabla}_R \Psi\|_{L_p(\mathbb{S}_R)} \leq C \, \|\vec{\nabla}_R \Psi\|_{L^{1-2/p}_2(\mathbb{S}_R)} \leq C \, \|(1-\Delta_R)^{1/2} \, \Psi\|_{L^{1-2/p}_2(\mathbb{S}_R)} \leq C' \, \|\Psi\|_{L^{2-2/p}_2(\mathbb{S}_R)}.$$

2.4 Functional analysis

Let H and E be two separable Hilbert spaces which are equipped with the norms $\|\bullet\|_H$ and $\|\bullet\|_E$, respectively, $\mathcal{L}(H)$ denote the vector space of all bounded linear operators from H to H and $T : \mathrm{Dom}(T) \subset H \to H$ be a self-adjoint operator with the spectral measure $\mathcal{P}_T$ such that $T = \int_{\mathbb{R}} \lambda\, \mathcal{P}_T(\mathrm{d}\lambda)$.

Remark 2.4.1. *The following definitions and facts are standard:*

(A) *The operator T commutes with a bounded operator $S \in \mathcal{L}(H)$ if $R(z) = (T-z)^{-1}$ commutes with S.*

(B) *We denote by $\Phi : B(\sigma(T), \mathbb{C}) \ni F \mapsto F(T) \in \mathcal{L}(H)$ the standard functional calculus for self-adjoint operators, where $B(\sigma(T), \mathbb{C})$ is the space of Borel functions. We recall that if T commutes with some $S \in \mathcal{L}(H)$ then also $F(T)$ commute with this S.*

(C) *An operator $S \in \mathcal{L}(H)$ commutes with T if and only if S commutes with every spectral projection associated to T [85, Thm. 12.22].*

(D) *T is compact if and only if its spectrum consists of eigenvalues of finite multiplicity and forms a finite set or a sequence converging to zero.*

(F) *A bounded linear operator $K : H \to E$ is called Hilbert-Schmidt operator if for some orthonormal basis $\{e_\alpha\}$ in H it holds $\|K\|_{\mathrm{HS}} := \Big(\sum_\alpha \|K\, e_\alpha\|_E^2\Big)^{1/2} < \infty$. We call $\|\bullet\|_{\mathrm{HS}}$ the Hilbert-Schmidt norm of the operator K. Equivalently, if $K \in \mathcal{L}(H)$ is Hilbert-Schmidt, one has $\|K\|_{\mathrm{HS}}^2 = \mathrm{Tr}(KK^*) < \infty$.*

Lemma 2.4.2. *(Schur) Let (X, μ), (Y, ν) be $\sigma-$finite measure spaces and let $K : X \times Y \to \mathbb{R}$ be a measurable kernel for which there exist measurable functions $p : X \to (0, \infty)$, $q : Y \to (0, \infty)$ and constants $\alpha, \beta > 0$ such that for a.e. $x \in X$ and a.e. $y \in Y$ it holds that*

$$\int_Y |K(x,y)|\, q(y)\, \nu(\mathrm{d}y) \le \alpha\, p(x), \qquad \int_X |K(x,y)|\, p(x)\, \mu(\mathrm{d}x) \le \beta\, q(y)\,,$$

respectively. Then, the mapping $T : L_2(Y, \nu) \to L_2(X, \mu)$ defined by $Tf(x) = \int_Y K(x,y)\, f(y)\, \nu(\mathrm{d}y)$ is a bounded operator such that its norm is bounded from above by $\sqrt{\alpha\beta}$.

Definition 2.4.3. *A self-adjoint operator T on a Hilbert space H equipped with the inner product $\langle \bullet, \bullet \rangle_H$ is called dissipative if and only if for every $\phi \in \mathrm{Dom}(T)$ it holds $\mathrm{Re}\langle T\phi, \phi\rangle_H \le 0$, i.e., its spectrum is contained in $(-\infty, 0]$.*

Remark 2.4.4. *For $R \in \mathbb{N}_+$ and $k \in \mathbb{Z}$ it holds that*

$$\mathrm{Dom}\,(-\Delta_R)^k = \mathrm{Dom}(1-\Delta_R)^k = L_2^{2k}(\mathbb{S}_R), \quad \mathrm{Dom}\,(-\Delta)^k = \mathrm{Dom}(1-\Delta)^k = L_2^{2k}(\mathbb{R}^2)\,.$$

Remark 2.4.5. *Let H be a separable Hilbert space and S denote a C_0-semigroup on H generated by the self-adjoint operator T, i.e., $S(t) = \mathrm{e}^{-t\,T}$. Writing up the Taylor formula with remainder R_n results in*

$$S(s)f = \sum_{k=0}^{n} \frac{(s-t)^k}{k!}\,\frac{\mathrm{d}^k}{\mathrm{d}t^k}\,S(t)f + R_n(s,t)f\,,$$

where

$$R_n(s,t)f = \frac{1}{n!}\int_t^s (s-r)^n\,\frac{\mathrm{d}^{n+1}}{\mathrm{d}r^{n+1}}\,S(r)\,f\mathrm{d}r\,,$$

provided $f \in \mathrm{Dom}(T^{n+1})$.

Properties of $(-\Delta_R)$

Note that $(-\Delta_R)$ has discrete spectrum accumulating to infinity as was shown in Sec. 2.2. By Remark 2.4.1 Item (D) the operator $(1-\Delta_R)^{-1}$ is compact.

Remark 2.4.6. *Evoking the spectral calculus for the self-adjoint operator $(-\Delta_R)$ for any Borel measurable function $F : \mathrm{spect}(-\Delta_R) \to \mathbb{R}$ one defines the operator*

$$F(-\Delta_R) := \sum_{l=0}^{\infty} (2l+1)\,F(l(l+1)/R^2)\,\mathcal{P}_{R,l} = \sum_{l=0}^{\infty} (2l+1)\,\mathrm{Tr}(F(-\Delta_R)\mathcal{P}_{R,l})\,\mathcal{P}_{R,l}\,,$$

where $\mathcal{P}_{R,l}$ is given in Remark 2.2.3. In particular, $\mathrm{spec}(F(-\Delta_R)) = F(\mathrm{spec}(-\Delta_R))$. Moreover, one has $\|F(-\Delta_R)\| \leq \|F\|_\infty$, where $\|F\|_\infty = \max\{|F(l(l+1)/R^2)| : l \in \mathbb{N}_0\}$.

Remark 2.4.7. *Let $G_{R,N} := (1-\Delta_R)^{-1}(1-\Delta_R/N^2)^{-2}$ and $f \in L_2(\mathbb{S}_R)$ such that $(G_{R,N}f)(\mathrm{x}) = \int_{\mathbb{S}_R} G_{R,N}(\mathrm{x},\mathrm{x}')\,f(\mathrm{x}')\,\rho_R(\mathrm{d}\mathrm{x}')$. Using Remark 2.4.6 one has*

$$G_{R,N}(\mathrm{x},\mathrm{x}') = \frac{1}{4\pi R^2}\sum_{l=0}^{\infty} \frac{(2l+1)}{(1+l(l+1)/R^2)(1+l(l+1)/(NR)^2)^2}\,P_l(\frac{\mathrm{x}\cdot\mathrm{x}'}{R^2}).$$

Observe that for all $Q \in O(3)$ and all $\mathrm{x},\mathrm{x}' \in \mathbb{S}_R$ one has $G_{R,N}(\mathrm{x},\mathrm{x}') = G_{R,N}(Q\mathrm{x},Q\mathrm{x}')$. This is due to the invariance property of the $l-th$ Legendre polynomial P_l, which is *given in Remark 2.2.3. Furthermore, it holds that*

$$\mathrm{Tr}(G_{R,N}) = \sum_{l=0}^{\infty} \frac{(2l+1)}{(1+l(l+1)/R^2)(1+l(l+1)/R^2N^2)^2}\,,$$

where we used the facts that $P_l(1) = 1$ *and* $1/(4\pi R^2)\int_{\mathbb{S}_R} \rho_R(\mathrm{dx}) = 1$.

Remark 2.4.8. *Let G denote a fundamental solution for the positive self-adjoint operator $(1-\Delta)$, where Δ is the Laplacian on $\mathbb{R}^2$, i.e., $(1-\Delta)G(x,x') = \delta(x-x')$. One writes the resolvent of the operator $(1-\Delta)$ as a convolution, i.e., $(1-\Delta)^{-1}\phi = G *_{\mathbb{R}^2} \phi$ for all $\phi \in L_2(\mathbb{R}^2)$. Using the translational invariance property of G one determines the mapping $G : \mathbb{R}^2 \to (0,\infty)$*

$$G(x) := \frac{1}{4\pi}\int_0^{\infty} \mathrm{e}^{-\pi|x|^2/t}\,\mathrm{e}^{-t/(4\pi)}\,\frac{\mathrm{d}t}{t}$$

such that $\|G\|_{L_1(\mathbb{R}^2)} = 1$. Furthermore, there are some constants $C, c \in (0,\infty)$ such that for all measurable sets $A, B \subset \mathbb{R}^2$ and all $\phi \in L_2(\mathbb{R}^2)$ with $\mathrm{supp}(\phi) \subset A$ it holds (cf. [42, Sec. 11.1])

$$\|(1-\Delta)^{-1}\,\phi\|_{L_2(B)} \le C\,\exp(-c\,\mathrm{dist}(A,B))\,\|\phi\|_{L_2(A)}\,.$$

On account of Remark A.3 as $\|x-x'\| \to \infty$ one infers that

$$G(x,x') = \frac{\mathrm{e}^{-\|x-x'\|}}{\sqrt{8\,\pi\,\|x-x'\|}}\,(1+o(1)).$$

Observe that $G(x,x')$ exhibits rapid decay at infinity. Using once again Remark A.3 as $\|x-x'\| \to 0$ it holds

$$G(x,x') = -\frac{1}{2\pi}\log\|x-x'\|\,(1+o(1)), \tag{2.4.1}$$

where $\|\bullet\|$ is the Euclidean distance between $x, x' \in \mathbb{R}^2$. The above asymptotic behaviour of the integral kernel G implies that the covariance of the Gaussian measure $\mathcal{N}(0,(1-\Delta)^{-1})$ in Sec. 3.1 diverges near the diagonal, i.e., $G(x,x')$ is $C^\infty(\mathbb{R}^2)$ except at $x = x'$.

Remark 2.4.9. *Note that $\mathbb{S}_R$ is a Riemannian manifold, hence, it must behave locally like a Euclidean space [8]. That is why one expects that the fundamental solution of the Laplace equation on $\mathbb{S}_R$ behaves locally like the one on $\mathbb{R}^2$. Let G_R*

[8] The $d-$dimensional sphere is locally homeomorphic to $\mathbb{R}^d$ at every point [19, Prop. 18.2.1].

denote the fundamental solution for the positive self-adjoint operator $(1 - \Delta_R)$*. Utilizing Eq.* (2.4.1) *with the Euclidean distance on* $\mathbb{S}_R \subset \mathbb{R}^3$ *combined with the fact that* $\frac{2}{\pi}\, d_R(\mathrm{x}, \mathrm{x}') \leq \|\mathrm{x} - \mathrm{x}'\| \leq d_R(\mathrm{x}, \mathrm{x}')$*, which is given in Remark 2.1.4, one surmises that* G_R *should have a singularity of the form* $\log(d_R(\mathrm{x}, \mathrm{x}'))$ *on the diagonal. This implies that for* θ *near the origin or equivalently at the coinciding points,* G_R *is diverging. Note that the explicit formula for the integral kernel corresponding to* $(1 - \Delta_1)$ *is obtained in [31, Eq. (1.8) and Thm. 4.1]. The latter result is generalized in [31, Sec. 4.3] for arbitrary radius* $R > 0$.

Lemma 2.4.10. *[37, Lemma B.1] Let* $\kappa \in (0, \infty)$*. There exists* $C \in (0, \infty)$ *such that for all* $R, N \in [1, \infty)$ *it holds*

$$-C \leq \sum_{l=0}^{\infty} \frac{(2l+1)}{2R^2\,(1 + l(l+1)/R^2)\,(1 + l(l+1)/(NR)^2)^{\kappa}} - \log(N+1) \leq C.$$

Proof. Observe that the expression in the statement of the lemma coincides with

$$\int_0^{\infty} \frac{(2\lfloor l \rfloor + 1)\,\mathrm{d}l}{2R^2\,(1 + \lfloor l \rfloor(\lfloor l \rfloor + 1)/R^2)\,(1 + \lfloor l \rfloor(\lfloor l \rfloor + 1)/(NR)^2)^{\kappa}} - \int_0^{\infty} \frac{\mathrm{d}l}{(1+l)(1 + (1+l)/N)}.$$

The absolute value of the above expression is bounded by

$$\int_0^{\infty} \left| \frac{(2\lfloor Rl \rfloor + 1)/R}{2(1 + \lfloor Rl \rfloor(\lfloor Rl \rfloor + 1)/R^2)\,(1 + \lfloor Rl \rfloor(\lfloor Rl \rfloor + 1)/(NR)^2)^{\kappa}} - \frac{1}{(1+l)(1 + (1+l)/N)} \right| \mathrm{d}l.$$

Using $0 \leq l - \lfloor Rl \rfloor / R \leq 1$ we show that there exists $\hat{C} \in (0, \infty)$ such that the above expression is bounded by

$$\hat{C} + \int_0^{\infty} \left| \frac{1}{(1+l)(1 + l^2/N^2)^{\kappa}} - \frac{1}{(1+l)(1 + (1+l)/N)} \right| \mathrm{d}l \leq C.$$

This finishes the proof. □

Remark 2.4.11. *For later use let*

$$\langle l \rangle = (1 + l(l+1)/R^2)(1 + l(l+1)/(N^2R^2))^2, \quad \ll l \gg := (1 + l(l+1)/R^2)\,.$$

Remark 2.4.12. *For all* $R, N \in \mathbb{N}_+$ *we let* $Q_{R,N} := (1 - \Delta_R)(1 - \Delta_R/N^2)^2$*, where* $Q_{R,N} : L_2^6(\mathbb{S}_R) \subset L_2(\mathbb{S}_R) \to L_2(\mathbb{S}_R)$ *is a self-adjoint and positive definite operator. Moreover, due to the positivity of the spectrum of the operators* $Q_{R,N}$ *for all* $R, N \in \mathbb{N}_+$*, one infers that* $(\mathrm{e}^{-t\,Q_{R,N}})_{t \in [0,\infty)}$ *is a* C_0*-semigroup of contractions on* $L_2^1(\mathbb{S}_R)$*. In particular, one has* $\mathrm{e}^{-t\,Q_{R,N}}\, L_2^1(\mathbb{S}_R) \subseteq L_2^1(\mathbb{S}_R)$.

Remark 2.4.13. *Using Remark 2.4.6, one shows that* $\mathrm{e}^{-t\,Q_{R,N}} = \sum_{l=0}^{\infty}(2l{+}1)\,\mathrm{e}^{-\langle l\rangle t}\,\mathcal{P}_{R,l}$, *where* $\langle l\rangle$ *was introduced in Remark 2.4.11. Hence, for fixed* $R, N \in \mathbb{N}_+$ *it holds*

$$\int_0^\infty \mathrm{Tr}(e^{-2t\,Q_{R,N}})\,\mathrm{d}t = \sum_{l=0}^{\infty}(2l+1)\int_0^\infty e^{-2t\,\langle l\rangle}\,\mathrm{d}t = \frac{1}{2}\,\mathrm{Tr}(G_{R,N}) < \infty\,.$$

The last bound follows from Remark 2.4.7 and Lemma 2.4.10.

Lemma 2.4.14. *For all* $R, N \in [1,\infty)$ *and* $t > 0$ *the operator* $\mathrm{e}^{-tQ_{R,N}} : L_2(\mathbb{S}_R) \to L_2^1(\mathbb{S}_R)$

(A) is bounded

(B) is a Hilbert–Schmidt operator.

Proof. Let $t > 0$. Note that for each $N \in \mathbb{N}_+$ there exists $C \in (0,\infty)$ such that for all $h \in L_2(\mathbb{S}_R)$ it holds

$$\|\mathrm{e}^{-tQ_{R,N}}h\|_{L_2^1(\mathbb{S}_R)} = \|(1-\Delta_R)^{1/2}\,\mathrm{e}^{-tQ_{R,N}}\,h\|_{L_2(\mathbb{S}_R)} \le C\,\|Q_{R,N}^{1/6}\,\mathrm{e}^{-tQ_{R,N}}\,h\|_{L_2(\mathbb{S}_R)}\,.$$

Utilizing Remark 2.2.4 one shows that there exists $M \in (0,\infty)$ depending on $N \in \mathbb{N}_+$ such that $\|\mathrm{e}^{-tQ_{R,N}}h\|_{L_2^1(\mathbb{S}_R)} \le M\,\|h\|_{L_2(\mathbb{S}_R)}$ with

$$M := \sup_{l\in\mathbb{N}_0}\,(\langle l\rangle^{1/3}\,\mathrm{e}^{-2t\langle l\rangle}\,)^{1/2} \le \sup_{x\ge 1}\,(x^{1/6}\,\mathrm{e}^{-tx}) \le \frac{C}{t^{1/6}}\,, \tag{2.4.2}$$

where $x := \langle l\rangle$, which was introduced in Remark 2.4.11 and $C \in (0,\infty)$ is some constant. The last bound follows from the fact that for all $t > 0$, the map $x \mapsto x^s\mathrm{e}^{-tx}$ can be bounded uniformly for all $x > 0$ by C/t^s. This verifies that the mapping $\mathrm{e}^{-t\,Q_{R,N}} : L_2(\mathbb{S}_R) \to L_2^1(\mathbb{S}_R)$ is bounded and concludes Item (A). To prove Item (B) one uses Remark 2.4.1 Item (F) with $H = L_2(\mathbb{S}_R)$, $E = L_2^1(\mathbb{S}_R)$ and $e_\alpha = Y_{lm}$. It holds

$$\begin{aligned}
\|\mathrm{e}^{-tQ_{R,N}}\|_{\mathrm{HS}} &= \left(\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\|\mathrm{e}^{-tQ_{R,N}}\,Y_{lm}\|^2_{L_2^1(\mathbb{S}_R)}\right)^{1/2}\\
&= \left(\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\|(1-\Delta_R)^{1/2}\,\mathrm{e}^{-tQ_{R,N}}\,Y_{lm}\|^2_{L_2(\mathbb{S}_R)}\right)^{1/2}\\
&= \left(\sum_{l=0}^{\infty}(2l+1)\,(1+\frac{l(l+1)}{R^2})\,\mathrm{e}^{-2t\,\langle l\rangle}\,\|Y_{lm}\|^2_{L_2(\mathbb{S}_R)}\right)^{1/2}\\
&= \left(\sum_{l=0}^{\infty}(2l+1)\,(1+\frac{l(l+1)}{R^2})\,\mathrm{e}^{-2t\,\langle l\rangle}\right)^{1/2}.
\end{aligned}$$

The sum is finite due to the damping factor $\mathrm{e}^{-2t\langle l\rangle}$. This finishes the proof of Item (B) and concludes the proof. $\square$

Remark 2.4.15. *In a similar manner, one verifies that the mapping* $\mathrm{e}^{-t\,Q_{R,N}} : L_2(\mathbb{S}_R) \to L_2^3(\mathbb{S}_R)$ *is bounded and there exists some constant* $C \in (0,\infty)$ *such that* $\|\mathrm{e}^{-t\,Q_{R,N}}\|_{L_2(\mathbb{S}_R)\to L_2^3(\mathbb{S}_R)} \leq C/(t^{1/2})$.

2.5 Probability

Let $\mathcal{X}$ be a complete separable metric space and $(\mathcal{X}, \mathrm{Borel}(\mathcal{X}))$ be the corresponding Borel measurable space. We will denote by $\mathcal{M}(\mathcal{X})$ the set of probability measures on $\mathrm{Borel}(\mathcal{X})$. Let $(\Omega, \mathcal{F}, \mathbb{P})$ denote a probability space, and let $\mathbb{E}$ stand for the integration with respect to the probability measure $\mathbb{P}$.

Definition 2.5.1. *[29, Eq. (1.4)] We define a random variable* X *valued in* $\mathcal{X}$ *as a map* $X : (\Omega, \mathcal{F}, \mathbb{P}) \to (\mathcal{X}, \mathrm{Borel}(\mathcal{X}))$ *such that it is a measurable function i.e.,* $X^{-1}(A) := \Big\{\omega \in \Omega \mid X(\omega) \in A\Big\} \in \mathcal{F}$ *for all* $A \in \mathrm{Borel}(\mathcal{X})$. *The distribution or the law of* X *is a probability measure* μ *defined on* $\mathrm{Borel}(\mathcal{X})$ *via* $\mu(A) = \mathbb{P}(X^{-1}(A))$.

Definition 2.5.2. *[23, Def. 2.2.2] Let* $(f_n)_{n\in\mathbb{N}_+}$ *be a sequence of random variables.*

(A) The sequence $(f_n)_{n\in\mathbb{N}_+}$ *is called fundamental or Cauchy in probability if for every* $\epsilon > 0$ *one has*

$$\lim_{N\to\infty} \sup_{n,k\geq N} \mathbb{P}\left(\omega : |f_n(\omega) - f_k(\omega)| \geq \epsilon\right) = 0\,.$$

(B) The sequence $(f_n)_{n\in\mathbb{N}_+}$ *is said to converge in probability to a random variable* f *if for every* $\epsilon > 0$ *one has*

$$\lim_{n\to\infty} \mathbb{P}\left(\omega : |f(\omega) - f_n(\omega)| \geq \epsilon\right) = 0\,.$$

Remark 2.5.3. *For all* $p \in [1,\infty)$ *the convergence in* $L_p(\Omega, \mathcal{F}, \mathbb{P})$ *implies convergence in probability. The latter property is preserved under composition with continuous functions [84, Corollary 6.3.1] or [23, Corollary 2.2.6]. Moreover, convergence in probability implies almost sure convergence of a subsequence [23, Thm 2.2.5].*

Definition 2.5.4. *[23, Def. 4.5.1] A set of random variables in* $A \subset L_1(\mathbb{P})$ *is called uniformly integrable if*

$$\lim_{C\to+\infty} \sup_{f\in A} \int_{\{|f(\omega)|>C\}} |f(\omega)|\,\mathbb{P}(\mathrm{d}\omega) = 0\,.$$

Remark 2.5.5. *Observe that if the set* $\{\mathbb{E}[|f_n|^p] : n \in \mathbb{N}_+\}$ *is bounded for some* $p > 1$*, then* $(f_n)_{n\in\mathbb{N}_+}$ *is uniformly integrable. To verify this claim suppose for some* $p > 1$ *and* $M > 0$ *that* $\mathbb{E}|f_n|^p \le M$ *for all* $n \in \mathbb{N}_+$. *Note that* $p - 1 > 0$ *and* $t \mapsto t^{p-1}$ *is increasing on* $(0, \infty)$*. Now, let* $|f_n(\omega)| > C$ *for some* $C > 0$*, which implies that* $|f_n(\omega)|^p = |f_n(\omega)|^{p-1}|f_n(\omega)| \ge C^{p-1}|f_n(\omega)|$*. Therefore, on the event* $\{\omega \in \Omega : |f_n(\omega)| > C\}$ *we obtain*

$$\mathbb{E}|f_n| \le \mathbb{E}\Big[|f_n|^p / C^{p-1}\Big] \le M/C^{p-1}.$$

Observe that the last bound is independent of $n \in \mathbb{N}_+$ *and converges to zero as* $C \to \infty$.

Theorem 2.5.6. *(Lebesgue-Vitali's theorem) [23, Thm. 4.5.4] Suppose that* f *is a random variable and* $(f_n)_{n\in\mathbb{N}_+}$ *is a sequence of integrable random variables. Then, the following assertions are equivalent:*

- *A) The sequence* $(f_n)_{n\in\mathbb{N}_+}$ *converges to* f *in probability and is uniformly integrable;*
- *B) The random variable* f *is integrable and the sequence* $(f_n)_{n\in\mathbb{N}_+}$ *converges to* f *in the space* $L_1(\mathbb{P})$.

Definition 2.5.7. *[86, Def. 6.7] Let* μ *and* ν *be two regular Borel probability measures on* $(\mathcal{X}, \mathrm{Borel}(\mathcal{X}))$*. We say* ν *is absolutely continuous with respect to* μ *if* $\nu(A) = 0$ *for every* $A \in \mathrm{Borel}(\mathcal{X})$ *for which* $\mu(A) = 0$*. Moreover, if there is a set* $A \in \mathrm{Borel}(\mathcal{X})$ *such that* $\mu(E) = \mu(A \cap E)$ *for every* $E \in \mathrm{Borel}(\mathcal{X})$*, we say that* μ *is concentrated on* A*. This is equivalent to saying that* $\mu(E) = 0$ *whenever* $E \cap A = \emptyset$.

Remark 2.5.8. *Let* μ *be a regular Borel probability measure on* $(\mathcal{X}, \mathrm{Borel}(\mathcal{X}))$*. The support of* μ *is the complement of the union of the* μ*-null open sets, i.e.,*

$$\operatorname{supp}\mu := \Big(\bigcup\{A \subset \mathcal{X} : A \text{ open and } \mu(A) = 0\}\Big)^c$$

One can say that $\operatorname{supp}\mu$ *is the smallest closed set on which* μ *is concentrated and if* μ *is concentrated on a non-empty set* $B \subset \mathcal{X}$*, then* $\operatorname{supp}\mu \subset \overline{B}$.

Theorem 2.5.9. *[104, Sec. 14.13] or [86, Thm 6.10] or [23, Thm. 3.2.2] Let* $(\Omega, \mathcal{F})$ *be a measure space with two probability measures* $\mathbb{Q}$ *and* $\mathbb{P}$*. The probability measure* $\mathbb{Q}$ *is absolutely continuous with respect to* $\mathbb{P}$ *if there exists* $V \in L_1(\Omega, \mathcal{F}, \mathbb{P})$ *such that for all* $A \in \mathcal{F}$ *it holds* $\mathbb{Q}(A) = \int_A V(\omega)\,\mathbb{P}(\mathrm{d}\omega)$.

Remark 2.5.10. *Le* ν_1, ν_2 *be two probability measures on* $(\mathcal{X}, \mathrm{Borel}(\mathcal{X}))$*. We denote by* $\nu_1 * \nu_2(A) := \int_H \nu_1(A - a)\,\nu_2(\mathrm{d}a)$ *for all* $A \in \mathrm{Borel}(\mathcal{X})$.

Lemma 2.5.11. *[11, Lemma A.7] or [61] Let $(\Omega, \mathcal{F}, \mu)$ be a probability space, $F : \Omega \to \mathbb{R}$ be a measurable function such that $\exp(F) \in L_1(\Omega, \mu)$ and*

$$\mu^F(\mathrm{d}\phi) := \frac{\exp(F(\phi))\, \mu(\mathrm{d}\phi)}{\int \exp(F(\phi))\, \mu(\mathrm{d}\phi)}.$$

It holds

$$\int \exp(F(\phi))\, \mu(d\phi) \leq \exp\left(\int F(\phi)\, \mu^F(d\phi) \right).$$

Proof. To verify the above statement first notice that by assumption

$$\int \exp\left(F(\phi)\right) \mu(\mathrm{d}\phi) \int \exp\left(-F(\phi)\right) \mu^F(\mathrm{d}\phi) = \int \mu(\mathrm{d}\phi) = 1.$$

Thus, $\int \exp\left(F(\phi)\right) \mu(\mathrm{d}\phi) = \left(\int \exp\left(-F(\phi)\right)\mu^F(\mathrm{d}\phi)\right)^{-1}$. By the Jensen inequality

$$\exp\left(-\int F(\phi)\mu^F(\mathrm{d}\phi)\right) \leq \int \exp\left(-F(\phi)\right) \mu^F(\mathrm{d}\phi).$$

Hence,

$$\int \exp\left(F(\phi)\right) \mu(\mathrm{d}\phi) \leq \exp\left(\int F(\phi)\mu^F(\mathrm{d}\phi)\right).$$

This finishes the proof. □

Definition 2.5.12. *[29, Sec. 2.1] Let $(\mu_n)_{n\in\mathbb{N}_+}$ be a sequence of probability measures defined on $(\mathcal{X}, \mathrm{Borel}(\mathcal{X}))$. The sequence $(\mu_n)_{n\in\mathbb{N}_+}$ is tight iff for every $\epsilon > 0$ there exists a compact set $K_\epsilon \subset \mathcal{X}$ such that $\mu_n(K_\epsilon) \geq 1 - \epsilon$ for all $n \in \mathbb{N}_+$. The sequence $(\mu_n)_{n\in\mathbb{N}_+}$ converges weakly if for every bounded $F \in C(\mathcal{X})$ the sequence of real numbers $(\mu_n(F))_{n\in\mathbb{N}_+}$ converges.*

Theorem 2.5.13. *[29, Thm. 2.3](Prokhorov's theorem) A sequence of probability measures $(\mu_n)_{n\in\mathbb{N}_+}$ on $(\mathcal{X}, \mathrm{Borel}(\mathcal{X}))$ is tight iff there exists a diverging sequence of natural numbers $(a_n)_{n\in\mathbb{N}_+}$ such that the sequence $(\mu_{a_n})_{n\in\mathbb{N}_+}$ converges weakly.*

Lemma 2.5.14. *[37, Lemma B.4] Let $\imath : \mathcal{X} \to \mathcal{Y}$ is a compact embedding and let $(\mu_n)_{n\in\mathbb{N}_+}$ be a sequence of probability measures on $(\mathcal{X}, \mathrm{Borel}(\mathcal{X}))$. Assume that there exists $M \in (0, \infty)$ such that $\int_{\mathcal{X}} \|x\|_{\mathcal{X}}\, \mu_n(\mathrm{d}x) \leq M$ for all $n \in \mathbb{N}_+$. Then the sequence of measures $(\nu_n)_{n\in\mathbb{N}_+}$ on $(\mathcal{Y}, \mathrm{Borel}(\mathcal{Y}))$ defined by*

$$\nu_n(A) := \mu_n(\imath^{-1}(A)), \qquad n \in \mathbb{N}_+, \quad A \in \mathrm{Borel}(\mathcal{Y}),$$

is tight.

Proof. Let $\epsilon > 0$, $L_\epsilon := \{x \in \mathcal{X} \mid \|x\|_{\mathcal{X}} \leq M/\epsilon\}$, which is a bounded subset of $\mathcal{X}$ and $K_\epsilon := \overline{\imath(L_\epsilon)}$. Observe that $K_\epsilon \subset \mathcal{Y}$ is compact. It holds

$$\mu_n\Big(\|x\|_{\mathcal{X}} > \frac{M}{\epsilon}\Big) \leq \epsilon/M \int_{\mathcal{X}} \|x\|_{\mathcal{X}}\, \mu_n(\mathrm{d}x) = \epsilon/M \int_{\mathcal{X}} \|x\|_{\mathcal{X}}\, \mu_n(\mathrm{d}x)\,.$$

Hence,

$$1 - \nu_n(K_\epsilon) \leq 1 - \mu_n(L_\epsilon) = \mu_n(\|x\|_{\mathcal{X}} > M/\epsilon) \leq \epsilon\,.$$

This concludes the proof. □

Remark 2.5.15. *Let* $(\mathcal{X}, \mathrm{Borel}(\mathcal{X}))$, $(\mathcal{Y}, \mathrm{Borel}(\mathcal{Y}))$ *be measure spaces and* $\tau : \mathcal{X} \to \mathcal{Y}$ *be measurable.*

(A) Let $\mu \in \mathcal{M}(\mathcal{X})$. *Then the push-forward measure* $\tau_*\mu \in \mathcal{M}(\mathcal{Y})$ *is defined by* $\tau_*\mu(A) := \mu(\tau^{-1}(A))$, *where* $A \in \mathrm{Borel}(\mathcal{Y})$. *It holds that* $\int_{\mathcal{X}} f \circ \tau \,\mathrm{d}\mu = \int_{\mathcal{Y}} f \,\mathrm{d}(\tau_*\mu)$ *for all* $f \in C(\mathcal{Y})$.

(B) Fix $\mu \in \mathcal{M}(\mathcal{X})$ *and* $\nu \in \mathcal{M}(\mathcal{Y})$. *Then we say that* τ *is a measure preserving map between the resulting probability spaces* $(\mathcal{X}, \mathrm{Borel}(\mathcal{X}), \mu)$, $(\mathcal{Y}, \mathrm{Borel}(\mathcal{Y}), \nu)$ *if* $\tau_*\mu = \nu$

(C) Suppose that τ *is invertible and the inverse is also measurable. Let* $\nu \in \mathcal{M}(\mathcal{X})$. *Then the pullback measure* $\tau^*\nu \in \mathcal{M}(\mathcal{X})$ *is defined by* $\tau^*\nu(A) = \nu(\tau(A))$ *for* $A \in \mathrm{Borel}(\mathcal{X})$.

2.6 Gaussian measures, Wiener processes and Wiener chaos

Let V be a locally convex topological vector space. We assume that the family of seminorms, defining the topology, separates points, so that the space is Hausdorff. In this situation the topological dual space is non-trivial and the following definition of a Gaussian measure is meaningful:

Definition 2.6.1. *A probability measure* ν *on* $(V, \mathrm{Borel}(V))$ *is said to be a Gaussian measure if and only if the law of an arbitrary continuous linear functional* F, *considered as a random variable on* $(V, \mathrm{Borel}(V), \nu)$, *is a Gaussian measure on* $(\mathbb{R}^1, \mathrm{Borel}(\mathbb{R}^1))$.

To answer the question of existence we need more structure. In the following two subsections we will discuss the nuclear spaces and Hilbert spaces. These two cases are distinct in infinite dimension.

Gaussian measures on nuclear spaces

Let us move on to the definition of nuclear spaces. For every continuous semi-norm p on V we denote by V_p the completion of $V/\ker p$ in the topology given by p, which is a Banach space. If, in addition, for every continuous semi-norm p on V there exists another continuous semi-norm q such that $p \leq q$ and the canonical map

$$\iota_{p,q} : V_q \to V_p$$

is a nuclear operator between Banach spaces, we say that V is nuclear. We recall that an operator $T : V_1 \to V_2$ between Banach spaces is nuclear, if there exist sequences $v^*_{1,n} \in V^*_1$ and $v_{2,n} \in V_2$, $n \in \mathbb{N}$, such that

$$Tv = \sum_{n\in\mathbb{N}} v^*_{1,n}(v)v_{2,n}, \quad v \in V,$$

and $\sum_{n\in\mathbb{N}} \|v^*_{1,n}\|_{V^*_1} \|v_{2,n}\|_{V_2} < \infty$.

Theorem 2.6.2. *(Bochner-Minlos) [102, Thms. 4.3, 4.4] Let V be a nuclear space. In order that a functional $S_\nu : V \to \mathbb{C}$ be the characteristic function of a probability measure ν on V', i.e.*

$$S_\nu(v) := \int_{V'} \mathrm{e}^{i\langle\phi,v\rangle}\nu(d\phi),$$

it is sufficient that S_ν be of positive type, continuous and $S_\nu(0) = 1$. If the space V is, in addition, metrizable, the condition is also necessary.

If $S_\nu(v) = \mathrm{e}^{-\frac{1}{2}\mathcal{G}(v)}$, where $\mathcal{G}$ is a continuous, positive quadratic form on V, then the measure ν is Gaussian in the sense of Def. 2.6.1. As the spaces of Schwartz class functions and smooth compactly supported functions on manifolds are nuclear, using Theorem 2.6.2, one can construct Gaussian measures on the corresponding spaces of distributions. However, as infinite dimensional Hilbert spaces are not nuclear, they require separate considerations.

Gaussian measures on Hilbert spaces

Let H be a separable Hilbert space with scalar product $\langle \cdot, \cdot\rangle_H$ and G be a positive operator on H with trivial kernel so that G^{-1} is also well defined as a self-adjoint operator. Following [83, p. 44], we define the scale of Hilbert spaces $H_\ell \subset H$, $\ell \in \mathbb{Z}$, as completions of $\mathrm{Dom}((G^{-1})^{\ell/2})$ in the norms coming from the scalar products $\langle \cdot, \cdot\rangle_\ell := \langle (G^{-1})^{\ell/2} \cdot, (G^{-1})^{\ell/2} \cdot\rangle_H$. We recall the identification $H_{-\ell} = H^*_\ell$ and set

$E := H_{-1}$ as this space will be particularly important in the following. We will identify E^* with H_1 and denote standard pairing between E and E^*, by $\langle \cdot , \cdot \rangle$. It coincides with the scalar product in H on arguments from this space. We denote by $\hat{f} := G^{-1} f$ the isometric embedding from E^* to E. Assuming that G is trace-class on E, there exists a measure ν on $(E, \mathrm{Borel}(E))$ whose characteristic function has the form

$$S_\nu(f) := \int_E \mathrm{e}^{i\langle \phi, f\rangle}\, \nu(\mathrm{d}\phi) = \int_E \mathrm{e}^{i\langle \phi, \hat{f}\rangle_E}\, \nu(\mathrm{d}\phi) = \mathrm{e}^{-\frac{1}{2}\langle G\hat{f}, \hat{f}\rangle_E} = \mathrm{e}^{-\frac{1}{2}\|f\|_H^2}, \quad f \in E^*. \tag{2.6.1}$$

It is called the (centered) Gaussian measure on E with covariance G and it is a Gaussian measure on E in the sense of Def. 2.6.1. The triple (H, E, ν) is called an abstract Wiener space, and H is called a reproducing kernel Hilbert space, cf. [91, Def. 1.2].

The existence of the measure ν above can be obtained from the following variant of the Bochner-Minlos theorem for Hilbert spaces:

Theorem 2.6.3. *[29, Thm. 2.27] Let E be a separable Hilbert space. Let $S_\nu : E \to \mathbb{C}$ be the characteristic function of a probability measure ν on $(E, \mathrm{Borel}(E))$, then S_ν is a continuous function of positive type such that $S_\nu(0) = 1$. Moreover, for arbitrary $\epsilon > 0$ there exists a nonnegative trace-class operator G_ϵ such that $1 - \mathrm{Re}\, S_\nu(f) \le \epsilon$ for all f such that $\langle G_\epsilon f, f\rangle \le 1$. Conversely, an arbitrary function $S_\nu : E \to \mathbb{C}$ satisfying the aforementioned conditions is the characteristic function of a probability measure ν on E.*

Wiener processes

Now let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $X \in L_2(\Omega, \mathcal{F}; E)$ be a random variable, taking values in the Hilbert space E, whose law is given by the Gaussian measure ν of Eq. (2.6.1). Then,

$$\mathbb{E}[\langle X, \hat{f}\rangle_E \, \langle X, \hat{g}\rangle_E] = \langle G\hat{f}, \hat{g}\rangle_E, \quad f, g \in E^*. \tag{2.6.2}$$

Now let $\{\mathcal{F}_t\}_{t\in[0,\infty)}$ be a filtration, that is an increasing family of σ-algebras, such that $\mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F}$ for $s \le t$ and let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,\infty)}, \mathbb{P})$ be the resulting filtered probability space. We assume the right-continuity and completeness of the filtration, i.e., $\mathcal{F}_t = \bigcap_{s>t} \mathcal{F}_s$ and $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets in $\mathcal{F}$.

Definition 2.6.4. *[28, Sec. 5.2] Let $\mu = \mathcal{N}(0, G)$ be a Gaussian measure on E with mean zero and a covariance operator G which is positive and trace-class. Then, an E-valued stochastic process $\{\tilde{W}(t)\}_{t\in[0,\infty)}$ on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,\infty)}, \mathbb{P})$ is called a G-Wiener process if:*

(A) $\tilde{W}(\bullet)$ has continuous paths and $\tilde{W}(0) = 0$.

(B) $\tilde{W}(\bullet)$ has independent increments.

(C) For arbitrary $t \geq 0$ and $h > 0$ the law of $h^{-1/2}(\tilde{W}(t+h) - \tilde{W}(t))$ equals $\mathcal{N}(0, G)$.

Furthermore, it is assumed that $\tilde{W}(\bullet)$ is adapted to the filtration, i.e., $\tilde{W}(t)$ is $\mathcal{F}_t$-measurable for every t.

It is well known that a process and filtration satisfying the requirements of Def. (2.6.4) exist. It is easy to see from this definition and Eq. (2.6.2) that

$$\mathbb{E}[\langle \tilde{W}(t), \hat{f}\rangle_E \langle \tilde{W}(s), \hat{g}\rangle_E] = \langle G\hat{f}, \hat{g}\rangle_E \, t \wedge s$$

for $f, g \in E^*$. By rewriting this formula in terms of the pairing $\langle \cdot , \cdot \rangle$ between E^* and E, we get

$$\mathbb{E}[\langle \tilde{W}(t), f\rangle \langle \tilde{W}(s), g\rangle] = \langle f, g\rangle_H \, t \wedge s$$

The expression $W(t, f) := \langle \tilde{W}(t), f\rangle$, $f \in E^*$, gives the cylindrical Wiener process, which is suitable for modeling spacetime white noise. This process can also be defined more abstractly, without reference to E, G.

Definition 2.6.5. *[81, Def. 7.11, Lemma 7.12] A cylindrical Wiener process on H, adapted to $\{\mathcal{F}_t\}_{t\in[0,\infty)}$, is a linear (in the second variable) mapping $W : [0, T] \times H \to L_2(\Omega, \mathcal{F}, \mathbb{P})$ satisfying the following conditions:*

(A) For each $f \in H$, $(W(t, f))_{t\geq 0}$ is a real-valued $\mathcal{F}_t$-adapted Wiener process (as in Def. 2.6.4 with $E = \mathbb{R}$, $G = 1$).

(B) For all $t, s \geq 0$ and $f, g \in H$ it holds that $\mathbb{E}[W(t, f)\, W(s, g)] = \langle f, g\rangle_H \, t \wedge s$.

To see how to define the stochastic integral with respect to the cylindrical Wiener process see [80, Sec. 2.5] or [26, Sec. 11.5].

Wiener chaos

Let $x \in \mathbb{R}$, $n \in \mathbb{N}$ and let $H_n(x) := (-1)^n \, \mathrm{e}^{x^2} \, (\mathrm{d}^n/\mathrm{d}x^n) \, \mathrm{e}^{-x^2}$ denote the Hermite polynomial of degree n with mean zero and unit variance [79, Def. 8.1.1] or [26, Sec. 9.2.1]. One has

$$H_m(x) = \sum_{k=0}^{\lfloor m/2 \rfloor} \frac{(-1)^k \, m!}{(m-2k)! \, k! \, 2^k} \, x^{m-2k} \, .$$

Observe that $H''_m(x) - x\, H'_m(x) = -m\, H_m(x)$ [8, Eq. (18.1) with different coefficent] or [26, Prop. 9.3], which implies that H_m is an eigenfunction of the one dimensional Ornstein-Uhlenbeck operator, i.e., $L := -1/2\, \Delta + x \circ \vec{\nabla}$, with eigenvalue $-m$ [57, Ex. 3.10]. Moreover, the Hermite polynomials with mean zero and variance c can be determined by $H_m(x, c) := (-c)^m\, \mathrm{e}^{x^2/2c}\, (\mathrm{d}^m/\mathrm{d}x^m)\, \mathrm{e}^{-x^2/2c}$ such that $c^{-m} H_m(c\, x, c^2) = H_m(x)$ [79, Sec. 8.1]. It holds

$$H_m(x, c) = \sum_{k=0}^{\lfloor m/2 \rfloor} \frac{(-1)^k\, m!}{(m-2k)!\, k!\, 2^k} c^k\, x^{m-2k}, \qquad c \geq 0. \tag{2.6.3}$$

Let $(\Omega, \mathcal{F}, \mathbb{P})$ denote a complete probability space and H be a real, separable Hilbert space equipped with scalar product $\langle \bullet\, , \bullet \rangle_H$. We say that a stochastic process $X = \{X(h) \,|\, h \in H\}$ defined in $(\Omega, \mathcal{F}, \mathbb{P})$ is a Gaussian process on H if X is a centered Gaussian family of random variables such that $\mathbb{E}\,[X(h)X(g)] = \langle h, g \rangle_H$ for $h, g \in H$. We let

$$H_m(X, c) := \sum_{k=0}^{\lfloor m/2 \rfloor} \frac{(-1)^k\, m!}{(m-2k)!\, k!\, 2^k}\, c^k\, X^{m-2k}. \tag{2.6.4}$$

Definition 2.6.6. *We denote by $\mathcal{H}_n$ the closed linear subspace of $L_2(\Omega, \mathcal{F}, \mathbb{P})$ generated by random variables $\{H_n(X(h)), h \in H, \|h\|_H = 1\}$ and call it the Wiener chaos of order n such that due to the Wiener-Itô decomposition [26, Sec. 9.3] or [64, Thm. 2.6] it holds $L_2(\Omega, \mathcal{F}, \mathbb{P}) = \bigoplus_{l=0}^{\infty} \mathcal{H}_l$. We call the subspace $\bigoplus_{\ell=0}^{n} \mathcal{H}_\ell$ the inhomogeneous Wiener chaos of order n.*

The Hermite polynomials form an orthogonal system with respect to the Gaussian measure in Euclidean space. Consequently, we have the following lemma.

Lemma 2.6.7. *Let X, Y be two random variables with joint Gaussian distribution such that $\mathbb{E}(X) = \mathbb{E}(Y) = 0$ and $\mathbb{E}(X^2) = \mathbb{E}(Y^2) = 1$. Then, for all n, m we have*

$$\mathbb{E}\,[H_n(X) H_m(Y)] = \delta_{n,m}\, n!\, (\mathbb{E}\,[XY])^n.$$

Proof. See [75, Lemma 1.1.1] or [26, Lem. 9.5] (with different coefficients). □

Remark 2.6.8. *Let $\delta c := c_2 - c_1$. From Eq. (2.6.4) one has*

$$
\begin{aligned}
H_n(X, c_1) &:= \sum_{m=0}^{\lfloor n/2 \rfloor} \frac{(-1)^m\, n!}{(n-2m)!\, m!\, 2^m}\, c_1^m\, X^{n-2m} \\
&= \sum_{m=0}^{\lfloor n/2 \rfloor} \frac{(-1)^m\, n!}{(n-2m)!\, m!\, 2^m}\, (c_2 - \delta c)^m\, X^{n-2m} \qquad (2.6.5)\\
&= \sum_{m=0}^{\lfloor n/2 \rfloor} \frac{(-1)^m\, n!}{(n-2m)!\, m!\, 2^m} \sum_{h=0}^{m} \frac{m!}{(m-h)!\, h!}\, c_2^{m-h}\, (-\delta c)^h\, X^{m-2k}.
\end{aligned}
$$

Observe that $0 \le h \le m \le \lfloor n/2 \rfloor$. Letting $m - h = k$ with $0 \le k \le \lfloor (n-h)/2 \rfloor$ one rewrites Eq. (2.6.5)

$$
\begin{aligned}
H_n(X, c_1) &= \sum_{h=0}^{\lfloor n/2 \rfloor} \sum_{m=h}^{\lfloor n/2 \rfloor} \frac{n!\, (-1)^m}{(n-2m)!\, m!\, 2^m}\, \frac{m!}{(m-h!)\, h!}\, c_2^{m-h}\, (-\delta c)^h\, X^{n-2m} \\
&= \sum_{h=0}^{\lfloor n/2 \rfloor} \frac{n!}{(n-2h)!\, 2^h}\, (\delta c)^h \sum_{k=0}^{\lfloor (n-h)/2 \rfloor} \frac{(-1)^k\, (n-2h)!}{(n-2h-2k)!\, k!\, 2^k}\, c_2^k\, X^{n-2h-2k}.
\end{aligned}
$$

This verifies that [50, Eq. (9.1.12)]

$$
H_n(X, c_1) = \sum_{h=0}^{\lfloor n/2 \rfloor} \frac{n!}{(n-2h)!\, h!\, 2^h}\, (\delta c)^h\, H_{n-2h}(X, c_2).
$$

Lemma 2.6.9. *(Nelson's estimate) For every random variable X in an inhomogeneous Wiener chaos of order $n \in \mathbb{N}_+$, cf. [75], and every $p \in [2, \infty)$ it holds*

$$
\mathbb{E}\big[|X|^p\big]^{\frac{1}{p}} \le \sqrt{n+1}\, (p-1)^{\frac{n}{2}}\, \mathbb{E}\big[X^2\big]^{\frac{1}{2}}, \qquad \mathbb{E} \exp\left(\frac{n\, |X|^{2/n}}{6\, \mathbb{E}\big[X^2\big]^{\frac{1}{n}}} \right) < \infty.
$$

Proof. The first bound follows from the observation that for a homogeneous Wiener chaos it holds, cf. [60, Eq. (7.2)],

$$
\mathbb{E}\, [\, |X|^p]^{1/p} \le\, (p-1)^{n/2}\, \mathbb{E}\, [\, |X|^2]^{1/2}\,.
$$

For an inhomogeneous Wiener chaos of order n we estimate

$$
\begin{aligned}
\mathbb{E}\, [\, |\sum_{i=0}^{n} X_i|^p]^{1/p} \le \sum_{i=0}^{n} \mathbb{E}\, [\, |X_i|^p]^{1/p} &\le (p-1)^{n/2} \sum_{i=0}^{n} \mathbb{E}\, [\, |X_i|^2]^{1/2} \\
&\le \sqrt{n+1}\, (p-1)^{n/2}\, \mathbb{E}\, [\sum_{i=0}^{n} |X_i|^2]^{1/2},
\end{aligned}
$$

where in the first step we used the triangle inequality for the L_p norm and in the last step the Cauchy-Schwarz inequality. To prove the second relation, consider

$$\mathbb{E}\exp\left(\frac{n|X|^{2/n}}{6\mathbb{E}\big[X^2\big]^{\frac{1}{n}}}\right) = \sum_{i=0}^{\infty} \frac{n^i \mathbb{E}|X|^{2i/n}}{i!\,6^i\,[\mathbb{E}\,|X|^2]^{i/n}}$$
$$= \sum_{i=0}^{n-1} \frac{n^i \mathbb{E}\,|X|^{2i/n}}{i!\,6^i\,[\mathbb{E}\,|X|^2]^{i/n}} + \sum_{i=n}^{\infty} \frac{n^i \mathbb{E}\,|X|^{2i/n}}{i!\,6^i\,[\mathbb{E}\,|X|^2]^{i/n}}.$$

The first sum is finite. For the second sum using the first bound in the statement with $p = (2i/n) \in [2, \infty)$ one gets

$$R := \sum_{i=n}^{\infty} \frac{n^i \mathbb{E}\,|X|^{2i/n}}{i!\,6^i\,[\mathbb{E}\,|X|^2]^{i/n}} \leq \sum_{i=n}^{\infty} \frac{n^i\,(n+1)^{i/n}\,(2i/n)^i\,[\mathbb{E}\,|X|^2]^{i/n}}{i!\,6^i\,[\mathbb{E}\,|X|^2]^{i/n}}$$
$$\leq \sum_{i=n}^{\infty} \frac{1}{i!}\left(\frac{(n+1)^{1/n}\,i}{3}\right)^i.$$

Let us calculate the ratio of R_{i+1}/R_i

$$\lim_{i\to\infty} \frac{R_{i+1}}{R_i} = \lim_{i\to\infty} \frac{(i+1)^{i+1}}{(1+i)!}\,\frac{i!}{i^i}\left(\frac{1}{3}\right)^{i+1} 3^i\,\frac{((n+1)^{1/n})^{i+1}}{((n+1)^{1/n})^i}$$
$$= \lim_{i\to\infty} \frac{1}{3}\Big(1+\frac{1}{i}\Big)^i (n+1)^{1/n} = \frac{e}{3}\left((n+1)^{1/n}\right).$$

We can assume without loss that n is sufficiently large, because $\mathcal{H}_n \subset \mathcal{H}_{n'}$, $n' \geq n$. Now we use the fact that the function $f(n) = (n+1)^{1/n}$ is decreasing and tends to 1 as $n \to \infty$. This gives convergence of the sum by the d'Alambert criterion. □

2.7 Stochastic differential equations and their invariant measures

In this section we introduce, in general terms, the class of stochastic differential equations we are interested in. Let H be the Hilbert space carrying a cylindrical Wiener process, as in Def. 2.6.5, and let $\tilde{H} \subset H$ be a Hilbert space continuously embedded in H (in our case $\tilde{H} = L_2^1(\mathbb{S}_R)$, $H = L_2(\mathbb{S}_R)$, $E = L_2^{-\kappa}(\mathbb{S}_R)$, for some $\kappa > 0$).

Definition 2.7.1. *[26, Sec. 11.5.2] We denote by $C([0,T]; L^2(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}; \tilde{H}))$ the space consisting of all stochastic processes $X(t) \in L^2(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}; \tilde{H})$ for all*

$t \in [0,T]$ *such that the mapping* $[0,T] \to L^2(\Omega, \mathcal{F}, \mathbb{P}; \tilde{H})$ *defined by* $t \mapsto X(t)$ *is continuous. Moreover, it is a Banach space endowed with the norm*

$$\|X\|_{C([0,T];L^2(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq 0},\mathbb{P};\tilde{H}))} := \sup_{t\in[0,T]} \left(\mathbb{E}\,\|X(t)\|^2_{\tilde{H}}\right)^{1/2}.$$

Consider the following stochastic differential equation

$$\begin{cases} \mathrm{d}X(t) = (-Q\,X(t) + F(X(t)))\,\mathrm{d}t + \sqrt{2}\,\mathrm{d}W(t), \\ X(0) = \mathrm{x} \in L^2(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}; \tilde{H}), \end{cases} \tag{2.7.1}$$

where $Q : \mathrm{Dom}(Q) \subset \tilde{H} \to \tilde{H}$ is the generator of a C_0-semigroup $(\mathrm{e}^{-t\,Q})_{t\geq 0}$ of contractions on $\tilde{H}$, the map $F : \tilde{H} \to H$ be a possibly non-linear function with properties specified in Lemma 2.7.4 below and $(W(t,\bullet))_{t\geq 0}$ be the cylindrical Wiener process on H adapted to $\{\mathcal{F}^W_t\}_{t\geq 0}$ given in Def. 2.6.5. A stochastic process $X \in C([0,T]; L^2(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}; \tilde{H}))$ is said to be a mild solution of Eq. (2.7.1) if for every $T \geq t \geq 0$ it satisfies the following integral equation

$$X(t) = \mathrm{e}^{-t\,Q}\,\mathrm{x} + \int_0^t \mathrm{e}^{-(t-s)\,Q}\,F(X(s))\,\mathrm{d}s + \sqrt{2}\int_0^t \mathrm{e}^{-(t-s)\,Q}\,\mathrm{d}W(s)\,. \tag{2.7.2}$$

We are going to consider such $F : \tilde{H} \to H$ that the corresponding integral above takes values in $\tilde{H}$.

Let us recall briefly the standard construction of $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ used above: Let $(\Omega^0, \mathcal{F}^0, \mathbb{P}^0)$ be the original probability space of the initial conditions and we complete it if necessary. (In applications the initial conditions may be defined, to start with, e.g. on the probability space $(\mathcal{D}'(\mathbb{S}_R), \mathrm{Borel}(\mathcal{D}'(\mathbb{S}_R)), \nu_R)$, which is incomplete). Next, let $(\Omega^{\mathrm{W}}, \mathcal{F}^W, \{\mathcal{F}^W\}_{t\geq 0}, \mathbb{P}^{\mathrm{W}})$ be the canonical filtered probability space of the cylindrical Wiener process. We form the product space

$$(\Omega, \mathcal{F}, \mathcal{P}) := (\Omega^0 \times \Omega^W, \mathcal{F}^0 \otimes \mathcal{F}^W, \mathbb{P}^0 \otimes \mathbb{P}^W).$$

Proceeding to the filtration, we first define

$$\mathcal{G}_t := \mathcal{F}^0 \otimes \mathcal{F}^W_t\,.$$

Now, we just have to complete it to ensure right continuity. For this purpose, we set

$$\bar{\mathcal{G}}_t := \sigma(\mathcal{G}_t \cup \mathcal{N}_{\mathbb{P}}), \quad \mathcal{F}_t := \bigcap_{s>t} \bar{\mathcal{G}}_s,$$

where $\sigma(\mathcal{G}_t \cup \mathcal{N}_\mathbb{P})$ is the smallest σ-algebra generated by $\mathcal{G}_t$ and the null sets of $\mathbb{P}$. This completes the construction of the filtered probability space, which carries the Wiener process and the initial conditions as independent random variables. We remark that we will sometimes write, for brevity, $X(0) \in \tilde{H}$, rather than $X(0) \in L^2(\Omega, \mathcal{F}, \mathbb{P}; \tilde{H})$, even if the initial data is random.

Let us now move on to the concept of an invariant measure for the above class of stochastic differential equations.

Definition 2.7.2. *Let $(X(t))_{t\geq 0}$ be a continuous stochastic process on a probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ with values in $\tilde{H}$. For all $A \in \mathrm{Borel}(\tilde{H})$, $t \geq 0$ and $X(0) =: \mathrm{x} \in \tilde{H}$, we denote the law of $X(t)$ by $P(t, \mathrm{x}, A) := \mathbb{P}(X(t) \in A \,|\, X(0) = \mathrm{x}) \in \mathcal{M}(\tilde{H})$. Moreover, for all $f \in B_\mathrm{b}(\tilde{H})$ we define its corresponding transition semigroup $\mathcal{P}_t$ by*

$$\mathcal{P}_t f(\mathrm{x}) := \mathbb{E}\big[f(X(t)) \,|\, X(0) = \mathrm{x}\big] = \int_{\tilde{H}} f(\mathrm{y})\, P(t, \mathrm{x}, \mathrm{dy}) \tag{2.7.3}$$

such that $P(t, \mathrm{x}, A) = \mathcal{P}_t \mathbb{1}_A(\mathrm{x}) = \int_{\tilde{H}} \mathbb{1}_A(\mathrm{y})\, P(t, \mathrm{x}, \mathrm{dy})$ for all $A \in \mathrm{Borel}(\tilde{H})$.

Definition 2.7.3. *A probability measure $\mu \in \mathcal{M}(\tilde{H})$ is invariant for the semigroup $\mathcal{P}_t$ if for all $f \in B_\mathrm{b}(\tilde{H})$ and all $t \geq 0$ it holds*

$$\int_{\tilde{H}} \mathcal{P}_t f(\mathrm{x})\, \mu(\mathrm{dx}) = \int_{\tilde{H}} f(\mathrm{x})\, \mu(\mathrm{dx}) = \int_{\tilde{H}} \mathbb{E}\left[f(X(t, \mathrm{x}))\right] \mu(\mathrm{dx}).$$

Let us now consider F in Eq. (2.7.2), which is a gradient of a certain function U, as described below:

Lemma 2.7.4. *[28, Hypothesis 8.4] Consider Eq. (2.7.1) with $F = U'$, where U is a polynomial with appropriate properties and assume that*

(1) Q is a self-adjoint positive definite operator on H such that Q^{-1} is of trace-class.

(2) For any $t \geq 0$, $\tilde{H}$ is an invariant subspace of $S(t) = \mathrm{e}^{-tQ}$. Moreover for any $f \in \tilde{H}$ the mapping $S(\bullet)f$ is continuous on $[0, +\infty)$ with the topology of $\tilde{H}$.

(3) The Ornstein–Uhlenbeck process $W_Q(t) := \sqrt{2} \int_0^t \mathrm{e}^{-(t-s)Q}\, \mathrm{d}W(s)$, where W is the cylindrical Wiener process on H, has a version concentrated on $\tilde{H}$.

(4) $U : \tilde{H} \to \mathbb{R}$ is continuous and there exists the directional derivative $DU(x : h)$ of U at any point $x \in \tilde{H}$ and at any direction $h \in \tilde{H}$. Moreover, there exists a mapping $F : \tilde{H} \to H$ such that $DU(x : h) = \langle F(x), h\rangle_H$. In this case we write $F = U'$.

(5) U is bounded from above, and $F : \tilde{H} \to H$ is a locally Lipschitz mapping: for arbitrary $r > 0$ there exists $k_r > 0$ such that $\|F(x) - F(y)\|_H \leq k_r \|x - y\|_{\tilde{H}}$.

(6) For arbitrary $x \in \tilde{H}$ there exists an $\tilde{H}$-continuous solution of the integral equation

$$X(t) = \mathrm{e}^{-t\,Q}\,x + \int_0^t \mathrm{e}^{-(t-s)\,Q}\,F(X(s))\,\mathrm{d}s + W_Q(t), \quad t \geq 0.$$

Now let $\nu = \mathcal{N}(0, Q^{-1})$. Under the above assumptions the measure

$$\mu_F(\mathrm{d}\phi) = \mathrm{e}^{U(\phi)}\,\nu(\mathrm{d}\phi), \quad \phi \in \tilde{H}$$

is invariant for Eq. (2.7.1) with $F = U'$ [28, Thm. 8.6.3] and the random field X is called a gradient process. One also shows that the measure μ_F given by the above expression, is absolutely continuous with respect to the measure ν. Similar result can be found in [81, Sec. 17.5].

Finite volume measure

We would like to construct the dynamical $P(\Phi)_2$ model of Euclidean quantum field theory, where $P : \mathbb{R} \to \mathbb{R}$ is a polynomial of the form

$$P(\tau) = \frac{1}{n}\,\tau^n + \sum_{m=0}^{n-1} a_m\,\tau^m,\ n \in 2\mathbb{N}_+,\ \text{and } a_m \in \mathbb{R}\ \ \forall\, m \in \{0, \cdots, n-1\}. \quad (3.0.1)$$

Observe that such polynomials are bounded from below, but may not be globally Lipschitz continuous.

3.1 $P(\Phi)_2$ measure on the plane

The measure of the dynamical $P(\Phi)_2$ on $\mathscr{S}'(\mathbb{R}^2)$ is, heuristically, given by

$$\mu(\mathrm{d}\phi) := \frac{1}{\mathcal{Z}} \exp\left(- \int_{\mathbb{R}^2} \lambda\, P(\phi(x))\,\mathrm{d}x \right) \nu(\mathrm{d}\phi), \quad (3.1.1)$$

where $\lambda \in (0, \infty)$ is the coupling constant, $\mathcal{Z} \in (0, \infty)$ is the normalization factor, which might not be finite due to the emerging dependence on the volume of the underlying space, $\nu(\mathrm{d}\phi)$ indicates the free field Gaussian measure with covariance $G := (1 - \Delta)^{-1}$, which is a bounded operator on $L_2(\mathbb{R}^2)$. Moreover, $(-\Delta)$ is the Laplacian on $\mathbb{R}^2$. Observe that the operator $(-\Delta)$ does not have a compact resolvent and has a purely continuous spectrum on $[0, \infty)$[1]. We define G on $L_2^{-2}(\mathbb{R}^2) \subset \mathscr{S}'(\mathbb{R}^2)$.

Remark 3.1.1. *We denote the centred Gaussian measure with the covariance G by $\nu := \mathcal{N}(0, G)$. The corresponding characteristic functional, i.e., $S_\nu(f)$ can be determined via*

$$\mathscr{S}(\mathbb{R}^2) \ni f \mapsto S_\nu(f) = \int_{\mathscr{S}'(\mathbb{R}^2)} \mathrm{e}^{i\langle \phi, f\rangle}\,\nu(\mathrm{d}\phi) = \mathrm{e}^{-\frac{1}{2}\langle Gf, f\rangle}\,,$$

[1]The operator $(-\Delta)^{-1}$ is unbounded as a bilinear form $\mathscr{S}(\mathbb{R}^2) \times \mathscr{S}(\mathbb{R}^2) \to \mathbb{R}$, since $|k|^{-2}$ is not locally integrable.

where $f \in \mathscr{S}(\mathbb{R}^2)$, $\langle \bullet, \bullet \rangle$ denote $\mathscr{S}(\mathbb{R}^2) - \mathscr{S}'(\mathbb{R}^2)$ duality pairing. The existence of the measure follows from Theorem 2.6.2. Note that $\langle Gf, f\rangle_{L_2(\mathbb{R}^2)} = \|(1-\Delta)^{-1/2} f\|^2_{L_2(\mathbb{R}^2)} = \|f\|^2_{L_2^{-1}(\mathbb{R}^2)}$. This implies that the measure ν is supported on the space of distributions, in sense of Remark 2.5.8. One may find a discussion on the support property of the Gaussian measure in [82].

On account of Remark 2.4.8 one infers that the measure of the dynamical $P(\Phi)_2$ model as given in Eq. (3.1.1) is ill-defined. Firstly, a typical field ϕ in the support of the Gaussian measure $\nu(\mathrm{d}\phi)$ lacks integrability, i.e., it does not have sufficient decay at infinity. Hence, it might contain some infrared divergences, since in Eq. (3.1.1) we are performing the integral over $\mathbb{R}^2$. Secondly, such a field does not have enough regularity as oftentimes it is a distribution. Hence, the product terms inside of $P(\phi(x))$ might not be well-defined. To circumvent these two problems, we first construct the measure $P(\Phi)_2$ on a two dimensional sphere $\mathbb{S}_R$ with finite volume and we shall introduce an appropriate UV cutoff to regularize the covariance of the Gaussian measure $\nu(\mathrm{d}\phi)$.

Remark 3.1.2. *The isometry group of $\mathbb{R}^2$ consists of all affine transformations of the form $Q(x) = Ax + b$, where $A \in O(2)$ and $b \in \mathbb{R}^2$ and it is isomorphic to the Euclidean group $E(2) = T(2) \rtimes O(2)$ with the multiplication rule $(t_1, R_1) \cdot (t_2, R_2) = (t_1 + R_1 t_2, R_1 R_2)$ for $t_i \in T(2)$ and $R_i \in O(2)$ for $i = 1, 2$. The orthogonal group $O(2)$ is the symmetry group of $\mathbb{R}^2$ fixing the origin with $\dim O(2) = 1$. Moreover, $T(2)$ is the two dimensional subgroup of the group $E(2)$ consisting of translations. One has $\dim E(2) = 3$, which is equal to the dimension of the symmetry group of $\mathbb{S}_R$, i.e., $O(3)$ as well as the group consists of its rotations, i.e., $SO(3)$, see Remark 2.1.2.*

3.2 $P(\Phi)_2$ measure on the sphere

The measure of the dynamical $P(\Phi)_2$ on $\mathscr{D}'(\mathbb{S}_R)$ is given by

$$\mu_R(\mathrm{d}\phi) := \frac{1}{\mathcal{Z}_R} \exp\Big(- \int_{\mathbb{S}_R} :P(\phi(\mathrm{x})): \, \rho_R(\mathrm{dx}) \Big) \nu_R(\mathrm{d}\phi),$$

where $\nu_R := \mathcal{N}(0, (1-\Delta_R)^{-1})$ indicates the free field Gaussian measure with covariance $G_R := (1-\Delta_R)^{-1}$, which is invariant under the action of $O(3)$, see Remark 2.4.7, and ρ_R is the Riemannian volume measure, which is invariant under the action of $SO(3)$. For all $R \in \mathbb{N}_+$ the finite volume $P(\Phi)_2$ measure, i.e., μ_R, is absolutely continuous with respect to the free field measure ν_R, see Remark 3.4.7. Its construction, which we give below, is based on Lemma 2.6.9 (Nelson's estimate), which in turn follows from a hypercontractivity estimate [32, 74], [64, Ch. 5]. In contrast to the Φ^4 model in three dimensions, where the mass renormalization is needed, to treat the $P(\Phi)_2$ model one only needs to take into account the vacuum energy renormalization via the Wick ordering [2], see [22, Sec. 4]. To this end, we work with a regularized covariance for the free field measure, which is the resolvent of an elliptic operator of degree six, by introducing the UV cut-off $N \in \mathbb{N}_+$ in the frequency space. Then, we shall study the limit as $N \to \infty$. Observe that such a choice for the UV cutoff breaks the reflection positivity of the Gaussian free field measure [7], but preserves its rotational invariance property. We restore the reflection positivity in Sec. 6.3 using a different regularization.

Remark 3.2.1. *For all $R \in [1, \infty)$ the measure μ_R is invariant under the action of the orthogonal group $O(3)$, see Remark 6.4.20.*

3.3 Regularized $P(\Phi)_2$ measure

For all $R, N \in \mathbb{N}_+$, consider the bounded operators $G_R, K_{R,N} : L_2(\mathbb{S}_R) \to L_2(\mathbb{S}_R)$ defined by

$$G_R := (1 - \Delta_R)^{-1}, \qquad K_{R,N} := (1 - \Delta_R/N^2)^{-1}.$$

Observe that from Remark 2.4.9 the integral kernel of the operator $G_R = (1 - \Delta_R)^{-1}$ has a logarithmic singularity in terms of the geodesics distance at the coinciding points. For all $R, N \in [1, \infty)$ we set $G_{R,N} := K_{R,N} G_R K_{R,N}$, which for all $R, N \in \mathbb{N}_+$ is trace-class operator, see Remark 2.4.7, and denote its inverse by $Q_{R,N} := (1 - \Delta_R)(1 - \Delta_R/N^2)^2$, where $Q_{R,N} : L_2^6(\mathbb{S}_R) \subset L_2(\mathbb{S}_R) \to L_2(\mathbb{S}_R)$ is a self-adjoint, positive definite operator and generates C_0-semigroup of contractions on $L_2^1(\mathbb{S}_R)$, see Remark 2.4.12.

We denote the centered Gaussian measure with the covariance $G_{R,N}$ by $\nu_{R,N} := \mathcal{N}(0, G_{R,N})$. The associated characteristic functional for all $f \in \mathscr{D}(\mathbb{S}_R)$ has the form

$$f \ni \mathscr{D}(\mathbb{S}_R) \mapsto S_{\nu_{R,N}}(f) = \int_{\mathscr{D}'(\mathbb{S}_R)} \mathrm{e}^{i\langle \phi, f\rangle}\, \nu_{R,N}(\mathrm{d}\phi) = \mathrm{e}^{-\frac{1}{2}\langle G_{R,N} f, f\rangle},$$

where $\langle \bullet, \bullet \rangle$ denote $\mathscr{D}(\mathbb{S}_R) - \mathscr{D}'(\mathbb{S}_R)$ duality pairing. The existence of the measure follows from Theorem 2.6.2.

One writes a class of regularized probability measures on $\mathscr{D}'(\mathbb{S}_R)$ capturing the $P(\Phi)_2$

[2] See [50, Table. 14.1] for a comparison between different theories.

interaction as follows

$$\mu_{R,N}(\mathrm{d}\phi) := \frac{1}{\mathcal{Z}_{R,N}} \exp\left(-\int_{\mathbb{S}_R} P(\phi(\mathrm{x}), c_{R,N})\,\rho_R(\mathrm{dx})\right) \nu_{R,N}(\mathrm{d}\phi),$$

where

$$\begin{aligned} c_{R,N} := \int_{\mathscr{D}'(\mathbb{S}_R)} \phi(\mathrm{x})^2\,\nu_{R,N}(\mathrm{d}\phi) &= \frac{1}{4\pi R^2}\int_{\mathbb{S}_R} G_{R,N}(\mathrm{x},\mathrm{x})\,\rho_R(\mathrm{dx}) \\ &= \mathrm{Tr}(G_{R,N})/4\pi R^2 \,. \end{aligned} \tag{3.3.1}$$

We call $c_{R,N}$ a counterterm, and it is independent of the spatial variables and the metric tensor g_R. For all $\mathrm{x} \in \mathbb{S}_R$ one has,

$$P(\phi(\mathrm{x}), c_{R,N}) := \sum_{m=0}^{n} a_m\, H_m(\phi(\mathrm{x}), c_{R,N}) := \sum_{m=0}^{n} a_m\, \phi(\mathrm{x})^{:m:}\,,$$

where $H_m(\phi(\mathrm{x}), c)$ is the $m-$th Hermite polynomial introduced in Eq. (2.6.3). We also have the well-known Wiener-It$\hat{o}$ chaos decomposition $L^2(\mathscr{D}'(\mathbb{S}_R), \nu_{R,N}) = \bigoplus_{l=0}^{\infty} \mathcal{H}_l$ [64, Thm. 2.6]. In particular, one has $P(\phi, c_{R,N}) \in \bigoplus_{l=0}^{n} \mathcal{H}_l$, which coincides with the inhomogeneous Wiener chaos introduced in Def. 2.6.6. Moreover, from Lemma 2.6.7 it holds $\int_{\mathscr{D}'(\mathbb{S}_R)} P(\phi(\mathrm{x}), c_{R,N})\,\nu_{R,N}(\mathrm{d}\phi) = a_0$. Using Remark 2.4.7 and Lemma 2.4.10 one deduces the existence of some $C \in (0, \infty)$ such that for all $N, R \in \mathbb{N}_+$ it holds

$$|c_{R,N} - \frac{1}{2\pi}\log N| \le C\,. \tag{3.3.2}$$

In order to control UV divergencies, it is essential to establish the local regularity properties for the regularized convariance. This is the content of the next lemma. For all $N, M \in \mathbb{N}_+$, define $G_{R,N,M} := (1-\Delta_R/M^2)^{-1}(1-\Delta_R)^{-1}(1-\Delta_R/N^2)^{-1}$.

Lemma 3.3.1. *Let $R \in \mathbb{N}_+$. For any $n \in [4, \infty)$ there exists $C \in (0, \infty)$ such that for all $N, M \in \mathbb{N}_+$ with $(N \wedge M) = N$ it holds*

(A) $\sup_{\mathrm{x}\in\mathbb{S}_R} \|G_{R,N,N}(\mathrm{x}, \bullet)\|_{L_n(\mathbb{S}_R)} \le C$.

(B) $\|G_{R,N,N}(\bullet,\bullet) - G_{R,N,M}(\bullet,\bullet)\|_{L_n(\mathbb{S}_R^2)} \le C\,(N^{-1/n})$.

(C) $\sup_{\mathrm{x}\in\mathbb{S}_R} G_{R,N,N}(\mathrm{x},\mathrm{x}) \le C\,(\log N)$.

Proof. Note that a similar result can be found in [32, Thm. 1]. Our proof relies on the rotational invariance property of the covariance $G_{R,N,N}$ and Lemma 2.3.14. There exist $\hat{C}, C \in (0, \infty)$ such that for all $N \in \mathbb{N}_+$ it holds

$$\begin{aligned} \|G_{R,N,N}(\mathrm{x},\bullet)\|^2_{L_n(\mathbb{S}_R)} &\le \hat{C}\,\|G_{R,N,N}(\mathrm{x},\bullet)\|^2_{L_2^{(n-2)/n}(\mathbb{S}_R)} \\ &= (4\pi R^2)^{-1}\,\hat{C}\,\left(\mathrm{Tr}\left[G_R^{1+2/n} K_{R,N}^4\right]\right). \end{aligned}$$

Observe that

$$\mathrm{Tr}\left[G_R^{(n+2)/n} K_{R,N}^4\right] \;=\; \sum_{l=0}^{\infty} \frac{(2l+1)}{(1+l(l+1)/R^2)^{1+2/n}} \left(\frac{1}{1+l(l+1)/(RN)^2}\right)^4 \;\leq\; C$$

for some constants $C, \hat{C} \in (0,\infty)$ independent of N. This finishes the proof of Item (A). To prove Item (B)

$$\begin{aligned}\|(G_{R,N,N} - G_{R,N,M})(\mathrm{x},\bullet)\|^2_{L_n(\mathbb{S}_R)} &\leq \hat{C}\, \|(G_{R,N,N} - G_{R,N,M})(\mathrm{x},\bullet)\|^2_{L_2^{(n-2)/n}(\mathbb{S}_R)} \\ &= (4\pi R^2)^{-1}\, \hat{C}\, \big(\mathrm{Tr}\big[G_R^{(n+2)/n} K_{R,N}^2 (K_{R,N} - K_{R,M})^2\big]\big) \leq C\,(N \wedge M)^{-2/n}.\end{aligned}$$

See the proof of Lemma B.6 for more details. One infers Item (C) using Lemma 2.4.10 and Eq. (3.3.2). This finishes the proof. □

Auxiliary $P(\Phi)_2$ measure on the sphere

Let $g \in C^\infty(\mathbb{S}_R)$ such that

$$\|g\|^n_{L_{n/(n-1)}(\mathbb{S}_R)} \leq 1/2, \qquad \|\Delta_R\, g\|^n_{L_{n/(n-1)}(\mathbb{S}_R)} \leq 1/2\,. \tag{3.3.3}$$

From now on we would like to work deliberately with a more general class of probability measures defined by

$$\mu^g_{R,N}(\mathrm{d}\phi) := \frac{1}{\mathcal{Z}^g_{R,N}} \exp\left(\phi(g)^n/n\right) \mu_{R,N}(\mathrm{d}\phi). \tag{3.3.4}$$

Observe that if $g = 0$, for all $R, N \in \mathbb{N}_+$ the measure $\mu^g_{R,N}$ coincides with the measure $\mu_{R,N}$. Our particular choice will turn out advantageous for proving the non-Gaussianity property of the measure $\mu_{R,N}$ in Ch. 6 using Lemma 2.5.11.

Lemma 3.3.2. *There exists $A \in (0,\infty)$ depending only on the coefficients of the polynomial $u \mapsto P(\tau)$ such that for all $\tau \in \mathbb{R}$ and $c \in (1,\infty)$ it holds $P(\tau, c) \geq \tau^n/2n - A\, c^{n/2}$.*

Proof. Let us start with the following consideration: Let $\delta \in (0,1)$, and $p, q \in [1,\infty)$ such that $1/p + 1/q = 1$. By the Young inequality, it holds that

$$\alpha\, a\, b \leq |a|\,(|\alpha|\,|b|) \leq (p\,\delta)^{1/p}\,|a|\,(p\,\delta)^{-1/p}\,(|\alpha|\,|b|) \leq \delta\,|a|^p + (1/q)\,(p\,\delta)^{-q/p}\,(|\alpha|\,|b|)^q\,.$$

This implies that $-\delta\,|a|^p - (1/q)\,(p\,\delta)^{-q/p}\,(|\alpha|\,|b|)^q \leq \alpha\,a\,b$. Assuming that $c \in (1,\infty)$, $0 \leq m \leq n-1$, $0 \leq k \leq \lfloor m/2 \rfloor$, $\alpha_k, \tau \in \mathbb{R}$, we apply the same reasoning with $1/p = (m-2k)/n$ and $1/q = (n-m+2k)/n$. We obtain

$$\begin{aligned}\alpha_k\,\tau^{m-2k}c^k &\leq |\tau^{m-2k}|\,(|\alpha_k|c^k)\\ &\leq \delta\,\tau^n + \frac{n-m+2k}{n}\,\big(\frac{n}{(m-2k)}\,\delta\big)^{-(n-2k)/(n-m+2k)}\,|\alpha_k|^{n/(n-m+2k)}\,c^{kn/(n-m+2k)}.\end{aligned}$$

Let $C := (n-m+2k)/n\,(n/(m-2k)\,\delta)^{-(n-2k)/(n-m+2k)}\,|\alpha_k|^{n/(n-m+2k)}$. Then, since $c^{n/2} \geq c^{kn/(n-m+2k)}$, we have $-\delta\,\tau^n - C\,c^{n/2} \leq \alpha_k\,c^k\,\tau^{m-2k}$. Using the above argument, one bounds all terms with powers τ^m, $m < n$, by τ^n:

$$\alpha\,\tau^{m-2k}c^k \geq -\delta\,\tau^n - C\,c^{\frac{n}{2}}\,. \tag{3.3.5}$$

To conclude we apply the above bound to all terms of the polynomial $P(\tau, c)$ but the term τ^n/n and choose $\delta \in (0,1)$ sufficiently small. A similar bound is obtained in [50, Prop. 8.6.3]. □

Remark 3.3.3. *We identify implicitly a function $\phi \in L_1(\mathbb{S}_R)$ with a distribution $\varphi \in \mathscr{D}'(\mathbb{S}_R)$ defined by $\varphi(f) \equiv \langle\varphi, f\rangle := \int_{\mathbb{S}_R} \phi(\mathrm{x})f(\mathrm{x})\,\rho_R(\mathrm{dx}) = \langle\phi, f\rangle_{L_2(\mathbb{S}_R)}$.*

Lemma 3.3.4. *For all $R, N \in \mathbb{N}_+$ and $g \in C^\infty(\mathbb{S}_R)$ such that $\|g\|^n_{L_{n/(n-1)}(\mathbb{S}_R)} \leq 1/2$ the measure $\mu^g_{R,N}$ is well-defined and both $\nu_{R,N}$ and $\mu^g_{R,N}$ are concentrated on $L^1_2(\mathbb{S}_R) \subset \mathscr{D}'(\mathbb{S}_R)$.*

Proof. From Lemma B.1 one infers that the measure $\nu_{R,N}$ is concentrated on $L^1_2(\mathbb{S}_R)$ in the sense of Def. 2.5.7. Moreover, using Lemma 2.3.14 it holds $L^1_2(\mathbb{S}_R) \subset L_n(\mathbb{S}_R)$ for all $n \geq 2$ and from the Hölder inequality and by assumption one has

$$\phi(g)^n = \Big(\int_{\mathbb{S}_R} \phi(\mathrm{x})\,g(\mathrm{x})\,\rho_R(\mathrm{dx})\Big)^n \leq \|\phi\|^n_{L_n(\mathbb{S}_R)}\,\|g\|^n_{L_{n/n-1}(\mathbb{S}_R)} \leq \frac{1}{2}\|\phi\|^n_{L_n(\mathbb{S}_R)}\,.$$

Recall that $P(\phi(\mathrm{x}), c_{R,N}) = \sum_{m=0}^n a_m\,\phi(\mathrm{x})^{:m:}$. Applying relation (3.3.5) to all terms in the sum with power $m < n$ one deduces that there exists $A \in (0,\infty)$ such that $P(\phi(\mathrm{x}), c_{R,N}) \geq -\phi^n/2n - A\,c^{n/2}_{R,N}$. Hence, by linearity one has

$$\frac{1}{n}\phi(g)^n - \int_{\mathbb{S}_R} P(\phi(\mathrm{x}), c_{R,N})\,\rho_R(\mathrm{dx}) \leq A\,c^{n/2}_{R,N}\,. \tag{3.3.6}$$

Consequently, the map $\mathcal{U}^g_{R,N}$ defined by

$$\mathcal{U}^g_{R,N}\,:\,L_n(\mathbb{S}_R) \ni \phi \mapsto \exp\left(\frac{1}{n}\phi(g)^n - \int_{\mathbb{S}_R} P(\phi(\mathrm{x}), c_{R,N})\,\rho_R(\mathrm{dx})\right) \in (0,\infty)$$

is bounded and continuous, hence, is measurable. Using the Jensen inequality, the Fubini theorem and the fact that $\int_{\mathscr{D}'(\mathbb{S}_R)} \phi(f)^{2n}\, \nu_{R,N}(\mathrm{d}\phi)$ is positive, one gets

$$\begin{aligned}
\mathcal{Z}^g_{R,N}\, \mathcal{Z}_{R,N} &= \int_{\mathscr{D}'(\mathbb{S}_R)} \exp\Big(\frac{1}{n}\phi(g)^n - \int_{\mathbb{S}_R} P(\phi(\mathrm{x}), c_{R,N})\, \rho_R(\mathrm{dx})\Big)\, \nu_{R,N}(\mathrm{d}\phi)\\
&\geq \exp\Big(\int_{\mathscr{D}'(\mathbb{S}_R)} \frac{1}{n}\phi(g)^n\, \nu_{R,N}(\mathrm{d}\phi) - \int_{\mathscr{D}'(\mathbb{S}_R)}\int_{\mathbb{S}_R} P(\phi(\mathrm{x}), c_{R,N})\, \rho_R(\mathrm{dx})\, \nu_{R,N}(\mathrm{d}\phi)\Big)\\
&\geq \exp\Big(\int_{\mathscr{D}'(\mathbb{S}_R)} \frac{1}{n}\phi(g)^n\, \nu_{R,N}(\mathrm{d}\phi) - \int_{\mathbb{S}_R}\int_{\mathscr{D}'(\mathbb{S}_R)} P(\phi(\mathrm{x}), c_{R,N})\, \nu_{R,N}(\mathrm{d}\phi)\, \rho_R(\mathrm{dx})\Big)\\
&\geq \exp\Big(-a_0\,(4\pi R^2)\Big),
\end{aligned}$$

where we evoked the fact that expectations of Wick monomials of non-zero order vanish. Hence, $\mathcal{Z}_{R,N}\mathcal{Z}^g_{R,N} = \|\mathcal{U}^g_{R,N}\|_{L_1(\mathscr{D}'(\mathbb{S}_R),\nu_{R,N})} \in (0,\infty)$. This finishes the proof. □

3.4 Existence of the UV limit

Consider the probability space $(\mathscr{D}'(\mathbb{S}_R), \mathrm{Borel}(\mathscr{D}'(\mathbb{S}_R)), \nu_R)$ and define for all $R, N \in [1,\infty)$ the random Gaussian variables $X_R, X_{R,N} := K_{R,N}X_R$ valued almost surely in the Hilbert space $L^1_2(\mathbb{S}_R)$. It holds $\mathrm{Law}(X_R) = \nu_R$ with covariance $\mathbb{E}\, X_R(\mathrm{x})\otimes X_R(\mathrm{x}') = G_R(\mathrm{x},\mathrm{x}')$ and $\mathrm{Law}(X_{R,N}) = \nu_{R,N}$ with covariance $\mathbb{E}X_{R,N}(\mathrm{x})\otimes X_{R,N}(\mathrm{x}') = G_{R,N}(\mathrm{x},\mathrm{x}')$ for all $\mathrm{x},\mathrm{x}' \in \mathbb{S}_R$. On account of Lemma B.2 Item (A) one deduces that X_R is of regularity $-\kappa$ for any $\kappa > 0$, i.e., takes values in $L^{-\kappa}_2(\mathbb{S}_R)$, almost surely w.r.t. ν_R.

Using Eq. (2.6.4) one has $X^{:m:}_{R,N}(\mathrm{x}) = c^{m/2}_{R,N} H_m(X_{R,N}(\mathrm{x})/c^{1/2}_{R,N})$. Note that from Lemma B.1 Item (A) it follows that for all $R, N \in \mathbb{N}_+$ and all $m \in \{1, 2\cdots, n-1\}$ the Wick-ordered random variables $X^{:m:}_{R,N}$ can be bounded almost surely in a space of regularity $2-\kappa$ for any $\kappa > 0$. Now, for all $h \in C^\infty(\mathbb{S}_R)$ we set

$$X^{:m:}_{R,N}(h) := \int_{\mathbb{S}_R} X^{:m:}_{R,N}(\mathrm{x})\, h(\mathrm{x})\, \rho_R(\mathrm{dx})\,.$$

Using the preceding expression with $h = \mathbb{1}_{\mathbb{S}_R}$ one defines

$$\begin{aligned}
Y_{R,N} &:= \sum_{m=0}^{n} a_m\, X^{:m:}_{R,N}(\mathbb{1}_{\mathbb{S}_R}), \qquad Y^g_{R,N} := Y_{R,N} - X_{R,N}(g)^n/n\,,\\
Y_R &:= \sum_{m=0}^{n} a_m\, X^{:m:}_R(\mathbb{1}_{\mathbb{S}_R}), \qquad Y^g_R := Y_R - X_R(g)^n/n\,.
\end{aligned} \tag{3.4.1}$$

Combining Lemma B.1 and Lemma 2.3.14 one gets $X_{R,N} \in L_2^{2-\kappa}(\mathbb{S}_R) \subset L_n(\mathbb{S}_R)$ almost surely. Consequently, $Y_{R,N}$ and $Y^g_{R,N}$ are well-defined.

Remark 3.4.1. *Note that $\nu_{R,N}, \mu_{R,N} \in \mathcal{M}(L_2^1(\mathbb{S}_R))$ in the sense explained in Sec. 2.5. This implies that for all bounded and continuous $F : \mathscr{D}'(\mathbb{S}_R) \to \mathbb{R}$ we have*

$$\int F(\phi)\,\nu_{R,N}(\mathrm{d}\phi) = \mathbb{E}\,F(X_{R,N}), \qquad \int F(\phi)\,\mu^g_{R,N}(\mathrm{d}\phi) = \frac{\mathbb{E}\,F(X_{R,N})\exp(-Y^g_{R,N})}{\mathbb{E}\exp(-Y^g_{R,N})}\,.$$

It follows form the fact that for all $R, N \in \mathbb{N}_+$ one has $\mathrm{Law}(X_{R,N}) = \nu_{R,N}$.

Lemma 3.4.2. *For every $R \in \mathbb{N}_+$ there are some constants C, $c \in (0,\infty)$ such that for all $N \in \mathbb{N}_+$ and all $t > 0$ it holds*

$$\mathbb{P}(-\,Y^g_{R,N} > t) \le C\,\exp(-\exp(c\,t^{2/n})). \tag{3.4.2}$$

Proof. By Lemma 3.3.2 for every $R \in \mathbb{N}_+$ there exists $A \in (0,\infty)$ such that for all $M \in \mathbb{N}_+$ it holds

$$Y^g_{R,M} \ge -A\,c^{n/2}_{R,M}. \tag{3.4.3}$$

Consequently, one has

$$\begin{aligned}\mathbb{P}(-Y^g_{R,N} > 2A\,c^{n/2}_{R,M}) &= \mathbb{P}(-Y^g_{R,M} - (Y^g_{R,N} - Y^g_{R,M}) > 2A\,c^{n/2}_{R,M})\\ &\le \mathbb{P}(|Y^g_{R,N} - Y^g_{R,M}| > A\,c^{n/2}_{R,M})\,.\end{aligned}$$

Let $(N \wedge M) = M$ and $c \in (0,\infty)$. One writes

$$\begin{aligned}\mathbb{P}(|Y^g_{R,N} - Y^g_{R,M}| > A\,c^{n/2}_{R,M}) &= \mathbb{P}\Big(\exp(c\,|Y^g_{R,N} - Y^g_{R,M}|^{2/n}) > \exp(c\,A^{2/n}\,c_{R,M})\Big)\\ &= \mathbb{P}\Big(\exp(c|Y^g_{R,N} - Y^g_{R,M}|^{2/n}\,M^{1/n^2}) > \exp(c\,A^{2/n}\,c_{R,M}\,M^{1/n^2})\Big)\\ &\le \frac{\mathbb{E}\,\exp(c\,|Y^g_{R,N} - Y^g_{R,M}|^{2/n}\,M^{1/n^2})}{\exp(c\,A^{2/n}\,c_{R,M}\,M^{1/n^2})}\\ &\le \exp(-c\,A^{2/n}\,c_{R,M}\,M^{1/n^2})\,\mathbb{E}\,\exp\Big(c\,M^{1/n^2}\,|Y^g_{R,N} - Y^g_{R,M}|^{2/n}\Big),\end{aligned} \tag{3.4.4}$$

where we used the Markov inequality in the third step. Evoking Lemma 2.6.9 with $X = |Y^g_{R,N} - Y^g_{R,M}|$ and letting $c = n/(6\,C^{2/n})$, where C is the constant from Lemma B.6 Item (B), one deduces that for some $C' \in (0,\infty)$ depending on $\mathbb{R} \in \mathbb{N}_+$ it holds that

$$\begin{aligned}&\mathbb{E}\,\exp\Big(c\,M^{1/n^2}\,|Y^g_{R,N} - Y^g_{R,M}|^{2/n}\Big)\\ &\qquad\le \mathbb{E}\,\exp\Big(\frac{n\,|Y^g_{R,N} - Y^g_{R,M}|^{2/n}\,(\mathbb{E}|Y^g_{R,N} - Y^g_{R,M}|^2)^{1/n}\,M^{1/n^2}}{6\,\mathbb{E}\,(|Y^g_{R,N} - Y^g_{R,M}|^2\,C^{2/n})^{1/n}}\Big) < C'\,.\end{aligned} \tag{3.4.5}$$

We used Lemma B.6 Item (B), which gives $\mathbb{E}\,|Y^g_{R,N} - Y^g_{R,M}|^2)^{1/n}\, M^{1/n^2} \le C^{2/n}$, where $C^{2/n} \in (0,\infty)$ does depend on $R \in \mathbb{N}_+$. Now from Eq. (3.3.2) one has $-C + (2\pi)^{-1}\log M \le c_{R,M} \le C + (2\pi)^{-1}\log M$. For now we restrict attention to $M \ge M_0$ such that $-C + (2\pi)^{-1}\log M > 0$. This implies that

(A) $\log M \le 2\pi(c_{R,M} + C)$. Hence, $M \le \exp\left(2\pi(c_{R,M}+C)\right)$.

(B) $c_{R,M} \le C + \frac{1}{2\pi}\log M$. Thus, $\exp\left(2\pi(c_{R,M}-C)\right) \le M$.

By Item (B) above, one rewrites Eq. (3.4.4) with the help of Eq. (3.4.5)

$$\begin{aligned}\mathbb{P}(-Y^g_{R,N} > 2A\, c^{n/2}_{R,M}) &\le C' \exp(-c\, A^{2/n}\, c_{R,M} M^{1/n^2}) \\ &\le C' \exp(-c\, A^{2/n}\, c_{R,M} \exp(\frac{2\pi\, c_{R,M} - C}{n^2}))\,.\end{aligned}$$

Recall that $A \in (0,\infty)$ was chosen so as to ensure $Y^g_{R,M} \ge -A\, c^{n/2}_{R,M}$. This latter bound remains valid if we increase A. Therefore, without loss, we can assume $cA^{2/n}\exp(-C/n^2) \ge 1$. By increasing M_0, if necessary, we can also ensure $c_{R,M} \ge 1$ and write

$$\mathbb{P}(-Y^g_{R,N} > 2A\, c^{n/2}_{R,M}) \le C' \exp(-\exp(\frac{2\pi\, c_{R,M}}{n^2})).$$

Setting $t_M := 2A\, c^{n/2}_{R,M}$, we obtain for some $C'' > 0$, $N \ge M \ge M_0$

$$\mathbb{P}(-Y^g_{R,N} > t_M) \le C' \exp(-\exp(C'' t_M^{2/n})). \tag{3.4.6}$$

Let us now eliminate the restriction $N \ge M$. Suppose that $(N \wedge M) = N$, which implies $\log N \le \log M$. It holds that

$$\begin{aligned}\{-Y^g_{R,N} > 2A\, c^{n/2}_{R,M}\} &\subseteq \{-Y^g_{R,N} > 2A\,(-C + (2\pi)^{-1}\log M)^{n/2}\} \\ &\subseteq \{-Y^g_{R,N} > 2A\,(-C + (2\pi)^{-1}\log N)^{n/2}\} \subseteq \{-Y^g_{R,N} > A\, c^{n/2}_{R,N}\}\end{aligned} \tag{3.4.7}$$

To justify the last inclusion, let us first write

$$-3C + (2\pi)^{-1}\log(N) \le c_{R,N} - 2C \le -C + (2\pi)^{-1}\log(N).$$

Then, by the second inequality above,

$$\begin{aligned}&\{-Y^g_{R,N} > 2A\,(-C + (2\pi)^{-1}\log N)^{n/2}\} \\ &\qquad\subseteq \{-Y^g_{R,N} > 2A(c_{R,N} - 2C)^{n/2}\} \subseteq \{-Y^g_{R,N} > A\, c^{n/2}_{R,N}\},\end{aligned}$$

provided that $N \geq N_0$, N_0 sufficiently large such that $2A(c_{R,N} - 2C)^{n/2} \geq A\, c_{R,N}^{n/2}$. This justifies (3.4.7). Using this relation and the fact that $B_1 \subseteq B_2$ implies $\mathbb{P}(B_1) \leq \mathbb{P}(B_2)$ one infers that

$$\mathbb{P}\left(-Y^g_{R,N} > 2\,A\,c_{R,M}^{n/2}\right) \leq \mathbb{P}\left(-Y^g_{R,N} > \,A\,c_{R,N}^{n/2}\right) = 0,$$

where in the last step we used Eq. (3.4.3). We conclude that Eq. (3.4.6) holds for $N \geq N_0$ and all $M \in \mathbb{N}_+$, and note that it suffices to prove the lemma under these conditions. In fact, for a finite number of cases, i.e. $N = 1, \ldots N_0 - 1$, estimate (3.4.2) trivially follows from the fact that $\mathbb{P}\left(-Y^g_{R,N} > t\right) = 0$ for t larger than some $t^{(N)}$, which is a consequence of Eq. (3.4.3). To extend Eq. (3.4.6) from the sequence $\{t_M\}_{M \in \mathbb{N}_+}$ to all $t \in \mathbb{R}_+$ we proceed as follows. For a given t, let t_{M_-} be the largest element of this sequence such that $t_{M_-} \leq t$. Let us show that for $\ell = \lceil \exp(4\pi C) \rceil$ we have $t_{M_-} \leq t_{\ell M_-}$, hence $t \leq t_{\ell M_-}$. This follows from

$$c_{R,M_-} \leq C + (2\pi)^{-1} \log M_- = (2\pi)^{-1} \log \ell M_- - C + (2C - (2\pi)^{-1} \log \ell) \leq c_{R,\ell M_-}.$$

Now, we can write, using Eq. (3.4.6)

$$\begin{aligned}\mathbb{P}(-Y^g_{R,N} > t) \leq \mathbb{P}(-Y^g_{R,N} > t_{M_-}) &\leq C' \exp(-\exp(C''(t_{M_-}/t_{\ell M_-})^{2/n} t_{\ell M_-}^{2/n}\,)) \\ &\leq C' \exp(-\exp(C''(t_{M_-}/t_{\ell M_-})^{2/n} t^{2/n}\,)).\end{aligned}$$

Finally, we note that

$$(t_{M_-}/t_{\ell M_-})^{2/n} = (c_{R,M_-}/c_{R,\ell M_-}) \geq \frac{(2\pi)^{-1} \log M_- - C}{(2\pi)^{-1} (\log \ell + \log M_-) + C}.$$

As we restricted attention to $M \geq M_0$ such that $(2\pi)^{-1} \log M_- - C > 0$, the last expression can be bounded from below by a strictly positive constant, independent of M_-. Thus, we conclude the proof for $t \geq t_0$, t_0 independent of N, enforced by the condition $M_- \geq M_0$. For $t \leq t_0$ we can ensure (3.4.2) by adjusting the constants so that the right hand side is larger that 1 on this set. □

Lemma 3.4.3. *Let X be a real-valued random variable such that $X \geq 0$. Suppose that the function $F : [0, \infty) \to [0, \infty)$ is continuously differentiable and such that $F(0) = 0$ and $F' \geq 0$. Then it holds*

$$\mathbb{E}\, F(X) = \int_0^\infty \mathbb{P}(X > t)\, F'(t)\, \mathrm{d}t.$$

Proof. Using the Fubini lemma one gets

$$\mathbb{E}\, F(X) = \mathbb{E} \int_0^X F'(t)\,\mathrm{d}t = \int_0^\infty \int_\Omega \mathbb{1}_{\{X>t\}}(\omega)\,\mathbb{P}(\mathrm{d}\omega)\,F'(t)\,\mathrm{d}t\,.$$

This finishes the proof. □

Lemma 3.4.4. *Let $F : \mathscr{D}'(\mathbb{S}_R) \to \mathbb{R}$ be continuous. Then, the sequence $(F(X_{R,N})\exp(-Y^g_{R,N}))_{N\in\mathbb{N}_+}$ converges in probability to $F(X_R)\exp(-Y^g_R)$.*

Proof. On account of Remark 2.5.3, it suffices to verify the $L_2(\Omega,\mathcal{F},\mathbb{P})$ convergence. In fact, for $\epsilon > 0$, $\kappa \in (0,\infty)$ and $\delta \in [0,2]$, one has

$$\mathbb{P}\Big(\|X_R - X_{R,N}\|^2_{L_2^{-\kappa-\delta}(\mathbb{S}_R)} \geq \epsilon\Big) \leq \epsilon^{-1}\,\mathbb{E}\,\|X_R - X_{R,N}\|^2_{L_2^{-\kappa-\delta}(\mathbb{S}_R)} \leq \epsilon^{-1}\,R^2\,C^2\,N^{-2\delta}.$$

Similarly,

$$\mathbb{P}\Big(|Y_R - Y_{R,N}|^2 \geq \epsilon\Big) \leq \epsilon^{-1}\,C^2\,N^{-1/n}\,,$$

where we used Lemma B.2 and Lemma B.6. From the above bounds one concludes that the sequence $(X_{R,N})_{N\in\mathbb{N}_+}$ and $(Y^g_{R,N})_{N\in\mathbb{N}_+}$ converge in probability as N approaches infinity to the random variables (X_R) and $(-Y^g_R)$. One deduces the statement using the fact that $F : \mathscr{D}'(\mathbb{S}_R) \to \mathbb{R}$ and $t \to \exp(t)$ are continuous. Hence, by Remark 2.5.3 the sequences $(F(X_{R,N}))_{N\in\mathbb{N}_+}$ and $(\exp(-Y^g_{R,N}))_{N\in\mathbb{N}_+}$ also converge in probability to $F(X_R)$ and $\exp(-Y^g_R)$. This finishes the proof. □

Lemma 3.4.5. *Let $F : \mathscr{D}'(\mathbb{S}_R) \to \mathbb{R}$ be bounded and continuous. Then, the sequence $(F(X_{R,N})\exp(-Y^g_{R,N}))_{N\in\mathbb{N}_+}$ is uniformly integrable.*

Proof. By Remark 2.5.5 it suffices to verify that the set $\{\mathbb{E}\,|\exp(-Y^g_{R,N})|^p\,;\, N\in\mathbb{N}_+\}$ is bounded for some $p > 1$. Observe that

$$\begin{aligned}\mathbb{E}\,\exp(-p\,Y^g_{R,N}) &= \mathbb{E}\,\exp(-p\,(Y^g_{R,N}\wedge 0))\,\mathbb{1}_{\{Y^g_{R,N}\leq 0\}} + \mathbb{E}\,\exp(-p\,(Y^g_{R,N}\vee 0))\,\mathbb{1}_{\{Y^g_{R,N}>0\}}\\ &\leq 1 + \mathbb{E}\,\exp(-p\,(Y^g_{R,N}\wedge 0))\,\mathbb{1}_{\{Y^g_{R,N}\leq 0\}}\,.\end{aligned}$$

Hence, using Lemma 3.4.3 with $F(X) = \exp(X) - 1$ and $X = -p\,(Y^g_{R,N}\wedge 0)$ yields

$$\begin{aligned}\mathbb{E}\,\exp(-p\,Y^g_{R,N}) &\leq 2 + \mathbb{E}\,(\exp(-p\,(Y^g_{R,N}\wedge 0)) - 1)\\ &= 2 + \int_0^\infty \mathbb{P}(-p\,(Y^g_{R,N}\wedge 0) > t)\,\exp(t)\,\mathrm{d}t\\ &= 2 + \int_0^\infty \mathbb{P}(-p\,Y^g_{R,N} > t)\,\exp(t)\,\mathrm{d}t\,.\end{aligned}$$

One utilizes Lemma 3.4.2 to ensure that the expression $\mathbb{P}(-p\,Y^g_{R,N} > t)$ has sufficient decay to counteract the growing behaviour of $\exp(t)$. This makes the last integral convergent and concludes the proof. □

Proposition 3.4.6. *Let $R \in \mathbb{N}_+$ and $g \in C^\infty(\mathbb{S}_R)$ satisfy the bounds (3.3.3). There exist random variables X_R and Y^g_R such that $\mathbb{E}\exp(-Y^g_R) < \infty$ and for all bounded and continuous $F : \mathscr{D}'(\mathbb{S}_R) \to \mathbb{R}$ it holds that*

$$\lim_{N\to\infty} \mathbb{E}\, F(X_{R,N}) \exp(-p\,Y^g_{R,N}) = \mathbb{E}\, F(X_R) \exp(-p\,Y^g_R).$$

Proof. On account of Lemmas 3.4.4 and 3.4.5 one infers the assumptions stated in Item (A) of the Vitali theorem 2.5.6. This concludes the statement. □

Remark 3.4.7. *From the Radon-Nikodym theorem 2.5.9 one infers that the finite volume measure μ^g_R is absolutely continuous with respect to the finite volume Gaussian measure ν_R, e.g., they have the same null sets, i.e., for all $A \in \mathrm{Borel}(\mathscr{D}'(\mathbb{S}_R))$ such that $\nu_R(A) = 0$ one has $\mu^g_R(A)$=0. However, in the limit as $R \to \infty$, the infinite volume $P(\Phi)_2$ measure on $\mathscr{S}'(\mathbb{R}^2)$ might not be absolutely continuous with respect to Gaussian measure see [87, p. 25].*

Stochastic quantization

Stochastic quantization is a quantization method introduced by Parisi and Wu [77]. It is based on the fact that the Euclidean Green functions can be seen as correlation functions of a statistical system in equilibrium. The equivalence of the stochastic quantization and the path integral quantization for a variety of field theories, in the heuristic sense, has been shown in [17,47]. In particular, one may find the equivalence of the perturbative stochastic quantization with the path integral quantization for a scalar field in [35, Sec. 3.3].

4.1 Stochastic quantization on the sphere

Implementing the setting described in Ch. 1, the parabolic stochastic quantization associated to the probability measure $\mu^g_{R,N}(\mathrm{d}\phi)$ can be read off as the following non-linear SPDE

$$\Big(\partial_t + Q_{R,N}\Big)\Phi^g_{R,N}(t,\mathrm{x}) = \sqrt{2}\,\xi(t,\mathrm{x}) - P'(\Phi^g_{R,N}(t,\mathrm{x}), c_{R,N}) + (\Phi^g_{R,N}(t,g))^{n-1}\,g\,, \quad (4.1.1)$$

where $(t,\mathrm{x}) \in \mathbb{R}_+ \times \mathbb{S}_R$, $Q_{R,N} := (1-\Delta_R)(1-\Delta_R/N^2)^2$, $P'(\tau,c) := \partial_\tau P(\tau,c)$ and $\xi(t,\mathrm{x}) := \frac{dW_R(t,\mathrm{x})}{dt}$, where W_R is the cylindrical Wiener process on $H = L_2(\mathbb{S}_R)$, cf. Def. 2.6.5. (By rescaling t, the factors $1/2$ appearing in Eq. (1.0.5) are traded for $\sqrt{2}$ in front of ξ).

We aim to prove that the measure $\mu^g_{R,N}(\mathrm{d}\phi)$ is invariant under the dynamics generated by Eq. (4.1.1), see Thm. 4.4.3. This implies that if $\Phi^g_{R,N}$ is a stationary solution to Eq. (4.1.1) with the initial condition $\Phi^g_{R,N}(0,\bullet) = \phi^g_{R,N}$ distributed according to the measure $\mu^g_{R,N}(\mathrm{d}\phi)$, then for all $t \in \mathbb{R}_+$ the random field $\Phi^g_{R,N}(t,\bullet)$ is also distributed according to $\mu^g_{R,N}(\mathrm{d}\phi)$.

The $P(\Phi)_2$ model is subcritical, i.e., the non-linear part in the RHS of Eq. (4.1.1) can be deemed as a perturbation of the linear equation. This implies that the renormalized non-linearity should have better regularity than the Gaussian space-time white noise [58, p. 417]. As a first step to construct the $P(\Phi)_2$ measure on $\mathscr{D}'(\mathbb{S}_R)$, we need to verify the existence of the global in time solution to Eq. (4.1.1) for any fixed $N \in \mathbb{N}_+$. To this end, we first prove the existence of the local in time stationary solution of Eq. (4.1.1) using the so-called Da Prato-Debussche decomposition [27] combined with a fixed point argument and with an appropriate Sobolev embedding. Then, we shall prove the existence of the global in time stationary solution of Eq. (4.1.1) using the energy method. Note that the negative sign in Eq. (4.1.1) is essential for proving the long time existence of the solution in Sec. 4.3.2.

Remark 4.1.1. *Here the extra (stochastic) time dimension is just a parameter and it differs from the physical time in Minkowski setting. It enables us to compute the correlation functions of the d–dimensional quantum field theory after the $(d+1)$–dimensional system has reached its equilibrium.*

Remark 4.1.2. *As a first orientation, let us analyze Eq. (4.1.1) heuristically, to identify function spaces in which one can try to solve it. Let us look at the corresponding linear equation*

$$\big(\partial_t + Q_{R,N}\big) Z_{R,N}(t,\mathrm{x}) = \sqrt{2}\,\xi(t,\mathrm{x}).$$

The solution, if it exists, has the form

$$Z_{R,N}(t) = \mathrm{e}^{-tQ_{R,N}}\, Z_{R,N}(0) + \sqrt{2}\int_0^t \mathrm{e}^{-(t-s)\,Q_{R,N}}\,\xi(s,\bullet)\,\mathrm{d}s\,. \tag{4.1.2}$$

Let us set $Z_{R,N}(0) \equiv 0$ for simplicity. The second term on the right hand side of Eq. (4.1.2) above is called a stochastic convolution and will be analyzed rigorously in Sec. 4.2 below. For now, let us provide a formal computation:

$$\begin{aligned}
\mathbb{E}\|Z_{R,N}(t)\|^2_{L_2^\alpha(\mathbb{S}_R)} &= 2\int_0^t\int_0^t \mathbb{E}\langle \mathrm{e}^{-(t-s)Q_{R,N}}\xi(s,\bullet), \mathrm{e}^{-(t-s')Q_{R,N}}\xi(s',\bullet)\rangle_{L_2^\alpha(\mathbb{S}_R)}\,\mathrm{d}s\,\mathrm{d}s' \\
&= 2\int_0^t\int_0^t \mathbb{E}\langle \xi(s,\bullet), \mathrm{e}^{-(2t-s-s')Q_{R,N}}(1-\Delta_R)^\alpha\xi(s',\bullet)\rangle_{L_2(\mathbb{S}_R)}\,\mathrm{d}s\,\mathrm{d}s' \\
&= 2\int_0^t\int_0^t \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \mathbb{E}\langle \xi(s,\bullet), Y_{l,m}\rangle_{L_2(\mathbb{S}_R)}\langle Y_{l,m}, \xi(s',\bullet)\rangle_{L_2(\mathbb{S}_R)}\times \\
&\qquad\qquad \times\,\mathrm{e}^{-(2t-s-s')\langle l\rangle}\big(1+\frac{l(l+1)}{R^2}\big)^\alpha \mathrm{d}s\,\mathrm{d}s',
\end{aligned}$$

where $\langle l\rangle \simeq l^6$ *was introduced in Remark 2.4.11. Now considering that* $\mathbb{E}\,(\xi(s,\mathrm{x})\xi(s',\mathrm{x}')) = \delta_R(\mathrm{x},\mathrm{x}')\delta(s-s')$*, where the Dirac delta on the sphere satisfies* $\int \delta_R(\mathrm{x},\mathrm{x}')\, f(\mathrm{x}')\, \rho_R(\mathrm{d}\mathrm{x}') = f(\mathrm{x})$*, we obtain*

$$\mathbb{E}\, \|Z_{R,N}(t)\|^2_{L^\alpha_2(\mathbb{S}_R)} = 2R^2 \sum_{l=0}^{\infty} (2l+1) \frac{\mathrm{e}^{-2t\langle l\rangle} - 1}{\langle l\rangle} (1 + \frac{l(l+1)}{R^2})^\alpha. \tag{4.1.3}$$

The above sum converges provided that $6 - (2\alpha + 1) > 1$*, i.e.* $\alpha < 2$*. (This is consistent with Lemma B.1). Accordingly, we will look for mild solutions of the regularized equation (4.1.1) in the space* $L^1_2(\mathbb{S}_R)$*, for which the non-linear terms in the equation make sense. On the other hand, if we removed the UV regularization, that is, set* $N \to \infty$*, the corresponding condition for convergence of the sum in Eq. (4.1.3) would be* $2 - (2\alpha + 1) > 1$*, i.e.,* $\alpha < 0$*. (This is consistent with Lemma B.2). As the corresponding spaces* $L^\alpha_2(\mathbb{S}_R)$ *contain distributions whose powers are undefined, the non-linear Eq. (4.1.1) would be singular without regularization.*

Throughout this chapter, we fix a stochastic basis $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,\infty)}, \mathbb{P})$ and we consider the cylindrical Wiener process $(W_R(t,\bullet))_{t\geq 0}$ on $L_2(\mathbb{S}_R)$ defined in Def. 2.6.5.

4.2 Linear equation

Let us first look at the linear equation corresponding to (4.1.1),

$$\mathrm{d}Z_{R,N}(t,\bullet) = \sqrt{2}\, \mathrm{d}W_R(t,\bullet) - Q_{R,N} Z_{R,N}(t,\bullet)\, \mathrm{d}t\,, \tag{4.2.1}$$

whose invariant measure is $\nu_{R,N}$.

Lemma 4.2.1. *The unique mild solution* $Z_{R,N} \in C([0,\infty), L^1_2(\mathbb{S}_R))$ *of Eq. (4.2.1) with the initial condition* $Z_{R,N}(0,\bullet) = z_{R,N} \in L^1_2(\mathbb{S}_R)$ *exists in the sense of Def. 2.7.1. It has the form*

$$Z_{R,N}(t,\bullet) = \mathrm{e}^{-t\, Q_{R,N}} z_{R,N} + \sqrt{2} \int_0^t \mathrm{e}^{-(t-s)\, Q_{R,N}}\, \mathrm{d}W_R(s,\bullet)\,. \tag{4.2.2}$$

Remark 4.2.2. *Actually, the lemma remains true if we replace* $L^1_2(\mathbb{S}_R)$ *with* $L^{2-\kappa}_2(\mathbb{S}_R)$ *for some* $0 < \kappa < 1$*. This can be checked by replacing in Eq. (4.2.5) the factor* $(1 + \frac{l(l+1)}{R^2})$ *with* $(1 + \frac{l(l+1)}{R^2})^{2-\kappa}$ *and choosing* $\alpha < \kappa/6$*. We will need this generalization in Remark 4.3.4 below.*

Proof of Lemma 4.2.1. The problem of existence of the mild solution (4.2.2) amounts to the existence of the stochastic integral. It exists almost surely, in particular, for (deterministic) Borel functions $\Phi : \mathbb{R}_+ \to \mathcal{T}_2(L_2(\mathbb{S}_R), L_2^1(\mathbb{S}_R))$, where the latter denotes the space of Hilbert-Schmidt operators from $L_2(\mathbb{S}_R)$ to $L_2^1(\mathbb{S}_R)$, with norm $\|\cdot\|_{\mathrm{HS}}$. Moreover, we have for $p \in (0,\infty)$, by the Burkholder-Davis-Gundy theorem [28, Thm 5.2.4]

$$\mathbb{E} \, \| \int_0^t \Phi(t) \, \mathrm{d}W(s) \, \|_{L_2^1(\mathbb{S}_R)}^p \le c_p \left(\mathbb{E} \int_0^t \|\Phi(t)\|_{\mathrm{HS}}^2 \, \mathrm{d}s \right)^{p/2} . \tag{4.2.3}$$

We note that for $p = 2$ this follows from the Itô isometry. For $p \ge 2$ it is a consequence of the Nelson's estimate (Lemma 2.6.9 above), since we consider deterministic Φ. We aim to modify [59, Thm. 6.10], which is based on [28, Thm 5.2.6] for our problem at hand. (We recall that [59, Thm. 6.10] considers stochastic convolutions of the form $\int_0^t S(t-s) \, Q \, \mathrm{d}W(s)$, where W is cylindrical Wiener process on $\mathcal{K}$, the operator $Q : \mathcal{K} \to \mathcal{H}$ is bounded and S is a semigroup on $\mathcal{H}$. We will check below that the argument still works if Q is omitted and S is considered as a family of operators from $\mathcal{K}$ to $\mathcal{H}$). Let $t \in [0,T]$ and define for $\alpha \in (0, 1/6)$

$$y(t,\bullet) := \int_0^t (t-s)^{-\alpha} \, \mathrm{e}^{-(t-s)Q_{R,N}} \, \mathrm{d}W_R(s,\bullet) \, .$$

To check if the integral is well defined, we estimate the Hilbert-Schmidt norm of the integrand: From Lemma 2.4.14 for arbitrary $T > 0$ one shows

$$\int_0^T s^{-2\alpha} \|\mathrm{e}^{-sQ_{R,N}}\|_{\mathrm{HS}}^2 \, \mathrm{d}s = \sum_{l=0}^{\infty} (2l+1) \, (1+\frac{l(l+1)}{R^2}) \int_0^T s^{-2\alpha} \, e^{-2s\langle l \rangle} \, \mathrm{d}s < \infty \, , \tag{4.2.4}$$

where $\langle l \rangle \simeq l^6$ was introduced in Remark 2.4.11. The preceding bound follows from the fact that for all $s \in [0,T]$ it holds

$$\int_0^T s^{-2\alpha} \, \mathrm{e}^{-2s\langle l \rangle} \mathrm{d}s \le \int_0^\infty s^{-2\alpha} \, \mathrm{e}^{-2s\langle l \rangle} \mathrm{d}s \; = \; (2\langle l \rangle)^{2\alpha - 1} \, \Gamma(1-2\alpha),$$

which is converging for fixed $l \in \mathbb{N}_0$ since $1 - 2\alpha > 0$. Consequently, Eq. (4.2.4) is finite if $\alpha < 1/6$. This follows from

$$\sum_{l=0}^{\infty} (2l+1) \, (1 + \frac{l(l+1)}{R^2}) \int_0^T s^{-2\alpha} \, e^{-2s\langle l \rangle} \, \mathrm{d}s \\ \le \; \Gamma(1-2\alpha) \sum_{l=0}^{\infty} (2l+1) \, (1 + \frac{l(l+1)}{R^2}) \, (2\langle l \rangle)^{2\alpha - 1} \, . \tag{4.2.5}$$

Making use of Eqs. (4.2.3), (4.2.4), we write

$$\mathbb{E}\int_0^t \|y(s,\bullet)\|^p_{L^1_2(\mathbb{S}_R)} \mathrm{d}s \;\le\; C_p \int_0^t \left|\int_0^s \|(s-r)^{-\alpha}\mathrm{e}^{-(s-r)Q_{R,N}}\|^2_{\mathrm{HS}}\,\mathrm{d}r\right|^{p/2} \mathrm{d}s \;<\; \infty, \tag{4.2.6}$$

Let $c_\alpha = \frac{\sin(2\pi\alpha)}{\pi}$ such that $c_\alpha \int_s^t (t-r)^{\alpha-1}\,(r-s)^{-\alpha}\,\mathrm{d}r = 1$ holds for every $0 < s < t$. Note that the integral over the set $\{(s,r) : 0 < s < r < t\}$ can be written in two equivalent ways $\int_0^t \int_s^t \ldots \mathrm{d}r\,\mathrm{d}W(s) = \int_0^t \int_0^r \ldots \mathrm{d}W(s)\,\mathrm{d}r$. Hence, implementing the preceding identity into Eq. (4.2.1) one gets

$$\begin{aligned} Z_{R,N}(t,\bullet) &= \mathrm{e}^{-tQ_{R,N}} z_{R,N} + c_\alpha \int_0^t \int_s^t (t-r)^{\alpha-1}\,(r-s)^{-\alpha}\,\mathrm{e}^{-(t-s)Q_{R,N}}\,\mathrm{d}r\,\mathrm{d}W_R(s,\bullet) \\ &= \mathrm{e}^{-tQ_{R,N}} z_{R,N} + c_\alpha \int_0^t \int_0^r (t-r)^{\alpha-1}\,(r-s)^{-\alpha}\,\mathrm{e}^{-(t-s)Q_{R,N}}\,\mathrm{d}W_R(s,\bullet)\,\mathrm{d}r. \end{aligned}$$

Using $\mathrm{e}^{-(t-s)Q_{R,N}} = \mathrm{e}^{-(t-r)Q_{R,N}}\,\mathrm{e}^{-(r-s)Q_{R,N}}$ one has

$$\begin{aligned} &\int_0^t \int_0^r (t-r)^{\alpha-1}\,(r-s)^{-\alpha}\,\mathrm{e}^{-(t-s)Q_{R,N}}\,\mathrm{d}W_R(s,\bullet)\,\mathrm{d}r \\ &\qquad = \int_0^t \mathrm{e}^{-(t-r)Q_{R,N}} \int_0^r (r-s)^{-\alpha}\,\mathrm{e}^{-(r-s)Q_{R,N}}\,\mathrm{d}W_R(s,\bullet)\,(t-r)^{\alpha-1}\,\mathrm{d}r. \end{aligned}$$

Hence,

$$Z_{R,N}(t,\bullet) = \mathrm{e}^{-tQ_{R,N}} z_{R,N} + c_\alpha \int_0^t \mathrm{e}^{-(t-r)Q_{R,N}}\,y(r,\bullet)\,(t-r)^{\alpha-1}\,\mathrm{d}r\,.$$

To proceed, consider the map $y \mapsto F_y$, where $F_y(t,\bullet) := \int_0^t \mathrm{e}^{-(t-r)Q_{R,N}}\,(t-r)^{\alpha-1}\,y(r,\bullet)\,\mathrm{d}r$. The statement will follow from Eq. (4.2.6) if we show that for every $\alpha \in (0,1/6)$ there exists $p > 0$ such that $y \mapsto F_y$ maps $L_p([0,\infty), L^1_2(\mathbb{S}_R))$ into $C([0,\infty), L^1_2(\mathbb{S}_R))$. Note that the semigroup $t \mapsto \mathrm{e}^{-tQ_{R,N}}$ is uniformly bounded in the operator norm on $L^1_2(\mathbb{S}_R)$ on any time interval due to the positivity of spectrum of the elliptic operator $Q_{R,N}$ and the map $t \mapsto (t-r)^{\alpha-1}$ belongs to $L_q([0,t))$ for $q \in [1, 1/(1-\alpha))$. From the Hölder inequality one infers that there exists a constant $C_T > 0$ such that for all $p > 1/\alpha$ it holds

$$\begin{aligned} \|F_y(t,\bullet)\|_{L^1_2(\mathbb{S}_R)} &\le \int_0^t (t-r)^{\alpha-1}\,\|y(r,\bullet)\|_{L^1_2(\mathbb{S}_R)}\,\mathrm{d}r \\ &\le \left(\int_0^t (t-r)^{q(\alpha-1)}\,\mathrm{d}r\right)^{1/q} \left(\int_0^t \|y(r,\bullet)\|^p_{L^1_2(\mathbb{S}_R)}\,\mathrm{d}r\right)^{1/p}. \end{aligned}$$

Consequently,

$$\sup_{t\in[0,T]} \|F_y(t,\bullet)\|^p_{L^1_2(\mathbb{S}_R)} \le C_T \int_0^T \|y(r,\bullet)\|^p_{L^1_2(\mathbb{S}_R)}\,\mathrm{d}r\,,$$

where $C_T := (\frac{T^{q(\alpha-1)+1}}{q(\alpha-1)+1})^{p/q}$. To proceed, we shall verify that the map $t \mapsto F_y(t)$ is continuous for every continuous function y with $y(0) = 0$. Fix $t > 0$. Using the semigroup property of $\mathrm{e}^{-(t+\Delta t-r)\,Q_{R,N}}$ one has

$$\begin{aligned} &\|F_y(t+\Delta t,\bullet) - F_y(t,\bullet)\|_{L^1_2(\mathbb{S}_R)} \\ &\quad = \| \int_0^t \left[(t+\Delta t-r)^{\alpha-1}\,\mathrm{e}^{-\Delta t Q_{R,N}} - (t-r)^{\alpha-1}\right]\,\mathrm{e}^{-(t-r)Q_{R,N}}\,y(r,\bullet)\,\mathrm{d}r \\ &\qquad + \int_t^{t+\Delta t} (t+\Delta t-r)^{\alpha-1}\,\mathrm{e}^{-(t+\Delta t-r)Q_{R,N}}\,y(r,\bullet)\,\mathrm{d}r\|_{L^1_2(\mathbb{S}_R)}. \end{aligned} \tag{4.2.7}$$

Consider the first integrand in the RHS of Eq. (4.2.7). It converges to zero as $\Delta t \to 0$ thanks to strong continuity of the semigroup $\mathrm{e}^{-(t-r)\,Q_{R,N}}$. Moreover, it is bounded by $C(t-r)^{\alpha-1}\,\|y(r,\bullet)\|_{L^1_2(\mathbb{S}_R)}$, which is integrable. Hence, by the dominated convergence theorem as $\Delta t \to 0$ this integral tends to zero. Consider the second integrand in the RHS of Eq. (4.2.7). One has

$$\begin{aligned} &\| \int_t^{t+\Delta t} (t+\Delta t-r)^{\alpha-1}\,\mathrm{e}^{-(t+\Delta t-r)\,Q_{R,N}}\,y(r,\bullet)\,\mathrm{d}r\|_{L^1_2(\mathbb{S}_R)} \\ &\qquad = \| \int_0^\infty \chi_{[0,\Delta t]}(r)\,r^{\alpha-1}\,\mathrm{e}^{-r\,Q_{R,N}}\,y(t+\Delta t-r,\bullet)\,\mathrm{d}r\|_{L^1_2(\mathbb{S}_R)}\,. \end{aligned}$$

Observe that as $\Delta t \to 0$ the integrand tends to zero, since $\chi_{[0,\Delta t]}(r)$ goes to zero. Moreover,

$$\begin{aligned} \|\chi_{[0,\Delta t]}(r)\,r^{\alpha-1}\,\mathrm{e}^{-rQ_{R,N}}\,y(t+\Delta t-r,\bullet)\|_{L^1_2(\mathbb{S}_R)} &\le C\,\chi_{[0,1]}(r)\,r^{\alpha-1}\,\|y(t+\Delta t-r,\bullet)\|_{L^1_2(\mathbb{S}_R)} \\ &\le C'\,\chi_{[0,1]}(r)\,r^{\alpha-1}\,(1+\|y(t-r,\bullet)\|_{L^1_2(\mathbb{S}_R)}), \end{aligned}$$

where we made use of continuity of y. As the last expression is integrable, by the dominated convergence theorem this integral tends to zero as $\Delta t \to 0$. This shows that $y \mapsto F_y$ is right-continuous for every continuous function y with $y(0) = 0$. For the proof of left-continuity, which is similar, we refer to [59, Thm. 6.10]. Noting that continuous functions are dense in L_p-space for all $1 \le p < \infty$ one concludes the proof of continuity of paths for almost all $\omega \in \Omega$. It now follows by dominated convergence and Eq. (4.2.3) that the solution belongs to $L_2(\Omega, \mathcal{F}, \{\mathcal{F}\}_{t\ge 0}, \mathbb{P})$. □

4.3 Non-linear equation

Let us come back to the stochastic quantization equation (4.1.1) whose invariant measure is $\mu^g_{R,N}$. It holds that

$$\mathrm{d}\varPhi^g_{R,N}(t,\bullet) = \sqrt{2}\,\mathrm{d}W_R(t,\bullet) - Q_{R,N}\,\varPhi^g_{R,N}(t,\bullet)\,\mathrm{d}t \\ - P'(\varPhi^g_{R,N}(t,\bullet), c_{R,N})\,\mathrm{d}t + (\varPhi^g_{R,N}(t,g))^{n-1} g\,\mathrm{d}t\,, \quad (4.3.1)$$

which is well-defined for fixed $R, N \in \mathbb{N}_+$. However, it becomes singular in the limit $N \to \infty$ due to Remark 4.1.2, i.e., its solution, if exists, will be bounded almost surely in a Bessel potential space of negative regularity. Evoking the Da Prato-Debussche trick [27], we write, for all $t \in [0,\infty)$, the random field $\varPhi^g_{R,N}(t,\bullet)$ as

$$\varPhi^g_{R,N}(t,\bullet) = \varPsi^g_{R,N}(t,\bullet) + Z_{R,N}(t,\bullet), \quad (4.3.2)$$

where $Z_{R,N}(t,\bullet)$ is given in Eq. (4.2.1) and $\varPsi^g_{R,N}(t,\bullet)$ is the mild solution to the following PDE

$$\partial_t \varPsi^g_{R,N} = -Q_{R,N}\varPsi^g_{R,N} - P'(\varPsi^g_{R,N} + Z_{R,N}, c_{R,N}) + ((\varPsi^g_{R,N} + Z_{R,N})(\bullet, g))^{n-1} g\,. \quad (4.3.3)$$

We choose the initial data $z_{R,N} \in L^1_2(\mathbb{S}_R)$, $\phi^g_{R,N} \in L^1_2(\mathbb{S}_R)$ of the solutions $\{Z_{R,N}(t,\bullet)\}_{t\geq 0}$, $\{\varPhi^g_{R,N}(t,\bullet)\}_{t\geq 0}$ to Eqs (4.2.1), and (4.3.1) as random variables with laws given by the respective measures $\nu_{R,N}$, $\mu^g_{R,N}$. This is legitimate, because we verified in Lemma 3.3.4 that these measures are concentrated on $L^1_2(\mathbb{S}_R) \subset \mathscr{D}'(\mathbb{S}_R)$. We will verify in Sec. 4.4 that these are invariant measures for the respective equations. Consequently, the resulting solutions are stationary, if they exist, cf. Lemmas 4.2.1, 4.3.2. However, $\varPsi_{R,N}(t,\bullet)$ might not be stationary.

Remark 4.3.1. *On account of Remark 4.1.2, $Z_{R,N}(t,\bullet)$ is of regularity $2-\kappa$ for all $\kappa > 0$. Using smoothing property of the heat kernel associated to the operator $Q_{R,N}$, we expect a better regularity for the remainder $\varPsi^g_{R,N}(t,\bullet)$. This property is of importance in Sec. 4.3.2, where we apply the energy method to Eq. (4.3.3) for proving the existence of the global solution corresponding to Eq. (4.3.3).*

Lemma 4.3.2. *The mild solution $\varPhi^g_{R,N}(t,\bullet) \in C([0,\infty), L^1_2(\mathbb{S}_R))$ to Eq. (4.3.1) with initial condition $\phi^g_{R,N} \in L^1_2(\mathbb{S}_R)$ exists in the sense of Def. 2.7.1.*

Proof. From Eq. (4.3.2) it suffices to verify the existence and the uniqueness of random fields $Z_{R,N}(t,\bullet)$ and $\varPsi^g_{R,N}(t,\bullet)$. By Lemma 4.2.1 the mild solution $Z_{R,N}(t,\bullet) \in C([0,\infty), L^1_2(\mathbb{S}_R))$ exists and unique. The existence and uniqueness of the local and global mild solution to Eq. (4.3.3), i.e., $\varPsi^g_{R,N}(t,\bullet) \in C([0,\infty), L^1_2(\mathbb{S}_R))$ are shown in Sec. 4.3.1. and Sec. 4.3.2. This finishes the proof. □

4.3.1 Local existence in time

This subsection is devoted to proving the existence of the local mild solution to Eq. (4.3.3) using a fixed point argument in the $L^1_2(\mathbb{S}_R)$-space. To this end, we shall adapt the setting presented in [98, Sec. 15.1]. A similar idea is implemented in [105, Sec. 3.2]. The proof is deterministic and relies mainly on two facts. First, for all $m \in \{1, 2, \cdots, n-1\}$, $p \in [2, \infty)$ and fixed $R, N \in \mathbb{N}_+$ the renormalized free field, $Z^{:m:}_{R,N}(t, \bullet)$ has L_p norms bounded in $t \in [0, T]$ with probability 1, cf. Remark 4.3.4 below. Second, the non-linearity $P'(\Phi^g_{R,N}(t, \bullet), c_{R,N}) + ((\Phi^g_{R,N})(t, g))^{n-1}\, g(\bullet) \in C([0, \infty), L_2(\mathbb{S}_R))$ is locally Lipschitz continuous map $L^1_2(\mathbb{S}_R) \to L_2(\mathbb{S}_R)$ for all $t \geq 0$, which is a consequence of a suitable Sobolev embedding and the Hölder inequality.

In Ch. 3 we introduced a set of real coefficient $(a_m)_{m\in\{0,\dots,n\}}$, which are fixed throughout the thesis. Now, we define a new set of coefficients $a_{m,l} := -\,a_{m+1}\,(m+1)!/(m-l)!l!$, for $l \in \{0, \dots, n-2\}$ and $m \in \{l, \dots, n-1\}$. Note that for some $C \in (0, \infty)$ and for all $0 \leq l \leq n-2$, $l \leq m \leq n-1$ it holds $|a_{m,l}| \leq C$. Moreover, from Eq. (2.6.4) for all $m \in \{1, \cdots, n-1\}$ one has

$$Z^{:m:}_{R,N} = \sum_{k=0}^{\lfloor m/2 \rfloor} \frac{(-1)^k\, m!}{(m-2k)!\, k!\, 2^k}\, (c_{R,N})^k\, Z^{m-2k}_{R,N}\,.$$

By definition $Z^{:0:}_{R,N} = 1$. Recall that $P'(\tau, c) := \partial_\tau P(\tau, c)$. It holds

$$P'(\Phi^g_{R,N}, c_{R,N}) = P'(\Psi^g_{R,N} + Z_{R,N}, c_{R,N}) = (\Psi^g_{R,N})^{n-1} - \sum_{l=0}^{n-2}\sum_{m=l}^{n-1} a_{m,l}\, Z^{:m-l:}_{R,N}\, (\Psi^g_{R,N})^l.$$

Let

$$\begin{aligned} &F(\Psi^g_{R,N}(s, \bullet)) \\ &\qquad = -P'((\Psi^g_{R,N} + Z_{R,N})(s, \bullet), c_{R,N}) + ((\Psi^g_{R,N} + Z_{R,N})(s, g))^{n-1}\, g(\bullet) \end{aligned} \tag{4.3.4}$$

and extend this definition from $s \mapsto \Psi^g_{R,N}(s, \bullet)$ to arbitrary $\Psi \in C([0, T]; L^1_2(\mathbb{S}_R))$.

Lemma 4.3.3. *Consider the mapping $\mathcal{K}$ defined by*

$$\mathcal{K}(\Psi(t, \bullet)) := \mathrm{e}^{-t\, Q_{R,N}}\, \psi^g_{R,N} - \Gamma(\Psi(t, \bullet)), \quad \Gamma(\Psi(t, \bullet)) = \int_0^t \mathrm{e}^{-(t-s)\, Q_{R,N}}\, F(\Psi(s, \bullet))\, \mathrm{d}s,$$

where $\psi^g_{R,N} := \phi^g_{R,N} - z_{R,N} \in L^1_2(\mathbb{S}_R)$. It holds that

$$\mathcal{K} \,:\, C([0, T]; L^1_2(\mathbb{S}_R)) \to C([0, T]; L^1_2(\mathbb{S}_R))$$

and there exists a constant $C \in (0,\infty)$ such that for every $\Psi_1, \Psi_2 \in C([0,T];L_2^1(\mathbb{S}_R))$ and all $g \in C^\infty(\mathbb{S}_R)$ it holds that

$$\|\mathcal{K}(\Psi_1) - \mathcal{K}(\Psi_2)\|_{C([0,T];L_2^1(\mathbb{S}_R))} \leq C\, T^{5/6}\, D(\Psi_1,\Psi_2)\, \|\Psi_1 - \Psi_2\|_{C([0,T];L_2^1(\mathbb{S}_R))}, \tag{4.3.5}$$

where $D(\Psi_1,\Psi_2) := \Big[2 + \sum_{l=1}^{n-2}\Big(\|\Psi_1\|^l_{C([0,T];L_2^1(\mathbb{S}_R))} + \|\Psi_2\|^l_{C([0,T];L_2^1(\mathbb{S}_R))}\Big)\Big]$.

Remark 4.3.4. *The constant C in the RHS of Eq. (4.3.5) depends on the quantities $\sup_{0\leq s\leq T}\|Z_{R,N}^{:m-l:}(s,\bullet)\|_{L_p(\mathbb{S}_R)}$. To justify that they are finite for any $T \geq 0$, we recall that we are now looking for solutions at a fixed UV cut-off N, thus we can express the Wick powers as polynomials in $Z_{R,N}$. For any monomial $Z_{R,N}^{\tilde m}$, $\tilde m \in \mathbb{N}$, we can write*

$$\|Z_{R,N}^{\tilde m}(s,\bullet)\|_{L_p(\mathbb{S}_R)} = \|Z_{R,N}(s,\bullet)\|^{\tilde m}_{L_{\tilde m p}(\mathbb{S}_R)} \leq c\|Z_{R,N}(s,\bullet)\|^{\tilde m}_{L_2^1(\mathbb{S}_R)}, \tag{4.3.6}$$

where we applied the Sobolev inequality given in Lemma 2.3.14. (We assumed here that $p \geq 2$, which can be ensured using the Hölder inequality). For almost all $\omega \in \Omega$, the RHS of Eq. (4.3.6) is bounded in $s \in [0,T]$ on compact sets by the continuity statement in Lemma 4.2.1. The argument can be generalized to treat $\|\vec\nabla_R Z_{R,N}^{:m-l:}(0,\bullet)\|_{L_p(\mathbb{S}_R)}$. Instead of (4.3.6), we are led to

$$\|\vec\nabla_R Z_{R,N}(s,\bullet) Z_{R,N}^{\tilde m}(s,\bullet)\|_{L_p(\mathbb{S}_R)} \leq \|Z_{R,N}(s,\bullet)\|_{L_2^{2-\kappa}(\mathbb{S}_R)}\|Z_{R,N}^{\tilde m}(s,\bullet)\|_{L_{p'}(\mathbb{S}_R)}$$

with some $0<\kappa<1$, where we applied the Hölder inequality with $1/p = 1/q' + 1/p'$, $q' \geq 2$, and then Remark 2.3.15. The second factor on the RHS is now estimated as in Eq. (4.3.6). For the finiteness of the first factor we refer to Remark 4.2.2.

Proof of Lemma 4.3.3. Observe that if $\psi^g_{R,N} \in L_2^1(\mathbb{S}_R)$ by Remark 2.4.12 one infers that $\mathrm{e}^{-t\,Q_{R,N}}\psi^g_{R,N}$ belongs to $L_2^1(\mathbb{S}_R)$-space as well. Furthermore, by estimate (2.4.2), for all $t \in [0,T]$ it holds that

$$\begin{aligned}\|\Gamma(\Psi)(t,\bullet)\|_{L_2^1(\mathbb{S}_R)} &\leq C\int_0^t \frac{1}{|t-s|^{1/6}}\,\|F(\Psi)(s,\bullet)\|_{L_2(\mathbb{S}_R)}\,\mathrm{d}s,\\ &\leq C\sup_{s\in[0,T]}\|F(\Psi)(s,\bullet)\|_{L_2(\mathbb{S}_R)}\int_0^t\frac{\mathrm{d}s}{|t-s|^{1/6}},\end{aligned} \tag{4.3.7}$$

where

$$\begin{aligned}\|F(\Psi)(s,\bullet)\|_{L_2(\mathbb{S}_R)} \leq\ & \|P'\big((\Psi+Z_{R,N})(s,\bullet),c_{R,N}\big)\|_{L_2(\mathbb{S}_R)}\\ &+\|\big((\Psi+Z_{R,N})(s,g)\big)^{n-1}g(\bullet)\|_{L_2(\mathbb{S}_R)} := R^{(1)}(s)+R^{(2)}(s)\,.\end{aligned} \tag{4.3.8}$$

By the triangle inequality one has

$$R^{(1)}(s) = \|P'((\varPsi + Z_{R,N})(s,\bullet), c_{R,N})(s,\bullet)\|_{L_2(\mathbb{S}_R)} \\ \leq \sum_{l=0}^{n-1}\sum_{m=l}^{n-1} |a_{m,l}|\, \|Z_{R,N}^{:m-l:}(s,\bullet)\,\varPsi(s,\bullet)^l\|_{L_2(\mathbb{S}_R)}\,.$$

From Remark 4.3.4 above one infers that the random field $Z_{R,N}^{:m-l:}(s,\bullet)$ is L_p-integrable with probability 1 for all $1 \leq m \leq n-1$ and for all $p \in [2,\infty)$. Using the Hölder inequality one gets $\|Z_{R,N}^{:m-l:}\,\varPsi^l\|_{L_2(\mathbb{S}_R)} \leq \|\varPsi^l\|_{L_{2q}(\mathbb{S}_R)}\,\|Z_{R,N}^{:m-l:}\|_{L_{2p}(\mathbb{S}_R)}$. Hence, from the assumption on the coefficients $|a_{m,l}|$ one verifies that there are $C, \hat{C}, \check{C} \in (0,\infty)$ such that

$$R^{(1)}(s) \leq C + \hat{C}\sum_{l=1}^{n-1}\|\varPsi(s,\bullet)\|^l_{L_{2ql}(\mathbb{S}_R)} \leq C + \check{C}\sum_{l=1}^{n-1}\|\varPsi(s,\bullet)\|^l_{L_2^1(\mathbb{S}_R)}\,, \tag{4.3.9}$$

where C depends on $\|Z_{R,N}^{:m-l:}\|_{L_{2p}(\mathbb{S}_R)}$, cf. Remark 4.3.4. Moving on to $R^{(2)}(s)$, from the Hölder inequality with $1/n + (n-1)/n = 1$ one gets

$$|\varPsi(s,g)| \leq \|\varPsi(s,\bullet)\|_{L_n(\mathbb{S}_R)}\,\|g\|_{L_{n/(n-1)}(\mathbb{S}_R)},$$
$$|Z_{R,N}(s,g)| \leq \|Z_{R,N}(s,\bullet)\|_{L_n(\mathbb{S}_R)}\,\|g\|_{L_{n/(n-1)}(\mathbb{S}_R)}.$$

Hence, for $C, \tilde{C}, C' \in (0,\infty)$ one has

$$\begin{aligned} R^{(2)}(s) &= \|((\varPsi + Z_{R,N})(s,g))^{n-1}g\|_{L_2(\mathbb{S}_R)} = \|g\|_{L_2(\mathbb{S}_R)}\,|(\Psi + Z_{R,N})(s,g)^{n-1}| \\ &\leq \|g\|_{L_\infty(\mathbb{S}_R)}\sum_{k=0}^{n-1}\binom{n-1}{k}|\varPsi(s,g)|^k\,|Z_{R,N}(s,g)|^{n-1-k} \\ &\leq C + C'\sum_{k=1}^{n-1}\binom{n-1}{k}\|\varPsi(s,\bullet)\|^k_{L_n(\mathbb{S}_R)}\,\|Z_{R,N}(s,\bullet)\|^{n-1-k}_{L_n(\mathbb{S}_R)} \\ &\leq C + \tilde{C}\sum_{k=1}^{n-1}\|\varPsi(s,\bullet)\|^k_{L_2^1(\mathbb{S}_R)}, \end{aligned} \tag{4.3.10}$$

where in the last step we used Remark 4.3.4 and Lemma 2.3.14. Combining Eqs. (4.3.9) and (4.3.10), one infers that $\Gamma(\varPsi(t,\bullet))$ takes values in $L_2^1(\mathbb{S}_R)$ if $\varPsi(t,\bullet) \in L_2^1(\mathbb{S}_R)$ for all $t \in [0,T]$. Now, we aim to verify that $\mathcal{K}(\varPsi(t,\bullet)) \in C([0,T]; L_2^1(\mathbb{S}_R))$ if $\varPsi \in C([0,T]; L_2^1(\mathbb{S}_R))$. Using Remark 2.4.12 it holds $\|\mathrm{e}^{-tQ_{R,N}}\,\psi_{R,N}\|_{L_2^1(\mathbb{S}_R)} \leq \|\psi_{R,N}\|_{L_2^1(\mathbb{S}_R)}$, which is due to positivity of the spectrum of the operator $Q_{R,N}$. It holds that

$$\lim_{\Delta t\to 0}\|\mathrm{e}^{-t\,Q_{R,N}}(\mathrm{e}^{-\Delta t\,Q_{R,N}} - 1)\,\psi^g_{R,N}\|_{L_2^1(\mathbb{S}_R)} = 0, \tag{4.3.11}$$

which is due to the strong continuity of the semigroup $t \mapsto \mathrm{e}^{-tQ_{R,N}}$. It remains to prove that $\Gamma(\Psi) \in C([0,T]; L^1_2(\mathbb{S}_R))$ if $\Psi \in C([0,T]; L^1_2(\mathbb{S}_R))$. To this end, we need to verify that the limit of

$$\|\Gamma(\Psi(t+\Delta t,\bullet)) - \Gamma(\Psi(t,\bullet))\|_{L^1_2(\mathbb{S}_R)} = \| \int_0^t [\mathrm{e}^{-(t+\Delta t-s)\, Q_{R,N}} - \mathrm{e}^{-(t-s)\, Q_{R,N}}]\, F(\Psi(s,\bullet))\, \mathrm{d}s + \int_t^{t+\Delta t} \mathrm{e}^{-(t+\Delta t-s)\, Q_{R,N}}\; F(\Psi(s,\bullet))\, \mathrm{d}s\, \|_{L^1_2(\mathbb{S}_R)} \quad (4.3.12)$$

tends to zero as $\Delta t \to 0$. Consider the first integral in the RHS of Eq. (4.3.12). Observe that $\|Q^{1/6}_{R,N}\, \mathrm{e}^{-(t-s)Q_{R,N}}\, (\mathrm{e}^{-\Delta t Q_{R,N}} - 1)\, F(\Psi(s,\bullet))\|_{L_2(\mathbb{S}_R)}$ is bounded by $(t - s)^{-1/6}\, \|F(\Psi(s,\bullet))\|_{L_2(\mathbb{S}_R)}$, which in integrable for any fixed for $s \neq t$. (The expression $\|F(\Psi(s,\bullet))\|_{L_2(\mathbb{S}_R)}$ is estimated as in the discussion (4.3.8)–(4.3.10) above). Moreover, for any fixed $s \neq t$ the following limit holds

$$\lim_{\Delta t \to 0} \|Q^{1/6}_{R,N}\, \mathrm{e}^{-(t-s)Q_{R,N}}\, (\mathrm{e}^{-\Delta t Q_{R,N}} - 1)\, F(\Psi(s,\bullet))\|_{L_2(\mathbb{S}_R)} = 0.$$

Thus, by dominated convergence theorem the first integral in the RHS of Eq. (4.3.12) converges to zero as $\Delta t \to 0$. Now consider the second integral in the RHS of Eq. (4.3.12). One has

$$\| \int_t^{t+\Delta t} \mathrm{e}^{-(t+\Delta t-s)Q_{R,N}}\; F(\Psi(s,\bullet))\, \mathrm{d}s \|_{L^1_2(\mathbb{S}_R)} \leq \| \int_0^\infty \chi_{[0,\Delta t]}(s)\, Q^{1/6}_{R,N}\, \mathrm{e}^{-sQ_{R,N}}\; F(\Psi(t+\Delta t - s,\bullet))\, \mathrm{d}s \|_{L_2(\mathbb{S}_R)}\,.$$

Note that

$$\|\chi_{[0,\Delta t]}(s)\, Q^{1/6}_{R,N}\, \mathrm{e}^{-s\, Q_{R,N}}\; F(\Psi(t+\Delta t - s,\bullet))\|_{L_2(\mathbb{S}_R)} \leq\; \chi_{[0,1]}(s)\, s^{-1/6}\, \|F(\Psi(t+\Delta t - s,\bullet))\|_{L_2(\mathbb{S}_R)},$$

where the expression $\|F(\Psi(t+\Delta t - s,\bullet))\|_{L_2(\mathbb{S}_R)}$ can be bounded as in the discussion (4.3.8)–(4.3.10) above by a constant independent of Δt. Furthermore, one has

$$\lim_{\Delta t \to 0} \|\chi_{[0,\Delta t]}(s)\, Q^{1/6}_{R,N}\, \mathrm{e}^{-s\, Q_{R,N}}\; F(\Psi(t+\Delta t - s,\bullet))\|_{L_2(\mathbb{S}_R)} = 0\,.$$

Having verified the requirements of the dominated convergence theorem one deduces that the second integral in the RHS of Eq. (4.3.12) converges to zero as $\Delta t \to 0$. This concludes that $\Gamma(\Psi) \in C((0,\infty); L^1_2(\mathbb{S}_R))$ if $\Psi \in C((0,\infty); L^1_2(\mathbb{S}_R))$. Combining the above results with Eq. (4.3.11) one shows that $\mathcal{K}(\Psi) \in C([0,T]; L^1_2(\mathbb{S}_R))$ if $\Psi \in C((0,\infty); L^1_2(\mathbb{S}_R))$. To verify the bound (4.3.5), let $\Psi_1, \Psi_2 \in C([0,T]; L^1_2(\mathbb{S}_R))$ and $t \in [0,T]$. Proceeding as in (4.3.7) above, one obtains

$$\begin{aligned}
&\|\Gamma(\Psi_1)(t,\bullet) - \Gamma(\Psi_2)(t,\bullet)\|_{L^1_2(\mathbb{S}_R)} \\
&\qquad \le C \int_0^t \frac{1}{|t-s|^{1/6}} \|F(\Psi_1)(s,\bullet) - F(\Psi_2)(s,\bullet)\|_{L_2(\mathbb{S}_R)} \, \mathrm{d}s \\
&\qquad\qquad \le C \sup_{s\in[0,T]} \|F(\Psi_1)(s,\bullet) - F(\Psi_2)(s,\bullet)\|_{L_2(\mathbb{S}_R)} \int_0^t \frac{\mathrm{d}s}{|t-s|^{1/6}} \,.
\end{aligned}$$

Performing the integral over $s \in [0,t]$ and taking the supremum over $0 \le t \le T$ of the resulting expression culminates in

$$\|\Gamma(\Psi_1) - \Gamma(\Psi_2)\|_{C([0,T];L^1_2(\mathbb{S}_R))} \le C\, T^{5/6} \, \|F(\Psi_1) - F(\Psi_2)\|_{C([0,T];L_2(\mathbb{S}_R))} \,, \tag{4.3.13}$$

where

$$\begin{aligned}
&\|F(\Psi_1(s,\bullet)) - F(\Psi_2(s,\bullet))\|_{L_2(\mathbb{S}_R)} \\
&\quad \le \|P'((\Psi_1 + Z_{R,N})(s,\bullet), c_{R,N}) - P'((\Psi_2 + Z_{R,N})(s,\bullet), c_{R,N})\|_{L_2(\mathbb{S}_R)} \\
&\quad + \|((\Psi_1 + Z_{R,N})(s,g))^{n-1} g(\bullet) - ((\Psi_2 + Z_{R,N})(s,g))^{n-1} g(\bullet)\|_{L_2(\mathbb{S}_R)} \\
&\qquad := R^{(11)}(s) + R^{(22)}(s) \,.
\end{aligned}$$

Using Remark B.5 combined with the Hölder inequality and with Remark 4.3.4, there is $\check{C}, C \in (0,\infty)$ such that it holds

$$\begin{aligned}
R^{(11)}(s) &\le \sum_{l=0}^{n-1} \sum_{m=l}^{n-1} |a_{m,l}| \, \|Z^{:m-l:}_{R,N}(s,\bullet)\|_{L_{2p}(\mathbb{S}_R)} \, \|(\Psi_1^l - \Psi_2^l)(s,\bullet)\|_{L_{2q}(\mathbb{S}_R)} \\
&\le C \, \|(\Psi_1 - \Psi_2)(s,\bullet)\|_{L_{4q}(\mathbb{S}_R)} \Big[2 + \sum_{l=1}^{n-2} \Big(\|\Psi_1(s,\bullet)\|^l_{L_{4ql}(\mathbb{S}_R)} + \|\Psi_2(s,\bullet)\|^l_{L_{4ql}(\mathbb{S}_R)} \Big) \Big] \\
&\le \check{C} \, \|(\Psi_1 - \Psi_2)(s,\bullet)\|_{L^1_2(\mathbb{S}_R)} \Big[2 + \sum_{l=1}^{n-2} \Big(\|\Psi_1(s,\bullet)\|^l_{L^1_2(\mathbb{S}_R)} + \|\Psi_2(s,\bullet)\|^l_{L^1_2(\mathbb{S}_R)} \Big) \Big],
\end{aligned} \tag{4.3.14}$$

where $p, q \ge 1$ with $1/p + 1/q = 1$ and in the last step we made use of Lemma 2.3.14. Utilizing the Hölder inequality, Remark 4.3.4, Remark B.5 and the triangle inequality for some $C, \hat{C} \in (0,\infty)$ one gets

$$\begin{aligned} R^{(22)}(s) \leq\ & \|g\|_{L_2(\mathbb{S}_R)} \sum_{k=0}^{n-1} \binom{n-1}{k} |Z_{R,N}(s,g)|^{n-1-k}\ |\Psi_1(s,g)^k - \Psi_2(s,g)^k| \\ & \leq C|\Psi_1(s,g) - \Psi_2(s,g)| \left[2 + \sum_{k=0}^{n-1} \left(|\Psi_1(s,g)|^k + |\Psi_2(s,g)|^k \right) \right] \\ & \leq C \,\|(\Psi_1 - \Psi_2)(s,\bullet)\|_{L_2(\mathbb{S}_R)} \left[2 + \sum_{k=1}^{n-2} \left(\|\Psi_1(s,\bullet)\|^k_{L_2(\mathbb{S}_R)} + \|\Psi_2(s,\bullet)\|^k_{L_2(\mathbb{S}_R)} \right) \right] \\ & \leq \hat{C} \,\|(\Psi_1 - \Psi_2)(s,\bullet)\|_{L^1_2(\mathbb{S}_R)} \left[2 + \sum_{k=1}^{n-2} \left(\|\Psi_1(s,\bullet)\|^k_{L^1_2(\mathbb{S}_R)} + \|\Psi_2(s,\bullet)\|^k_{L^1_2(\mathbb{S}_R)} \right) \right]. \end{aligned} \tag{4.3.15}$$

Combining Eqs. (4.3.14) and (4.3.15) and letting $C = 2\,(\check{C} \vee \hat{C})$ culminates in the following inequality

$$\begin{aligned} \|F(\Psi_1(s,\bullet)) - F(\Psi_2(s,\bullet))\|_{L_2(\mathbb{S}_R)} &\leq C \,\|(\Psi_1 - \Psi_2)(s,\bullet)\|_{L^1_2(\mathbb{S}_R)} \\ & \times \left[2 + \sum_{l=1}^{n-2} \left(\|\Psi_1(s,\bullet)\|^l_{L^1_2(\mathbb{S}_R)} + \|\Psi_2(s,\bullet)\|^l_{L^1_2(\mathbb{S}_R)} \right) \right]. \end{aligned} \tag{4.3.16}$$

The above bound implies that $F : L^1_2(\mathbb{S}_R) \to L_2(\mathbb{S}_R)$ is locally Lipschitz. One verifies Eq. (4.3.5) by taking the supremum over $s \in [0,T]$ of (4.3.16) and combining the resulting expression with Eq. (4.3.13). This finishes the proof. □

Theorem 4.3.5. *There exists $T > 0$ depending on the $L^1_2(\mathbb{S}_R)$-norm of $\psi^g_{R,N}$ such that Eq. (4.3.3) has a unique mild solution $\Psi^g_{R,N} \in C([0,T]; L^1_2(\mathbb{S}_R))$.*

Proof. The proof is an application of the standard contraction mapping argument. Let $\|\psi^g_{R,N}\|_{L^1_2(\mathbb{S}_R)} = L$. Thanks to the positivity of the spectrum of the elliptic operator $Q_{R,N}$ one gets $\|\mathrm{e}^{-tQ_{R,N}}\, \psi^g_{R,N}\|_{L^1_2(\mathbb{S}_R)} \leq L$ for all $0 \leq t \leq T$. We choose

$$H := \{\Psi^g_{R,N} \in C([0,T]; L^1_2(\mathbb{S}_R)) \ :\ \|\Psi^g_{R,N}\|_{C([0,T];L^1_2(\mathbb{S}_R))} \leq 2L\}\,.$$

Using Lemma 4.3.3 one infers that $\mathcal{K} : H \to H$ if we choose $T > 0$ such that

$$C\,T^{5/6}\,(2 + 2\sum_{l=1}^{n-2} L^l) =: \gamma\,, \tag{4.3.17}$$

where $0 < \gamma < 1$. Hence, for every $\Psi_1, \Psi_2 \in H$ it holds

$$\|\mathcal{K}(\Psi_1) - \mathcal{K}(\Psi_2)\|_{C([0,T];L^1_2(\mathbb{S}_R))} \leq \gamma \, \|\Psi_1 - \Psi_2\|_{C([0,T];L^1_2(\mathbb{S}_R))} .$$

Now, the existence of the unique mild solution $\Psi^g_{R,N} \in H \subset C([0,T]; L^1_2(\mathbb{S}_R))$ follows from the contraction mapping theorem. Observe that Eq. (4.3.17) implies that the existence-time depends on the $L^1_2(\mathbb{S}_R)$-norm of the initial data $\psi^g_{R,N}$. This concludes the proof. □

4.3.2 Global existence in time

In this section, we aim to prove the existence of the global mild solution to Eq. (4.3.3) in the $L^1_2(\mathbb{S}_R)$-space. In general, the $L^1_2(\mathbb{S}_R)$-norm of the solution might explode after a finite time interval. However, if one has an a priori estimate for $\|\Psi^g_{R,N}(t,\bullet)\|_{L^1_2(\mathbb{S}_R)}$-norm that is global in time, then the global existence of the solution follows from the local existence [56, p. 8]. To obtain such a priori estimate, we shall apply the energy method to Eq. (4.3.18) in the $L^1_2(\mathbb{S}_R)$-space. The proof relies mainly on the fact that for all $m \in \{1, 2, \cdots, n-1\}$, $p \in [2, \infty)$ and all $R, N \in \mathbb{N}_+$ the renormalized free fields $Z^{:m:}_{R,N}(t,\bullet)$ and their derivatives $\vec{\nabla}_R \, Z^{:m:}_{R,N}(t,\bullet)$ have L_p norms bounded in $t \in [0,T]$ with probability 1, cf. Remark 4.3.4. Moreover, we also make use of the negative sign in the RHS of Eq. (4.1.1), which is included in the definition of the coefficients $a_{m,l}$ introduced in the beginning of Sec. 4.3.1. One rewrites Eq. (4.3.3) as follows

$$\begin{aligned}(\partial_t + Q_{R,N}) \, \Psi^g_{R,N} &+ (\Psi^g_{R,N})^{n-1} \\ &= \sum_{l=0}^{n-2} \sum_{m=l}^{n-1} a_{m,l} \, Z^{:m-l:}_{R,N} \, (\Psi^g_{R,N})^l + ((\Psi^g_{R,N} + Z_{R,N})(\bullet, g))^{n-1} g \, . \qquad (4.3.18)\end{aligned}$$

Proposition 4.3.6. *For all $R, N \in \mathbb{N}$ there exist $C \in (0,\infty)$ and $p \in [2,\infty)$ such that for all $t \in (0,\infty)$ and all $g \in C^\infty(\mathbb{S}_R)$ it holds*

$$\begin{aligned}\partial_t \|\Psi^g_{R,N}(t,\bullet)\|^2_{L^1_2(\mathbb{S}_R)} &+ \|\Psi^g_{R,N}(t,\bullet)\|^2_{L^4_2(\mathbb{S}_R)} + \|\Psi^g_{R,N}(t,\bullet)\|^n_{L_n(\mathbb{S}_R)} \\ &+ \int_{\mathbb{S}_R} (\vec{\nabla}_R \Psi^g_{R,N})(t,\bullet)^2 \, (\Psi^g_{R,N})(t,\bullet)^{n-2} \, \rho_R(\mathrm{dx}) \\ &\leq C \sum_{k=0}^{n-1} \Big(\|Z^{:k:}_{R,N}(t,\bullet)\|^p_{L_p(\mathbb{S}_R)} + \|\vec{\nabla}_R Z^{:k:}_{R,N}(t,\bullet)\|^p_{L_p(\mathbb{S}_R)} \Big) .\end{aligned}$$

Remark 4.3.7. *On account of Remark 4.3.1 one deduces that for all $m \in \mathbb{N}_+$ and all $R, N \in \mathbb{N}_+$ the renormalized free fields $Z^{:m:}_{R,N}(t,\bullet)$ is of regularity $2-\kappa$ for all*

$\kappa > 0$. *Now from the smoothing property of the heat kernel associated to the operator* $Q_{R,N}$*, we expect that the solution to Eq.* (4.3.18) *to be of regularity* $8-\kappa$ *for all* $\kappa > 0$*. In particular,* $\|\Psi^g_{R,N}(t,\bullet)\|^2_{L^4_2(\mathbb{S}_R)}$ *is well-defined. The former fact implies that all product terms present in Eq.* (4.3.18) *can be bounded in some Sobolev spaces with positive regularity cf. [10, Sec. 2.7–8]. Hence, one can apply the energy method to Eq.* (4.3.18)*.*

Proof of Prop. 4.3.6. Let

$$R_1 := \sum_{l=0}^{n-2}\sum_{m=l}^{n-1} a_{m,l}\, Z^{:m-l:}_{R,N}(\Psi^g_{R,N})^l, \quad R_2 := ((\Psi^g_{R,N} + Z_{R,N})(\bullet, g))^{n-1} g(\bullet)\,.$$

To lighten the notation, let us denote $\Psi^g_{R,N}(t,\bullet) := \Psi^g(\bullet)$ and $Z^{:m:}_{R,N}(t,\bullet) := Z^{:m:}(\bullet)$. Multiplying both sides of Eq. (4.3.18) by $(1-\Delta_R)\Psi^g$ and integrating over $\mathbb{S}_R$ using the rotational invariance property of the measure $\rho_R(\mathrm{dx})$ we obtain the following terms

(A) $\langle (1-\Delta_R)\Psi^g, \partial_t \Psi^g\rangle_{L_2(\mathbb{S}_R)} = \frac{1}{2}\partial_t \|\Psi^g\|^2_{L^1_2(\mathbb{S}_R)}$.

(B) $\langle (1-\Delta_R)\Psi^g, Q_{R,N}\,\Psi^g\rangle_{L_2(\mathbb{S}_R)} \geq \frac{1}{N^4}\|\Psi^g\|^2_{L^4_2(\mathbb{S}_R)}$.

(C) $\langle (1-\Delta_R)\Psi^g, (\Psi^g)^{n-1}\rangle_{L_2(\mathbb{S}_R)} \leq \|\Psi^g\|^n_{L_n(\mathbb{S}_R)} + (n-1)\int_{\mathbb{S}_R} |\nabla_R \Psi^g(x)|^2 |\Psi^g(x)|^{n-2}\rho_R(\mathrm{dx})$.

(D) $\langle (1-\Delta_R)\Psi^g, R_1\rangle_{L_2(\mathbb{S}_R)} = \sum_{l=0}^{n-2}\sum_{m=l}^{n-1} a_{m,l}\,\langle (1-\Delta_R)\Psi^g,\, Z^{:m-l:}(\Psi^g)^l\rangle_{L_2(\mathbb{S}_R)}$.

(E) $\langle (1-\Delta_R)\Psi^g, R_2\rangle_{L_2(\mathbb{S}_R)} = \langle (1-\Delta_R)\Psi^g, ((\Psi^g + Z)(\bullet, g))^{n-1} g(\bullet)\rangle_{L_2(\mathbb{S}_R)}$.

The general strategy of the proof is to estimate Items (D) and (E) with Items (B) and (C) in order to find an upper bound in terms of the L_p-norms of $Z^{:m:}$ and $\vec{\nabla}_R Z^{:m:}$. To this end, we shall make use of the fact that $(1-\Delta_R)$ is a self-adjoint and positive operator repeatedly, see Remark 2.1.6. Consider Item (D). One has

$$\begin{aligned}\Big|\sum_{l=0}^{n-2}\sum_{m=l}^{n-1} a_{m,l}\,\langle (1-\Delta_R)\Psi^g, Z^{:m-l:}(\Psi^g)^l\rangle_{L_2(\mathbb{S}_R)}\Big| &\leq \Big|\sum_{l=0}^{n-2}\sum_{m=l}^{n-1} a_{m,l}\,\langle \Psi^g, Z^{:m-l:}(\Psi^g)^l\rangle_{L_2(\mathbb{S}_R)}\Big| \\ &+ \Big|\sum_{l=0}^{n-2}\sum_{m=l}^{n-1} a_{m,l}\,\langle\, (\vec{\nabla}_R\Psi^g), (\vec{\nabla}_R Z^{:m-l:})(\Psi^g)^l\rangle_{L_2(\mathbb{S}_R)}\Big| \\ &+ \Big|\sum_{l=0}^{n-2}\sum_{m=l}^{n-1} a_{m,l}\,\langle (\vec{\nabla}_R\Psi^g), (Z^{:m-l:})(\vec{\nabla}_R(\Psi^g)^l)\rangle_{L_2(\mathbb{S}_R)}\Big|\,. \end{aligned} \tag{4.3.19}$$

To find upper bounds for the three sums present in the RHS of Eq. (4.3.19), we shall take advantage of the fact that $Z^{:m-l:}$ and $\vec{\nabla}_R Z^{:m-l:}$ have L_p norms bounded in $t \in [0,T]$ with probability one for all $m \in \{1,2,\cdots,n-1\}$, $p \in [2,\infty)$ and all

$R, N \in \mathbb{N}_+$, repeatedly, see Remark 4.3.4. As a result, we apply Hölder's and Young's inequality with $1/\tilde{q}+1/p = 1$ such that $p \in [2,\infty)$ is very large and $\tilde{q} \in (1,\infty)$ is close to one. Consider the first sum in the above expression. Setting $(l+1)/q + 1/p = 1$, $q > n-1$, one infers that for all sufficiently small $\delta_{d_1} \in (0,\infty)$ there exists $C_{d_1} \in (0,\infty)$ such that it holds

$$\begin{aligned}\Big|\sum_{l=0}^{n-2}\sum_{m=l}^{n-1} a_{m,l}\,\langle \Psi^g, Z^{:m-l:}(\Psi^g)^l\rangle_{L_2(\mathbb{S}_R)}\Big| &\leq \sum_{l=0}^{n-2}\sum_{m=l}^{n-1} |a_{m,l}|\, \|\Psi^g\|^{l+1}_{L_q(\mathbb{S}_R)}\, \|Z^{:m-l:}\|_{L_p(\mathbb{S}_R)} \\ &\leq \sum_{l=0}^{n-2}\sum_{m=l}^{n-1} |a_{m,l}| \left(\delta'_{d_1} \|\Psi^g\|^q_{L_q(\mathbb{S}_R)} + C'_{d_1} \|Z^{:m-l:}\|^p_{L_p(\mathbb{S}_R)}\right) \\ &\leq \delta_{d_1} \|\Psi^g\|^n_{L_n(\mathbb{S}_R)} + C_{d_1} \sum_{k=0}^{n-1} \|Z^{:k:}\|^p_{L_p(\mathbb{S}_R)}, \end{aligned} \tag{4.3.20}$$

where we set $q = n$ in the last line. Consider the second sum in Eq. (4.3.19). Let $1/q+1/p = 1$ such that $p \in [2,\infty)$ is very large and $q \in (1,\infty)$ is close to one. Applying the Hölder and the Young inequalities, for every sufficiently small $\delta_{d_2} \in (0,\infty)$ there exists $C_{d_2} \in (0,\infty)$ such that

$$\begin{aligned}&\Big|\sum_{l=0}^{n-2}\sum_{m=l}^{n-1} |a_{m,l}|\, \langle (\vec{\nabla}_R \Psi^g), (\vec{\nabla}_R Z^{:m-l:})(\Psi^g)^l\rangle_{L_2(\mathbb{S}_R)}\Big| \\ &\quad\leq \sum_{l=0}^{n-2}\sum_{m=l}^{n-1} |a_{m,l}| \left(\int_{\mathbb{S}_R} |\vec{\nabla}_R \Psi^g(\mathrm{x})|^q\, |\Psi^g(\mathrm{x})|^{lq} \rho_R(\mathrm{dx})\right)^{1/q}\Big|\, \|(\vec{\nabla}_R Z^{:m-l:})\|_{L_p(\mathbb{S}_R)} \\ &\quad\leq \delta_{d_2} \sum_{l=0}^{n-2} \int_{\mathbb{S}_R} |\vec{\nabla}_R \Psi^g(\mathrm{x})|^q\, |\Psi^g(\mathrm{x})|^{lq} \rho_R(\mathrm{dx}) + C_{d_2} \sum_{k=1}^{n-1} \|(\vec{\nabla}_R Z^{:k:})\|^p_{L_p(\mathbb{S}_R)}\,. \end{aligned} \tag{4.3.21}$$

Let us fix a very large $p \in [2,\infty)$, then $q \in (1,\infty)$ will be fixed as well via $1/p+1/q = 1$ such that it is close to one. We need to study the following two cases:

I) $0 \leq l < n/2$. Let $1/2 + l/n = 1/q$. Using Hölder's and Young's inequality with $q/2 + ql/n = 1$ one obtains

$$\begin{aligned}\int_{\mathbb{S}_R} |\vec{\nabla}_R \Psi^g(\mathrm{x})|^q\, |\Psi^g(\mathrm{x})|^{lq}\, \rho_R(\mathrm{dx}) &\leq \|\vec{\nabla}_R \Psi^g\|^q_{L_2(\mathbb{S}_R)} \|\Psi^g\|^{lq}_{L_n(\mathbb{S}_R)} \\ &\leq \hat{\delta}\, \|\vec{\nabla}_R \Psi^g\|^2_{L_2(\mathbb{S}_R)} + \check{\delta}\, \|\Psi^g\|^n_{L_n(\mathbb{S}_R)} \leq \tilde{\delta}\, \|\Psi^g\|^2_{L^1_2(\mathbb{S}_R)} + \check{\delta}\, \|\Psi^g\|^n_{L_n(\mathbb{S}_R)}\,. \end{aligned} \tag{4.3.22}$$

The last bound follows from Remark 2.3.15.

II) $n/2 \le l < n-2$. One has $1 \le (2l-(n-2))\,q/(2-q) < n$ for $q \in (1,\infty)$ sufficiently closed to one. For some $\delta', \delta'' \in (0,\infty)$ one obtains

$$\begin{aligned}\int_{\mathbb{S}_R} |\vec{\nabla}_R \Psi^g(\mathrm{x})|^q\, |\Psi^g(\mathrm{x})|^{lq} \rho_R(\mathrm{dx}) &\le \Big(\int_{\mathbb{S}_R} |\vec{\nabla}_R \Psi^g(\mathrm{x})^2|\, |\Psi^g(\mathrm{x})|^{(n-2)}\, \rho_R(\mathrm{dx})\Big)^{q/2} \\ &\quad \times \Big(\int_{\mathbb{S}_R} |\Psi^g(\mathrm{x})|^{(2l-(n-2))q/(2-q)}\, \rho_R(\mathrm{dx})\Big)^{1-q/2} \\ &\le \delta' \Big(\int_{\mathbb{S}_R} |\vec{\nabla}_R \Psi^g(\mathrm{x})|^2\, |\Psi^g(\mathrm{x})|^{(n-2)}\, \rho_R(\mathrm{dx})\Big) + \delta''\, \|\Psi^g\|^n_{L_n(\mathbb{S}_R)} + \delta''. \quad (4.3.23)\end{aligned}$$

The above follows from the Hölder and the Young inequalities with $(1-\frac{q}{2})+\frac{q}{2}=1$ combined with the fact that for all $1 \le \gamma < n$ there are $C, \hat{C} \in (0,\infty)$ such that one has

$$\begin{aligned}\|\Psi^g\|^\gamma_{L_\gamma(\mathbb{S}_R)} &\le C\, \|\Psi^g\|^\gamma_{L_n(\mathbb{S}_R)} \\ &\le C(1+\|\Psi^g\|_{L_n(\mathbb{S}_R)})^\gamma \le C(1+\|\Psi^g\|_{L_n(\mathbb{S}_R)})^n \le \hat{C}(1+\|\Psi^g\|^n_{L_n(\mathbb{S}_R)}). \quad (4.3.24)\end{aligned}$$

Consider the third sum in Eq. (4.3.19). Let $1/q+1/p=1$ with $p \in [2,\infty)$ very large and $q \in (1,\infty)$ sufficiently close to one. Note that $\vec{\nabla}_R(\Psi)^l = l\,\Psi^{l-1}(\vec{\nabla}_R\Psi)$ and recall that $(\sum_{l=0}^n x^l)' = \sum_{l=1}^n l\, x^{l-l} = \sum_{l=0}^{n-1}(l+1)\, x^l$. For every small enough $\delta_{d_3} \in (0,\infty)$ there exists some $C_{d_3} \in (0,\infty)$ such that one has

$$\begin{aligned}&\Big| \sum_{l=0}^{n-2}\sum_{m=l}^{n-1} a_{m,l}\, \langle(\vec{\nabla}_R\Psi^g), (Z^{:m-l:})(\vec{\nabla}_R(\Psi^g)^l\rangle_{L_2(\mathbb{S}_R)} \Big| \\ &\quad \le \delta_{d_3} \sum_{l=1}^{n-2} \int_{\mathbb{S}_R} |\vec{\nabla}_R\Psi^g(\mathrm{x})|^{2q}\, |\Psi^g(\mathrm{x})|^{q(l-1)}\, \rho_R(\mathrm{dx}) + C_{d_3} \sum_{k=0}^{n-1} \|Z^{:k:}\|^p_{L_p(\mathbb{S}_R)}\,. \quad (4.3.25)\end{aligned}$$

The last estimate follows from the Hölder and the Young inequalities. Observe that $q = q(l-1)/(n-2)+r$, where $r = q(n-2-(l-1))/((n-2)$ such that for all $n \ge 4$ and for $q \in (1,\infty)$ sufficiently close to one it holds $r \ge 1$. Thus, for all $\hat{\delta} \in (0,\infty)$ there exists $\check{C}_{d_3} \in (0,\infty)$ such that it holds

$$\begin{aligned}\int_{\mathbb{S}_R} |\vec{\nabla}_R\Psi^g(\mathrm{x})|^{2q}\, |\Psi^g(\mathrm{x})|^{q(l-1)}\, \rho_R(\mathrm{dx}) &\le \Big(\int_{\mathbb{S}_R} |\vec{\nabla}_R\Psi^g(\mathrm{x})|^2\, |\Psi^g(\mathrm{x})|^{(n-2)} \rho_R(\mathrm{dx})\Big)^{q(l-1)/(n-2)} \\ &\quad \times \Big(\int_{\mathbb{S}_R} |\vec{\nabla}_R\Psi^g(\mathrm{x})|^{2q(n-2-(l-1))/((n-2)-q(l-1))} \rho_R(\mathrm{dx})\Big)^{1-q(l-1)/(n-2)} \\ &\le \hat{\delta} \int_{\mathbb{S}_R} |\vec{\nabla}_R\Psi^g(\mathrm{x})|^2\, |\Psi^g(\mathrm{x})|^{(n-2)}\, \rho_R(\mathrm{dx}) + \check{C}_{d_3} \int_{\mathbb{S}_R} |\vec{\nabla}_R\Psi^g(\mathrm{x})|^{2\tilde{r}}\, \rho_R(\mathrm{dx})\,. \quad (4.3.26)\end{aligned}$$

where $\tilde{r} := q(n-2-(l-1))/((n-2)-q(l-1))$. We used above the Hölder and the Young inequalities for $1/p' + 1/q' = 1$ with $1/p' = q(l-1)/(n-2)$ and $1/q' = 1 - q(l-1)/(n-2)$. To proceed, one uses Remark 2.3.15 to write $\int_{\mathbb{S}_R} |\vec{\nabla}_R \Psi^g)|^{2\tilde{r}}\, \rho_R(\mathrm{dx}) \leq C_{\tilde{r}}\, \|\Psi^g\|^{2\tilde{r}}_{L^2_2(\mathbb{S}_R)}$ for some $C_{\tilde{r}} \in (0,\infty)$. Utilizing Remark 2.2.4 one gets $\|\Psi^g\|^2_{L_2(\mathbb{S}_R)} = \sum_{l=0}^{\infty} \|\Psi^g_l\|^2_{L_2(\mathbb{S}_R)} = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} |\Psi^g_{lm}|^2$. Using the notation introduced in Remark 2.4.11 one obtains

$$\begin{aligned}
\|\Psi^g\|^2_{L^2_2(\mathbb{S}_R)} &= \sum_{l=0}^{\infty} \ll l \gg\, \|\Psi^g_l\|^2_{L_2(\mathbb{S}_R)} = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \ll l \gg^2\, |\Psi^g_{l,m}|^2 \\
&\leq \left(\sum_{l=0}^{\infty}\sum_{m=-l}^{l} \ll l \gg^3\, |\Psi^g_{l,m}|^2\right)^{2/3} \left(\sum_{l=0}^{\infty}\sum_{m=-l}^{l} |\Psi^g_{l,m}|^2\right)^{1/3} \qquad (4.3.27)\\
&\leq \|\Psi^g\|^{4/3}_{L^3_2(\mathbb{S}_R)}\, \|\Psi^g\|^{2/3}_{L_2(\mathbb{S}_R)} \leq \tilde{C}_{d_3} \|\Psi^g\|^{4/3}_{L^3_2(\mathbb{S}_R)}\, \|\Psi^g\|^{2/3}_{L_n(\mathbb{S}_R)},
\end{aligned}$$

where we used the Hölder inequality for sequences. Now using the Young inequality and the argument in Eq. (4.3.24) above, one obtains

$$\begin{aligned}
\int_{\mathbb{S}_R} |\vec{\nabla}_R \Psi^g)|^{2\tilde{r}}\, \rho_R(\mathrm{dx}) \leq\, C_{\tilde{r}}\, \|\Psi^g\|^{2\tilde{r}}_{L^2_2(\mathbb{S}_R)} &\leq C_{\tilde{r}}\, \tilde{C}^{\tilde{r}}_{d_3} \|\Psi^g\|^{(4/3)\tilde{r}}_{L^3_2(\mathbb{S}_R)}\, \|\Psi^g\|^{(2/3)\tilde{r}}_{L_n(\mathbb{S}_R)} \\
&\leq \delta' \|\Psi\|^2_{L^3_2(\mathbb{S}_R)} + \overline{\delta} \|\Psi\|^n_{L_n(\mathbb{S}_R)} + \overline{\delta}. \quad (4.3.28)
\end{aligned}$$

Consider Item (E). One gets

$$\begin{aligned}
&\langle (1-\Delta_R)\Psi^g, R_2\rangle_{L_2(\mathbb{S}_R)} \\
&\qquad = \langle \Psi^g, ((\Psi^g + Z)(g))^{n-1} g\rangle_{L_2(\mathbb{S}_R)} + \langle (-\Delta_R)\Psi^g, ((\Psi^g + Z)(g))^{n-1} g\rangle_{L_2(\mathbb{S}_R)}.
\end{aligned}$$

From the Hölder and the Young inequalities combined with the relation $\|g\|^n_{L_{n-1/n}} < 1/2$ given in Eq. (3.3.3) and with the fact that for all $k > 1$ and $f(a_i) \in \mathbb{R}_+$ it holds $(\sum_{i=1}^n f(a_i))^k \leq n^{k-1} \sum_{i=1}^n f(a_i)^k$ one obtains

$$\begin{aligned}
&\langle \Psi^g, ((\Psi^g + Z)(g))^{n-1} g\rangle_{L_2(\mathbb{S}_R)} \leq |\langle \Psi^g, g\rangle_{L_2(\mathbb{S}_R)}\, |((\Psi^g + Z)(g))^{n-1}| \\
&\quad \leq\, \|\Psi^g\|_{L_n(\mathbb{S}_R)}\, \|g\|_{L_{n/n-1}(\mathbb{S}_R)}\, \left[2^{n-2}\, \|g\|^{n-1}_{L_{n/n-1}(\mathbb{S}_R)}\right] \left[\|\Psi^g\|^{n-1}_{L_n(\mathbb{S}_R)} + \|Z\|^{n-1}_{L_n(\mathbb{S}_R)}\right] \\
&\quad \leq \delta_{e_1} \|\Psi^g\|^n_{L_n(\mathbb{S}_R)} + \delta_{e_2} \|Z\|^n_{L_n(\mathbb{S}_R)}\,, \quad (4.3.29)
\end{aligned}$$

where $\delta_{e_1}, \delta_{e_2} \in (0,\infty)$. Furthermore, from the Hölder and the Young inequalities combined with the assumption $\|\Delta_R\, g\|^n_{L_{n-1/n}} < 1/2$ given in Eq. (3.3.3) and with the fact that $(-\Delta_R)$ is self-adjoint one has

$$|\langle -\Delta_R \Psi^g, ((\Psi^g + Z)(\bullet, g))^{n-1} g\rangle_{L_2(\mathbb{S}_R)}| = |\langle \Psi^g, -\Delta_R\, g\rangle_{L_2(\mathbb{S}_R)}| \; \left|((\Psi^g + Z)(\bullet, g))^{n-1}\right|$$
$$\leq \|\Psi^g\|_{L_n(\mathbb{S}_R)} \, \|\Delta_R\, g\|_{L_{n-1/n}(\mathbb{S}_R)} \left[2^{n-2}\, \|g\|^{n-1}_{L_{n/n-1}(\mathbb{S}_R)}\right] \left[\|\Psi^g\|^{n-1}_{L_n(\mathbb{S}_R)} + \|Z\|^{n-1}_{L_n(\mathbb{S}_R)}\right]$$
$$\leq \delta_{e_4} \|\Psi^g\|^n_{L_n(\mathbb{S}_R)} + \delta_{e_5}\, \|Z\|^n_{L_n(\mathbb{S}_R)}, \quad (4.3.30)$$

where $\delta_{e_4}, \delta_{e_5} \in (0,\infty)$. Putting Items (A), (B), (C), and Eqs. (4.3.20), (4.3.21), (4.3.22), (4.3.23), (4.3.25), (4.3.26), (4.3.27), (4.3.28), (4.3.29) and (4.3.30) together one infers that for all $R, N \in \mathbb{N}$ there exist $C \in (0,\infty)$ and $p \in [2,\infty)$ such that for all $t \in (0,\infty)$ and all $g \in C^\infty(\mathbb{S}_R)$ it holds

$$\frac{1}{2}\, \partial_t \|\Psi^g_{R,N}(t,\bullet)\|^2_{L^1_2(\mathbb{S}_R)} + (1/N^4 - \delta_{d_2} \tilde{\delta}\, C_\sigma - \delta_{d_3} \check{C}_{d_3} \delta'\, C_\sigma)\, \|\Psi^g_{R,N}(t,\bullet)\|^2_{L^4_2(\mathbb{S}_R)}$$
$$+ (1 - \delta_{d_1} - \delta_{d_2}\delta'' - \delta_{d_2}\check{\delta} - \delta_{d_3}\check{C}_{d_3}\bar{\delta} - \delta_{e_1} - \delta_{e_4})\, \|\Psi^g_{R,N}(t,\bullet)\|^n_{L_n(\mathbb{S}_R)}$$
$$+ (n - 1 - \delta_{d_2}\delta' - \delta_{d_3}\hat{\delta}) \int_{\mathbb{S}_R} |\vec{\nabla}_R \Psi^g_{R,N}(t,\mathrm{x})|^2\, |\Psi^g_{R,N}(t,\mathrm{x})|^{n-2}\, \rho_R(\mathrm{dx})$$
$$\leq (\delta_{d_2}\delta'' + \delta_{d_3}\check{C}_{d_3}\bar{\delta}) + C \sum_{k=0}^{n-1} \left(\|Z^{:k:}_{R,N}(t,\bullet)\|^p_{L_p(\mathbb{S}_R)} + \|\vec{\nabla}_R Z^{:k:}_{R,N}(t,\bullet)\|^p_{L_p(\mathbb{S}_R)}\right),$$

where we used the fact that for $\sigma \in \{1,2,3\}$ there is $C_\sigma \in (0,\infty)$ such that $\|\bullet\|_{L^\sigma_2(\mathbb{S}_R)} \leq C_\sigma\, \|\bullet\|_{L^4_2(\mathbb{S}_R)}$ and we set $C = (C_{d_1} \vee C_{d_2} \vee C_{d_3} \vee \delta_{e_2} \vee \delta_{e_5})$ in the last line. Observe that the constant term $(\delta_{d_2}\delta'' + \delta_{d_3}\check{C}_{d_3}\bar{\delta})$ can be absorbed in the sum with $k = 0$. This finishes the proof. □

Theorem 4.3.8. *For any initial condition $\psi^g_{R,N} \in L^1_2(\mathbb{S}_R)$, all $g \in C^\infty(\mathbb{S}_R)$ the global in time solution $\Psi^g_{R,N} \in C([0,\infty), L^1_2(\mathbb{S}_R))$ to Eq. (4.3.18) exists in the sense of Def. 2.7.1.*

Proof. Using Prop. 4.3.6 one deduces that for all $R, N \in \mathbb{N}_+$, there exist $C \in (0,\infty)$ and $p \in [2,\infty)$ such that for all $T \in (0,\infty)$ and all $g \in C^\infty(\mathbb{S}_R)$, it holds

$$\|\Psi^g_{R,N}(T,\bullet)\|^2_{L^1_2(\mathbb{S}_R)} + \int_0^T \|\Psi^g_{R,N}(t,\bullet)\|^n_{L_n(\mathbb{S}_R)}\, \mathrm{d}t$$
$$+ \int_0^T \int_{\mathbb{S}_R} (\vec{\nabla}_R \Psi^g_{R,N})(t,\bullet)^2\, (\Psi^g_{R,N})(t,\bullet)^{n-2}\, \rho_R(\mathrm{dx})\, \mathrm{d}t \;\leq\; \|\psi^g_{R,N}\|^2_{L^1_2(\mathbb{S}_R)}$$
$$+ C \sum_{k=0}^{n-1} \int_0^T \left(\|Z^{:k:}_{R,N}(t,\bullet)\|^p_{L_p(\mathbb{S}_R)} + \|\vec{\nabla}_R Z^{:k:}_{R,N}(t,\bullet)\|^p_{L_p(\mathbb{S}_R)}\right) \mathrm{d}t\,.$$

From the preceding bound one infers that the solution to Eq. (4.3.18) cannot explode in a finite time and we have an explicit upper bound on its growth in term of $\|\psi^g_{R,N}\|^2_{L^1_2(\mathbb{S}_R)}$ and the L_p−norms of $Z^{:m:}_{R,N}(t,\bullet)$, and $\vec{\nabla}_R Z^{:m:}_{R,N}(t,\bullet)$. Noting that for all $m \in \{1,2,\cdots,n-1\}$, $p \in [2,\infty)$ and all $R,N \in \mathbb{N}_+$ the renormalized free fields $Z^{:m:}_{R,N}(t,\bullet)$ and their derivatives $\vec{\nabla}_R Z^{:m:}_{R,N}(t,\bullet)$ have L_p−norms bounded in $t \in [0,T]$ with probability 1, cf. Remark 4.3.4 one concludes the proof. □

Lemma 4.3.9. *It holds*

$$\Psi^g_{R,N} \in C([0,\infty), L^1_2(\mathbb{S}_R)) \cap C((0,\infty), L^3_2(\mathbb{S}_R)) \cap C^1((0,\infty), L^{-3}_2(\mathbb{S}_R))$$

and the following equality $\partial_t \Psi^g_{R,N} = -Q_{R,N}\Psi^g_{R,N} - P'(\Phi^g_{R,N}, c_{R,N}) + (\Phi^g_{R,N})(t,g)^{n-1} g$ *is satisfied in* $C((0,\infty), L^{-3}_2(\mathbb{S}_R))$.

Proof. See Appendix C. □

4.4 Invariant measures

The aim of this section is to study the long time behaviour of the semigroups associated to Eqs. (4.2.1) and (4.3.1) in the sense of Def. 2.7.2. We shall verify that for all $t \geq 0$ and all $R,N \in \mathbb{N}_+$ one has $\mathrm{Law}(Z_{R,N}(t,\bullet)) = \mathrm{Law}(z_{R,N}) = \nu_{R,N}$ and $\mathrm{Law}(\Phi^g_{R,N}(t,\bullet)) = \mathrm{Law}(\phi^g_{R,N}) = \mu^g_{R,N}$ on $\mathscr{D}'(\mathbb{S}_R)$, as anticipated in Sec. 4.3. Recall that we let $\nu_{R,N} := \mathcal{N}(0, G_{R,N})$ and $\mu^g_{R,N}$ was given in Eq. (3.3.4). Combining the intertwining property of the map $\jmath^*_R$ with the former result one deduces that for all $R,N \in \mathbb{N}_+$ and all $t \geq 0$ it holds $\mathrm{Law}(\jmath^*_R Z_{R,N}(t,\bullet)) = \mathrm{Law}(\jmath^*_R z_{R,N}) = \jmath^*_R \sharp \nu_{R,N}$ and $\mathrm{Law}(\jmath^*_R \Phi^g_{R,N}(t,\bullet)) = \mathrm{Law}(\jmath^*_R \phi_{R,N}) = \jmath^*_R \sharp \mu^g_{R,N}$ on $\mathscr{S}'(\mathbb{R}^2)$. All Hilbert space scalar products $\langle \bullet, \bullet \rangle$ in this section are in $\tilde{H} = L^1_2(\mathbb{S}_R)$.

Gaussian invariant measure

For all $R,N \in \mathbb{N}_+$ consider the linear operator $Q_t : L^1_2(\mathbb{S}_R) \to L^1_2(\mathbb{S}_R)$ defined by $Q_t\phi := 2\int_0^t \mathrm{e}^{-2s\,Q_{R,N}}\,\phi\,\mathrm{d}s$ for $\phi \in L^1_2(\mathbb{S}_R)$. On account of Remark 2.4.13 the operator Q_∞ is of trace-class, i.e., $\mathrm{Tr}(Q_\infty) = \mathrm{Tr}(G_{R,N}) < \infty$. Let $z \in L^1_2(\mathbb{S}_R)$ and $F \in B_\mathrm{b}(L^1_2(\mathbb{S}_R))$ be a bounded Borel function. One denotes the semigroup (2.7.3) associated to Eq. (4.2.1) by

$$\begin{aligned}\mathcal{R}_{R,t}\,F(z) :=& \mathbb{E}\big[F(Z_{R,N}(t,\bullet))\,|\,Z_{R,N}(0,\bullet) = z\big]\\ =& \int_{L^1_2(\mathbb{S}_R)} F(\phi)\,\mathcal{N}(\mathrm{e}^{-t\,Q_{R,N}}\,z, Q_t)(\mathrm{d}\phi).\end{aligned}$$

A probability measure ν on $\mathcal{M}(L^1_2(\mathbb{S}_R))$ is invariant for the semigroup $\mathcal{R}_{R,t}$ in the sense of Def. 2.7.3 if and only if $\widehat{\nu}(f) = \widehat{\nu}(\mathrm{e}^{-t\,Q_{R,N}} f)\,\mathrm{e}^{-\frac{1}{2}\langle Q_t f, f\rangle}$ for all $t \geq 0$ and $f \in L^1_2(\mathbb{S}_R)$. To see this, let $F_f(\phi) = \mathrm{e}^{i\langle f,\phi\rangle} \in B_{\mathrm{b}}(L^1_2(\mathbb{S}_R))$. Using Def. 2.6.1 one has

$$\mathcal{R}_{R,t}\,F_f(z) = \int_{L^1_2(\mathbb{S}_R)} \mathrm{e}^{i\langle f,\phi\rangle}\,\mathcal{N}(\mathrm{e}^{-t\,Q_{R,N}}\,z, Q_t)(\mathrm{d}\phi) = \mathrm{e}^{i\langle f,\mathrm{e}^{-t\,Q_{R,N}}\,z\rangle}\,\mathrm{e}^{-\frac{1}{2}\langle Q_t f,f\rangle}\,.$$

The preceding expression implies that $\mathcal{R}_{R,t}\,F_f(z)$ is the characteristic function of the measure $\mathcal{N}(\mathrm{e}^{-tQ_{R,N}}\,z, Q_t)$. It holds that

$$\int_{L^1_2(\mathbb{S}_R)} \mathcal{R}_{R,t}\,F_f(\phi)\,\nu(\mathrm{d}\phi) = \widehat{\nu}(\mathrm{e}^{-t\,Q_{R,N}} f)\,\mathrm{e}^{-\frac{1}{2}\langle Q_t f,f\rangle}. \tag{4.4.1}$$

Now from Def. 2.7.3 a measure ν is invariant for Eq. (4.2.1) if Eq. (4.4.1) is equal to $\widehat{\nu}(f) = \int_{L^1_2(\mathbb{S}_R)} \mathrm{e}^{i\langle f,\phi\rangle}\nu(\mathrm{d}\phi)$. Observe that the requirement $\widehat{\nu}(f) = \widehat{\nu}(\mathrm{e}^{-tQ_{R,N}} f)\,\mathrm{e}^{-\frac{1}{2}\langle Q_t f,f\rangle}$ is equivalent to saying that ν is invariant if and only if $\nu = \big(\mathrm{e}^{-t\,Q_{R,N}}\nu\big) * \mathcal{N}(0, Q_t)$ for all $t \geq 0$. It follows from the fact that $\mathcal{F}(f * g) = \mathcal{F}(f) \times \mathcal{F}(g)$, see Remark 2.5.10.

Theorem 4.4.1. *The following conditions are equivalent*

(A) $\sup_{t\geq 0} \operatorname{Tr} Q_t < \infty$

(B) $\nu = \mathcal{N}(0, Q_\infty)$ *is an invariant measure for Eq.* (4.2.1)*, i.e., if* $z_{R,N}$ *has a distribution* ν *and is independent of the cylindrical Wiener process* W_R*, then the* $L^1_2(\mathbb{S}_R)$*-valued stochastic process* $Z_{R,N}(t)$ *has distribution* ν *for any* $t \geq 0$.

Proof. Similar result can be found in [59, Thm. 6,22] and [28, Thm. 6.2.1]. Recall Eq. (4.2.1). One has

$$Z_{R,N}(t,\bullet) = \mathrm{e}^{-tQ_{R,N}}\,z_{R,N} + \sqrt{2}\int_0^t \mathrm{e}^{-tQ_{R,N}}\,\mathrm{d}W_R(t,\bullet)\,.$$

Using the fact that the distribution of $\sqrt{2}\int_0^t \mathrm{e}^{-tQ_{R,N}}\,\mathrm{d}W_R(t,\bullet)$ is $\mathcal{N}(0, Q_t)$, one infers that ν is invariant for Eq. (4.2.1) if $\nu = (\mathrm{e}^{-tQ_{R,N}}\nu) * \mathcal{N}(0, Q_t)$ for all $t \geq 0$. Taking the Fourier transform of both sides leads to

$$\widehat{\nu}(f) = (\widehat{\mathrm{e}^{-tQ_{R,N}}\nu})(f)\,\times \widehat{\mathcal{N}(0, Q_t)}(f) \tag{4.4.2}$$

for $f \in L^1_2(\mathbb{S}_R)$. It holds that

$$\begin{aligned}(\widehat{\mathrm{e}^{-tQ_{R,N}}\nu})(f) &= \int_{L^1_2(\mathbb{S}_R)} \mathrm{e}^{i\langle f,\phi\rangle}\,\left(\mathrm{e}^{-tQ_{R,N}}\nu\right)(\mathrm{d}\phi) = \int_{L^1_2(\mathbb{S}_R)} \mathrm{e}^{i\langle f,\mathrm{e}^{-tQ_{R,N}}\phi\rangle}\,\nu(\mathrm{d}\phi)\\ &= \int_{L^1_2(\mathbb{S}_R)} \mathrm{e}^{i\langle \mathrm{e}^{-tQ_{R,N}} f,\phi\rangle}\,\nu(\mathrm{d}\phi) = \widehat{\nu}(\mathrm{e}^{-tQ_{R,N}} f).\end{aligned} \tag{4.4.3}$$

Note that $\widehat{\mathcal{N}(0,Q_t)}(f) = \mathrm{e}^{-\frac{1}{2}\langle Q_t f,f\rangle}$. Now, combining Eqs (4.4.2) and (4.4.3) one has

$$\hat{\nu}(f) = \hat{\nu}(\mathrm{e}^{-t\,Q_{R,N}} f) \times \mathrm{e}^{-\frac{1}{2}\langle Q_t f,f\rangle} \quad \forall f \in L^1_2(\mathbb{S}_R)\,.$$

This implies that

$$\mathrm{e}^{\frac{1}{2}\langle Q_t f,f\rangle}\,\mathrm{Re}\,\hat{\nu}(f) = \mathrm{Re}\,\hat{\nu}(\mathrm{e}^{-t\,Q_{R,N}} f) \leq 1\,.$$

Hence, for all $f \in L^1_2(\mathbb{S}_R)$ one has

$$\langle Q_t f, f\rangle \leq 2\log\left(\frac{1}{\mathrm{Re}\,\hat{\nu}(f)}\right).$$

By the Bochner theorem (Theorem 2.6.3) for $\epsilon = 1/2$ there exists a trace-class operator G such that $\mathrm{Re}\,\hat{\nu}(f) \geq 1/2$ for all $f \in L^1_2(\mathbb{S}_R)$ with $\langle Gf, f\rangle \leq 1$. This gives rise to $\langle Q_t f, f\rangle \leq (2\log 2)$, which amounts to writing $0 \leq Q_t \leq (2\log 2)\,G$. Note that $\sup_{t\geq 0}\mathrm{Tr}\,Q_t \leq 2\log 2\,\mathrm{Tr}\,G < \infty$. This concludes that item (B) implies item (A).

For the converse direction, recall that $Q_t : L^1_2(\mathbb{S}_R) \to L^1_2(\mathbb{S}_R)$ is given by $Q_t := 2\int_0^t \mathrm{e}^{-2sQ_{R,N}}\,\mathrm{d}s$. Let $\nu = \mathcal{N}(0,Q_\infty)$, i.e., $\widehat{\nu}(f) = \mathrm{e}^{-\frac{1}{2}\langle Q_\infty f,f\rangle}$. This yields $\widehat{\nu}(\mathrm{e}^{-tQ_{R,N}} f) = \mathrm{e}^{-\frac{1}{2}\langle \mathrm{e}^{-tQ_{R,N}}\, Q_\infty\, \mathrm{e}^{-tQ_{R,N}} f,f\rangle}$. From the semigroup property one gets

$$\begin{aligned}\mathrm{e}^{-t\,Q_{R,N}} Q_\infty \mathrm{e}^{-t\,Q_{R,N}} =& 2\int_0^\infty \mathrm{e}^{-2(t+s)\,Q_{R,N}}\mathrm{d}s\\ &= 2\int_0^\infty \mathrm{e}^{-2r\,Q_{R,N}}\mathrm{d}r - 2\int_0^t \mathrm{e}^{-2r\,Q_{R,N}}\mathrm{d}r = Q_\infty - Q_t\,.\end{aligned}$$

This implies that $\hat{\nu}(f) = \hat{\nu}(\mathrm{e}^{-t\,Q_{R,N}} f)\,\mathrm{e}^{-\frac{1}{2}\langle Q_t f,f\rangle}$ and finishes the proof of the converse direction. In fact, as $t \to \infty$ one has $\mathrm{e}^{-\frac{1}{2}\langle Q_t f,f\rangle} \to \mathrm{e}^{-\frac{1}{2}\langle Q_\infty f,f\rangle} = \mathcal{N}(0,Q_\infty)$. □

Lemma 4.4.2. *For all $R,N \in [1,\infty)$ the measures $\nu_{R,N} = \mathcal{N}(0,G_{R,N})$ concentrated on $L^1_2(\mathbb{S}_R)$ are invariant for Eq. (4.2.1). In particular, for all $t \in [0,\infty)$ it holds* $\mathrm{Law}(Z_{R,N}(t,\bullet)) = \mathrm{Law}(z_{R,N}) = \nu_{R,N}$.

Proof. Noting that $\mathcal{N}(0,Q_\infty) = \mathcal{N}(0,G_{R,N}) = \nu_{R,N}$, the statement is immediate consequence of Thm. 4.4.1. □

Interacting measure

Let $F \in B_{\mathrm{b}}(L^1_2(\mathbb{S}_R))$. We denote by $\mathcal{P}_{R,t}$ the semigroup associated to Eq. (4.3.1)

$$\mathcal{P}_{R,t}\,F(\phi) := \mathbb{E}\left[F(\Phi^g_{R,N}(t,\bullet))\,|\,\Phi^g_{R,N}(0,\bullet) = \phi\right].$$

We aim to verify that for all $R,N \in \mathbb{N}_+$ the measures $\mu^g_{R,N}$ is invariant for $\mathcal{P}_{R,t}$ in the sense of Def. 2.7.3.

Theorem 4.4.3. *Let $g \in C^\infty(\mathbb{S}_R)$. For all $R, N \in [1,\infty)$ there exits a measure of the form*

$$\mu^g_{R,N}(\mathrm{d}\phi) = \frac{1}{\mathcal{Z}^g_{R,N}} \frac{1}{\mathcal{Z}_{R,N}} \exp\Big(\frac{1}{n}\phi(g)^n - \int_{\mathbb{S}_R} P(\phi(\mathrm{x}), c_{R,N})\, \rho_R(\mathrm{dx})\Big)\, \nu_{R,N}(\mathrm{d}\phi)\,,$$

concentrated on $L^1_2(\mathbb{S}_R)$, which is invariant for Eq. (4.3.1). In particular, for all $t \in [0,\infty)$ it holds $\mathrm{Law}(\Phi^g_{R,N}(t,\bullet)) = \mu^g_{R,N}$ *.*

Proof. We shall verify the assumptions of Lemma 2.7.4 with $\tilde{H} = L^1_2(\mathbb{S}_R)$ and $H = L_2(\mathbb{S}_R)$.

1) For all $R, N \in [1,\infty)$ the elliptic operator $Q_{R,N} : L^6_2(\mathbb{S}_R) \subset L_2(\mathbb{S}_R) \to L_2(\mathbb{S}_R)$ is a self-adjoint, positive definite operator. In particular, for all $R, N \in [1,\infty)$ it holds $\mathrm{Tr}(Q^{-1}_{R,N}) = \mathrm{Tr}(G_{R,N}) < \infty$. It follows from Remark 2.4.13.
2) On account of Remark 2.4.12 for all $R, N \in [1,\infty)$ the operator $Q_{R,N}$ generates a C_0-semigroup of contractions on $L^1_2(\mathbb{S}_R)$. This implies that for all $f \in L^1_2(\mathbb{S}_R)$ one has $\|\mathrm{e}^{-t\,Q_{R,N}} f\|_{L^1_2(\mathbb{S}_R)} \le \|f\|_{L^1_2(\mathbb{S}_R)}$. Consequently, $\mathrm{e}^{-tQ_{R,N}}\, L^1_2(\mathbb{S}_R) \subseteq L^1_2(\mathbb{S}_R)$ for all $t \ge 0$. This verifies that $L^1_2(\mathbb{S}_R)$ is an invariant subspace of $\mathrm{e}^{-tQ_{R,N}}$ for all $R, N \in \mathbb{N}_+$. To proceed, fix $t > 0$ and set $\Delta t > 0$. Using the semigroup property of $\mathrm{e}^{-t\,Q_{R,N}}$ one obtains

   $$\lim_{\Delta t \to 0} \|\mathrm{e}^{-(t+\Delta t)\,Q_{R,N}} f - \mathrm{e}^{-t\,Q_{R,N}} f\|_{L^1_2(\mathbb{S}_R)} = \lim_{\Delta t \to 0} \|\mathrm{e}^{-t\,Q_{R,N}}(\mathrm{e}^{-\Delta t\,Q_{R,N}} - 1) f\|_{L^1_2(\mathbb{S}_R)}$$

   is zero. It follows from the strong continuity of the semigroup $\mathrm{e}^{-tQ_{R,N}}$. This implies that $\mathrm{e}^{-t\,Q_{R,N}} f$ is right continuous. Similarly, one deduces that

   $$\lim_{\Delta t \to 0} \|\mathrm{e}^{-t\,Q_{R,N}} f - \mathrm{e}^{-(t-\Delta t)\,Q_{R,N}} f\|_{L^1_2(\mathbb{S}_R)} = 0\,,$$

   which implies that $\mathrm{e}^{-t\,Q_{R,N}} f$ is left continuous. Hence, for all $f \in L^1_2(\mathbb{S}_R)$ the map $t \mapsto \mathrm{e}^{-t\,Q_{R,N}} f$ is continuous in $L^1_2(\mathbb{S}_R)$.
3) Lemma 4.2.1 implies that the Ornstein-Uhlenbeck process $Z_{R,N}(t,\bullet) \in C([0,\infty), L^1_2(\mathbb{S}_R))$ exists.
4) Let $U^g_{R,N}(\phi) := \frac{1}{n}\,\phi(g)^n - \int_{\mathbb{S}_R} P(\phi(\mathrm{x}), c_{R,N})\,\rho_R(\mathrm{dx})$. It is obvious that $U^g_{R,N}$ is continuous as it is a polynomial. Let $\psi \in L^1_2(\mathbb{S}_R)$ and D_ϵ denote the derivative

with respect to $\varepsilon > 0$. By definition one gets

$$
\begin{aligned}
&D_\varepsilon\, U^g_{R,N}(\phi + \varepsilon\,\psi)\Big|_{\varepsilon=0} \\
&\quad = D_\varepsilon \left(- \int_{\mathbb{S}_R} P(\phi(\mathrm{x}) + \varepsilon\,\psi(\mathrm{x}), c_{R,N})\,\rho_R(\mathrm{dx}) + \frac{1}{n}\,((\phi + \varepsilon\,\psi)(g))^n \right)\Bigg|_{\varepsilon=0} \\
&\qquad\qquad\qquad = - \int_{\mathbb{S}_R} P'(\phi(\mathrm{x}), c_{R,N})\,\psi(\mathrm{x})\,\rho_R(\mathrm{dx}) \;+\; \phi(g)^{n-1}\,.
\end{aligned}
$$

Observe that $D_\varepsilon\, U^g_{R,N}(\phi)$ with $\phi \to \varPhi^g_{R,N}(t,\bullet)$ coincides with the RHS of the function F given by Eq. (4.3.4) with $\varPhi^g_{R,N}(t,\bullet) = \varPsi^g_{R,N}(t,\bullet) + Z_{R,N}(t,\bullet)$.

5) The fact that the mapping $U^g_{R,N}$ is bounded from above is shown in Eq. (3.3.6) in the proof of Lemma 3.3.4. Moreover, in the proof of Lemma 4.3.3, it is proven that $F\,:\,L^1_2(\mathbb{S}_R) \to L_2(\mathbb{S}_R)$ is locally Lipschitz.
6) The mild solution $\varPhi^g_{R,N} \in C([0,\infty), L^1_2(\mathbb{S}_R))$ of Eq. (4.3.1) with the initial condition $\varPhi^g_{R,N}(0,\bullet) = \phi^g_{R,N} \in L^1_2(\mathbb{S}_R)$ is given by

$$
\begin{aligned}
&\varPhi^g_{R,N}(t,\bullet) = \mathrm{e}^{-t\,Q_{R,N}}\,\phi^g_{R,N} \\
&+ \int_0^t \mathrm{e}^{-(t-s)\,Q_{R,N}} \left[\sqrt{2}\,\mathrm{d}W_R(s,\bullet) - P'(\varPhi^g_{R,N}(s,\bullet), c_{R,N})\,\mathrm{d}s + (\varPhi^g_{R,N}(s,g))^{n-1} g(\bullet)\,\mathrm{d}s\right].
\end{aligned}
$$

This is the content of Lemma 4.3.2.

Combining Items (1)-(6) obtained above culminates in the existence of an invariant measure $\mu^g_{R,N}$ for Eq. (4.3.1) on $L^1_2(\mathbb{S}_R)$-space, which is absolutely continuous with respect to the measure $\nu_{R,N}$, see Remark 3.4.7. This completes the proof. □

Infinite volume measure

In this section, we shall show the existence of a weak limit of a subsequence of the sequence of measures $(\jmath_R^*\sharp\mu_{R,N}^g)_{R,N\in\mathbb{N}_+}$ on $\mathscr{S}'(\mathbb{R}^2)$ as $R\to\infty$ using *tightness*. We call such a limit a $P(\Phi)_2$ measure on $\mathscr{S}'(\mathbb{R}^2)$. To this end, we need to take the limits $R\to\infty$ and $N\to\infty$ of the sequence of measures $(\jmath_R^*\sharp\mu_{R,N}^g)_{R,N\in\mathbb{N}_+}$ on $\mathscr{S}'(\mathbb{R}^2)$. The existence of the UV limit as $N\to\infty$ of the family of measures $(\mu_{R,N}^g)_{R,N\in\mathbb{N}_+}$ was the subject of Sec. 3.4. This result will remain true for the sequence $(\jmath_R^*\sharp\mu_{R,N}^g)_{R,N\in\mathbb{N}_+}$ on $\mathscr{S}'(\mathbb{R}^2)$. To show the existence of the limit as $R\to\infty$, we shall first obtain an appropriate a priori bound in Prop. 5.1.5 using the energy method in $L_2(\mathbb{R}^2,v_L^{1/2})$-space. The function $\nu_L\in L_1(\mathbb{R}^2)$ is an admissible weight introduced in Def. 5.1.1. Next, we shall prove tightness of the family of measures $(\jmath_R^*\sharp\mu_{R,N}^g)_{R,N\in\mathbb{N}_+}$ by averaging over the time of the expected value of $\jmath_R^*\Phi_{R,N}^g(t,\bullet)$ using the fact that $\mathrm{Law}(\jmath_R^*\Phi_{R,N}^g(t,\bullet))=\jmath_R^*\sharp\mu_{R,N}^g$ for all $t\in[0,\infty)$. This implies the existence of subsequential limits as $R\to\infty$ of the sequence of measures $(\jmath_R^*\sharp\mu_{R,N}^g)_{R,N\in\mathbb{N}_+}$. Indeed, a $P(\Phi)_2$ measure on $\mathscr{S}'(\mathbb{R}^2)$ can be deemed as the marginal distribution of the stationary solution associated to the counterpart of Eq. (4.1.1) on $\mathbb{R}^2$. Similar argument was first used in [5, 6, 54], where they proved the existence of a Φ^4 measure on $\mathbb{R}^3$.

5.1 A priori bound

Definition 5.1.1. *Let* $v_L:=1/(4\pi L^2)\,w_L^8$*, where* $L\in[1,\infty)$ *is fixed as in Lemma 5.1.3 and* $w_L(x):=16L^4/(4L^2+x_1^2+x_2^2)^2\in C^\infty(\mathbb{R}^2)$ *was given in Sec. 2.1 below Eq. (2.1.2). Observe that using the polar coordinates one gets*

$$\|v_L\|_{L_1(\mathbb{R}^2)}=\frac{1}{4L^2}\int_0^\infty\Big(\frac{16L^4}{(4L^2+r^2)^2}\Big)^8\,(2r)\,\mathrm{d}r=\frac{4^{16}L^{32}}{4L^2}\,\frac{1}{15(8^{15})L^{30}}=\frac{1}{15\,(2)^{15}}<1\,.$$

Remark 5.1.2. *The precise choice of the weight v_L is not of much importance. It is convenient to use a weight that decays polynomially and express it in terms of the* *function w_R introduced in Sec. 2.1. The prefactor $1/(4\pi L^2)$ guarantees that the $L_1(\mathbb{R}^2)$ norm of the weight is bounded by 1 and the decay rate is chosen so that the estimate stated in Remark 5.1.4 is true.*

Utilizing Remarks 2.1.8 and 2.1.9 one rewrites the Eq. (4.3.18) in the stereographic coordinates as follows

$$\begin{aligned}(\partial_t + (1-w_R^{-1}(x)\Delta)(1-w_R^{-1}(x)\Delta/N^2)^2)\jmath_R^*\Psi_{R,N}^g(t,x) + (\jmath_R^*\Psi_{R,N}^g(t,x))^{n-1} \\ = \sum_{l=0}^{n-2}\sum_{m=l}^{n-1} a_{m,l}\,\jmath_R^* Z_{R,N}^{:m-l:}(t,x)\,(\jmath_R^*\Psi_{R,N}^g(t,x))^l \\ + (\jmath_R^*(\Psi_{R,N}^g + Z_{R,N})(t,w_R\jmath_R^*g))^{n-1}\jmath_R^*g(x)\,. \end{aligned} \tag{5.1.1}$$

We shall apply the energy method in $L_2(\mathbb{R}^2, v_L^{1/2})$-space in order to obtain a priori bound. To this end, we multiply both sides of the above equation by $v_L\jmath_R^*\Psi_{R,N}^g$, and perform integration over $\mathbb{R}^2$. The resulting bound, which is uniform in $R,N\in\mathbb{N}_+$, will be used to prove tightness of the family of measures $(\jmath_R^*\sharp\mu_{R,N}^g)_{R,N\in\mathbb{N}_+}$. Note that we have already utilized the energy method in $L_2^1(\mathbb{S}_R)$-space in order to prove the global existence of the mild solution in Sec. 4.3.2 .

Lemma 5.1.3. *There exists $L\in[1,\infty)$ such that for all $R\in[L,\infty)$ it holds*

(A) $\langle\Psi, v_L(-w_R^{-1}\Delta)\Psi\rangle_{L_2(\mathbb{R}^2)} \geq 1/2\,\|\vec\nabla\Psi\|^2_{L_2(\mathbb{R}^2,w_R^{-1/2}v_L^{1/2})} - 1/8\,\|\Psi\|^2_{L_2(\mathbb{R}^2,w_R^{-1/2}v_L^{1/2})}$,

(B) $\langle\Psi, v_L(-w_R^{-1}\Delta)^2\Psi\rangle_{L_2(\mathbb{R}^2)} \geq 1/2\,\|\Delta\Psi\|^2_{L_2(\mathbb{R}^2,w_R^{-1}v_L^{1/2})} - 1/8\,\|\Psi\|^2_{L_2(\mathbb{R}^2,w_R^{-1}v_L^{1/2})}$,

(C) $\langle\Psi, v_L(-w_R^{-1}\Delta)^3\Psi\rangle_{L_2(\mathbb{R}^2)} \geq 1/2\,\|\vec\nabla\Delta\Psi\|^2_{L_2(\mathbb{R}^2,w_R^{-3/2}v_L^{1/2})} - 1/8\,\|\Psi\|^2_{L_2(\mathbb{R}^2,w_R^{-3/2}v_L^{1/2})}$.

Proof. There exists $C\in(0,\infty)$ such that for all $L\in[1,\infty)$, $R\in[L,\infty)$ it holds

$$|\vec\nabla w_R^{-1/2}| \leq (C/L)\,w_R^{-1/2}, \qquad |\vec\nabla v_L^{1/2}| \leq (C/L)\,v_L^{1/2}.$$

This gives readily (A) by integrating by parts in $\|\vec\nabla\Psi\|^2_{L_2(\mathbb{R}^2,w_R^{-1/2}v_L^{1/2})}$ applying the Leibniz rule and the Young inequality. Estimates (B) and (C) are obtained analogously, with the help of the following auxiliary inequalities

$$\|\vec\nabla\Psi\|^2_{L_2(\mathbb{R}^2,w_R^{-1}v_L^{1/2})} \leq 2\|\Delta\Psi\|^2_{L_2(\mathbb{R}^2,w_R^{-1}v_L^{1/2})} + 2\|\Psi\|^2_{L_2(\mathbb{R}^2,w_R^{-1}v_L^{1/2})}\,,$$

$$\|\vec\nabla\Psi\|^2_{L_2(\mathbb{R}^2,w_R^{-3/2}v_L^{1/2})} + \|\Delta\Psi\|^2_{L_2(\mathbb{R}^2,w_R^{-3/2}v_L^{1/2})} \leq 4\|\vec\nabla\Delta\Psi\|^2_{L_2(\mathbb{R}^2,w_R^{-3/2}v_L^{1/2})} + 4\|\Psi\|^2_{L_2(\mathbb{R}^2,w_R^{-3/2}v_L^{1/2})}$$

valid for sufficiently big $L\in[1,\infty)$ and all $R\in[L,\infty)$. The latter inequalities are proven by the same token as Item (A). For the complete proof see Appendix D. □

Remark 5.1.4. *For all $L \in [1,\infty)$, $R \in [L,\infty)$ and $p \in \{1,2,3\}$ it holds*

$$\|\Psi\|_{L_2(\mathbb{R}^2, w_R^{-p/2} v_L^{1/2})} = \|w_R^{-p/2}\, v_L^{1/2}\Psi\|_{L_2(\mathbb{R}^2)} \leq \|w_R^{-p/2} v_L^{(n-2)/2n}\|_{L_{2n/(n-2)}(\mathbb{R}^2)}\, \|\Psi\|_{L_n(\mathbb{R}^2, v_L^{1/n})}\,.$$

It follows from the Hölder inequality. Observe that there exists $C \in (0,\infty)$ such that

$$\|w_R^{-p/2} v_L^{(n-2)/2n}\|_{L_{2n/(n-2)}(\mathbb{R}^2)} \leq C\,,$$

where we used the facts that $n \geq 4, w_R \leq 1$ and $w_R \geq w_L$.

Proposition 5.1.5. *There exist $\kappa \in (0,\infty)$, $C \in (0,\infty)$, $p \in [1,\infty)$ and a ball $B \subset \mathscr{S}(\mathbb{R}^2)$ with respect to some Schwartz semi-norm centered at the origin such that for all $t \in (0,\infty)$ and $R, N \in \mathbb{N}_+$, $R \geq L$, as well as all $g \in C^\infty(\mathbb{S}_R)$, $w_R \jmath_R^* g \in B$, it holds*

$$8\,\partial_t \|\jmath_R^* \Psi^g_{R,N}(t,\bullet)\|^2_{L_2(\mathbb{R}^2, v_L^{1/2})} + \|\jmath_R^* \Psi^g_{R,N}(t,\bullet)\|^n_{L_n(\mathbb{R}^2, v_L^{1/n})} \leq C \sum_{k=0}^{n-1} \|\jmath_R^* Z^{:k:}_{R,N}(t,\bullet)\|^p_{L_p^{-\kappa}(\mathbb{R}^2, v_L^{1/p})}\,.$$

Proof. After multiplying both sides Eq. (5.1.1) by $v_L \jmath_R^* \Psi^g_{R,N}$, integrating over $\mathbb{R}^2$ and applying Lemma 5.1.3 and Remark 5.1.4 we obtain

$$\begin{aligned}
\frac{1}{2}\partial_t \|\jmath_R^* \Psi^g_{R,N}(t,\bullet)\|^2_{L_2(\mathbb{R}^2, v_L^{\frac12})} + \|\jmath_R^* \Psi^g_{R,N}(t,\bullet)\|^2_{L_2(\mathbb{R}^2, v_L^{\frac12})} + (\frac{1}{N^2} + \frac{1}{2}) \|\vec{\nabla} \jmath_R^* \Psi^g_{R,N}(t,\bullet)\|^2_{L_2(\mathbb{R}^2, v_L^{\frac12} w_R^{-\frac12})} \\
+ (1 - \frac{1}{4N^2} - \frac{1}{8N^4} - \frac{1}{8} - \frac{1}{4N^2} - \frac{1}{8N^4}) \|\jmath_R^* \Psi^g_{R,N}(t,\bullet)\|^n_{L_n(\mathbb{R}^2, v_L^{\frac1n})} \\
+ (\frac{1}{2N^4} + \frac{1}{N^2}) \|\Delta \jmath_R^* \Psi^g_{R,N}(t,\bullet)\|^2_{L_2(w_R^{-\frac12} v_L^{\frac12})} \\
+ \frac{1}{2N^4} \|\vec{\nabla}\Delta \jmath_R^* \Psi^g_{R,N}(t,\bullet)\|^2_{L_2(w_R^{-\frac32} v_L^{\frac12})} \leq R^{(1)}_{R,N}(t) + R^{(2)}_{R,N}(t),
\end{aligned}$$

where

$$\begin{aligned}
R^{(1)}_{R,N}(t) &= \sum_{l=0}^{n-2} \sum_{m=l}^{n-1} \int_{\mathbb{R}^2} a_{m,l}\, v_L(x)\, (\jmath_R^* Z^{:m-l:}_{R,N})\, (\jmath_R^* \Psi^g_{R,N}(t,x))^{l+1} \mathrm{d}x, \\
R^{(2)}_{R,N}(t) &= (\jmath_R^*(\Psi^g_{R,N} + Z_{R,N})(t, w_R \jmath_R^* g))^{n-1}\, (\jmath_R^* \Psi^g_{R,N})(t, v_L \jmath_R^* g)\,.
\end{aligned}$$

Observe that $w_R^{-1}(x) > 1$ and $N \in [1,\infty)$. Hence,

$$\begin{aligned}
\frac{1}{2}\partial_t \|\jmath_R^* \Psi^g_{R,N}(t,\bullet)\|^2_{L_2(\mathbb{R}^2, v_L^{\frac12})} + \frac{1}{2} \|\nabla \jmath_R^* \Psi^g_{R,N}(t,\bullet)\|^2_{L_2(\mathbb{R}^2, v_L^{\frac12})} \\
+ \frac{1}{8} \|\jmath_R^* \Psi^g_{R,N}(t,\bullet)\|^n_{L_n(\mathbb{R}^2, v_L^{\frac1n})} \leq |R^{(1)}_{R,N}(t) + R^{(2)}_{R,N}(t)|. \quad (5.1.2)
\end{aligned}$$

Moreover, one has

$$|R^{(1)}_{N,R}(t)| \leq \sum_{l=0}^{n-2}\sum_{m=l}^{n-1} |a_{m,l}|\, |\langle \jmath_R^* Z^{:m-l:}_{R,N}(t,x), (\jmath_R^* \Psi^g_{R,N}(t,x))^{l+1}\rangle_{L_2(\mathbb{R}^2,v_L)}|\,.$$

By Lemma 2.3.13 for every $\delta_1 \in (0,1)$ there exists $\hat{C} \in (0,\infty)$ such that

$$\begin{aligned}|\langle \jmath_R^* Z^{:m-l:}_{R,N}, (\jmath_R^* \Psi^g_{R,N})^{l+1}\rangle_{L_2(\mathbb{R}^2,v_L^{1/2})}| &\leq \hat{C}\|\jmath_R^* Z^{:m-l:}_{R,N}(t,\bullet)\|^p_{L^{-\kappa}_p(\mathbb{R}^2,v_L^{1/p})} \\ &+ \delta_1\, \|\nabla \jmath_R^* \Psi^g_{R,N}(t,\bullet)\|^2_{L_2(\mathbb{R}^2,v_L^{1/2})} + \delta_1\, \|\jmath_R^* \Psi^g_{R,N}(t,\bullet)\|^n_{L_n(\mathbb{R}^2,v_L^{1/n})} + \delta_1\,.\end{aligned}$$

By the assumption made in Sec. 4.3.1 on the coefficients $a_{m,l}$ there is $\check{C} \in (0,\infty)$ such that $|a_{m,l}| \leq \check{C}$. Thus, there exists $C, C_1 \in (0,\infty)$ such that

$$\begin{aligned}\sum_{l=0}^{n-2}\sum_{m=l}^{n-1} \hat{C}\, |a_{m,l}|\, \|\jmath_R^* Z^{:m-l:}_{R,N}(t,\bullet)\|^p_{L^{-\kappa}_p(\mathbb{R}^2,v_L^{1/p})} &= \sum_{l=0}^{n-2}\sum_{k=0}^{n-1-l} C\, \|\jmath_R^* Z^{:k:}_{R,N}(t,\bullet)\|^p_{L^{-\kappa}_p(\mathbb{R}^2,v_L^{1/p})} \\ &= \sum_{k=0}^{n-1}\sum_{l=0}^{(n-1-l)\wedge(n-2)} C\, \|\jmath_R^* Z^{:k:}_{R,N}(t,\bullet)\|^p_{L^{-\kappa}_p(\mathbb{R}^2,v_L^{1/p})} \\ &\leq C_1 \sum_{k=0}^{n-1} \|\jmath_R^* Z^{:k:}_{R,N}(t,\bullet)\|^p_{L^{-\kappa}_p(\mathbb{R}^2,v_L^{1/p})}\,.\end{aligned}$$

Combining all the above results culminates in

$$\begin{aligned}|R^{(1)}_{R,N}(t)| \leq C_1 \sum_{k=0}^{n-1} &\|\jmath_R^* Z^{:k:}_{R,N}(t,\bullet)\|^p_{L^{-\kappa}_p(\mathbb{R}^2,v_L^{1/p})} \\ &+ \delta_1\, \|\nabla \jmath_R^* \Psi^g_{R,N}(t,\bullet)\|^2_{L_2(\mathbb{R}^2,v_L^{1/2})} + \delta_1\, \|\jmath_R^* \Psi^g_{R,N}(t,\bullet)\|^n_{L_n(\mathbb{R}^2,v_L^{1/n})}, \qquad (5.1.3)\end{aligned}$$

where $k=0$ term of the sum above is a constant. Using the fact that for all $k>1$ and $f(a_i) \in \mathbb{R}_+$ it holds $(\sum_{i=1}^n f(a_i))^k \leq n^{k-1} \sum_{i=1}^n f(a_i)^k$ one obtains

$$\begin{aligned}|R^{(2)}_{R,N}(t)| \leq &2^{n-2}\, |(\jmath_R^*(\Psi^g_{R,N})(t, w_R \jmath_R^* g))^{n-1} (\jmath_R^* \Psi^g_{R,N})(t, v_L \jmath_R^* g)| \\ &+ 2^{n-2}\, |(\jmath_R^* Z_{R,N})(t, w_R \jmath_R^* g))^{n-1}\, (\jmath_R^* \Psi^g_{R,N})(t, v_L \jmath_R^* g)|.\end{aligned}$$

Observe that from the Hölder inequality with $1/n + (n-1)/n = 1$ one gets

$$\begin{aligned}|(\jmath_R^* \Psi^g_{R,N})(t, w_R \jmath_R^* g))| &\leq \|\jmath_R^*(\Psi^g_{R,N}(t,\bullet)\|_{L_n(\mathbb{R}^2,v_L^{1/n})}\, \|v_L^{-1/n} w_R \jmath_R^* g\|_{L_{n/(n-1)}(\mathbb{R}^2)} \\ |(\jmath_R^* Z_{R,N})(t, w_R \jmath_R^* g))| &\leq \|\jmath_R^* Z_{R,N}(t,\bullet)\|_{L^{-\kappa}_n(\mathbb{R}^2,v_L^{1/n})}\, \|v_L^{-1/n} w_R \jmath_R^* g\|_{L^{\kappa}_{n/(n-1)}(\mathbb{R}^2)} \\ |(\jmath_R^* \Psi^g_{R,N})(t, v_L \jmath_R^* g)| &\leq \|\jmath_R^*(\Psi^g_{R,N}(t,\bullet)\|_{L_n(\mathbb{R}^2,v_L^{1/n})}\, \|v_L^{(n-1)/n} \jmath_R^* g\|_{L_{n/(n-1)}(\mathbb{R}^2)}\,.\end{aligned}$$

The above expressions implies that there exists $C_2 \in (0,\infty)$ such that for all $\delta_2 \in (0,1)$ it holds

$$|R^{(2)}_{R,N}(t)| \leq C_2\, \delta_2^n\, \|\jmath_R^* \Psi^g_{R,N}(t,\bullet)\|^n_{L_n(\mathbb{R}^2,v_L^{1/n})} + C_2\, \delta_2^n\, \|\jmath_R^* Z_{R,N}\|^n_{L_n^{-\kappa}(\mathbb{R}^2,v_L^{1/n})} \tag{5.1.4}$$

provided that

$$\|v_L^{(n-1)/n} \jmath_R^* g\|_{L_{n/(n-1)}(\mathbb{R}^2)} \vee \|v_L^{-1/n} w_R \jmath_R^* g\|_{L_{n/(n-1)}(\mathbb{R}^2)} \vee \|v_L^{-1/n} w_R \jmath_R^* g\|_{L^\kappa_{n/(n-1)}(\mathbb{R}^2)} \leq \delta_2.$$

Now the statement follows by substituting Eq.'s (5.1.3) and (5.1.4) in Eq. (5.1.2) and choosing δ_1, δ_2 such that $\delta_1 \leq 1/2$ and $\delta_1 + C_2\, \delta_2^n \leq 1/16$. This guarantees that all coefficients in Eq. (5.1.2) are positive and finishes the proof. □

5.2 Tightness

Proposition 5.2.1. *Let $\kappa \in (0,\infty)$. There exists a ball $B \subset \mathscr{S}(\mathbb{R}^2)$ with respect to some Schwartz semi-norm centered at the origin and a constant $C \in (0,\infty)$ such that for all $R \in \mathbb{N}_+$, $R \geq L$, $N \in \mathbb{N}_+$ and all $g \in C^\infty(\mathbb{S}_R)$, $w_R \jmath_R^* g \in B$, it holds*

$$\int \|\jmath_R^* \phi\|^n_{L_n^{-\kappa}(\mathbb{R}^2,v_L^{1/n})}\, \mu^g_{R,N}(\mathrm{d}\phi) \leq C.$$

Proof. On account of Theorem 4.4.3 one has $\mathrm{Law}(\jmath_R^* \Phi^g_{R,N}(t,\bullet)) = \jmath_R^* \sharp \mu^g_{R,N}$ for all $t \in [0,\infty)$. Hence,

$$\int \|\phi\|^n_{L_n^{-\kappa}(\mathbb{R}^2,v_L^{1/n})}\, (\jmath_R^* \sharp \mu^g_{R,N})(\mathrm{d}\phi) = \int \|\jmath_R^* \phi\|^n_{L_n^{-\kappa}(\mathbb{R}^2,v_L^{1/n})}\, \mu^g_{R,N}(\mathrm{d}\phi) = \mathbb{E}\|\jmath_R^* \Phi^g_{R,N}(t,\bullet)\|^n_{L_n^{-\kappa}(\mathbb{R}^2,v_L^{1/n})}.$$

Using Lemma 4.4.2 one deduces that $\mathrm{Law}(\jmath_R^* X_{R,N}) = \mathrm{Law}(\jmath_R^* Z_{R,N}(t,\bullet))$ for all $t \in [0,\infty)$. Thus, from Lemma B.8 and Proposition 5.1.5 we have

$$8\, \partial_t \mathbb{E}\|\jmath_R^* \Psi^g_{R,N}(t,\bullet)\|^2_{L_2(\mathbb{R}^2,v_L^{1/2})} + \mathbb{E}\|\jmath_R^* \Psi^g_{R,N}(t,\bullet)\|^n_{L_n(\mathbb{R}^2,v_L^{1/n})} \leq C_1$$

for some constant $C_1 \in (0,\infty)$ independent of g, R, N and t. The above inequality implies that for all $T \in (0,\infty)$ it holds

$$\frac{1}{T}\int_0^T \mathbb{E}\|\jmath_R^* \Psi^g_{R,N}(t,\bullet)\|^n_{L_n(\mathbb{R}^2,v_L^{1/n})} \mathrm{d}t \leq C_1 - \frac{8}{T}\mathbb{E}\, \|\jmath_R^* \Psi^g_{R,N}(T,\bullet)\|^2_{L_2(\mathbb{R}^2,v_L^{1/2})} + \frac{8}{T}\mathbb{E}\|\jmath_R^* \Psi^g_{R,N}(0,\bullet)\|^2_{L_2(\mathbb{R}^2,v_L^{1/2})} \leq C_1 + \frac{C_{R,N}}{T}\,,$$

where

$$C_{R,N} := 8\,\mathbb{E}\|\jmath_R^*\Psi^g_{R,N}(0,\bullet)\|^2_{L_2(\mathbb{R}^2,v_L^{1/2})} \leq 8\,\mathbb{E}\|\Psi^g_{R,N}(0,\bullet)\|^2_{L_2(\mathbb{S}_R)} < \infty$$

for every $R,N\in\mathbb{N}_+$ and $R\geq L$. The last bound follows from Remark 2.1.8 and the fact that $w_R\geq w_L$. From the fact that $\jmath_R^*\Phi^g_{R,N}$ and $\jmath_R^*Z_{R,N}$ are stationary in time one deduces that

$$\begin{aligned}\mathbb{E}\|\jmath_R^*\Phi^g_{R,N}(0,\bullet)\|^n_{L_n^{-\kappa}(\mathbb{R}^2,v_L^{1/n})} &= \frac{1}{T}\int_0^T \mathbb{E}\|\jmath_R^*\Phi^g_{R,N}(t,\bullet)\|^n_{L_n^{-\kappa}(\mathbb{R}^2,v_L^{1/n})}\mathrm{d}t\\ &\leq c\,\mathbb{E}\|\jmath_R^*Z_{R,N}(0,\bullet)\|^n_{L_n^{-\kappa}(\mathbb{R}^2,v_L^{1/n})} + \frac{c}{T}\int_0^T \mathbb{E}\|\jmath_R^*\Psi^g_{R,N}(t,\bullet)\|^n_{L_n^{-\kappa}(\mathbb{R}^2,v_L^{1/n})}\mathrm{d}t\,,\end{aligned}$$

where $c=2^{n-1}$ and we used the fact that for all $n>1$ and $a_i\in\mathbb{R}_+$, $i\in\{1,\cdots,k\}$ it holds $(\sum_{i=1}^k a_i)^n \leq k^{n-1}\sum_{i=1}^k a_i^n$, which follows from convexity. By Lemma B.8 there exists $C_2\in(0,\infty)$ such that for all $R,N\in\mathbb{N}_+$ it holds

$$\mathbb{E}\|\jmath_R^*Z_{R,N}(0,\bullet)\|^n_{L_n^{-\kappa}(\mathbb{R}^2,v_L^{1/n})} \leq C_2.$$

Combining the bounds proved above yields

$$\mathbb{E}\|\jmath_R^*\Phi^g_{R,N}(0,\bullet)\|^n_{L_n^{-\kappa}(\mathbb{R}^2,v_L^{1/n})} \leq c\,C_1 + c\,C_2 + \frac{c\,C_{R,N}}{T}$$

for all $T\in(0,\infty)$. Choosing $T=C_{R,N}$ concludes the proof. □

Remark 5.2.2. *By Lemma B.8 one obtains*

$$\begin{aligned}&\Big|\frac{\mathbb{E}\,[\|\jmath_R^*X_{R,N}\|^n_{L_n^{-\kappa}(\mathbb{R}^2,\nu_L^{1/n})}\,\exp(-Y_{R,N})]}{\mathbb{E}\,[\exp(-Y_{R,N})]} - \frac{\mathbb{E}\,[\|\jmath_R^*X_R\|^n_{L_n^{-\kappa}(\mathbb{R}^2,\nu_L^{1/n})}\,\exp(-Y_{R,N})\,]}{\mathbb{E}\,[\exp(-Y_{R,N})\,]}\Big|\\ &\qquad\leq \Big|\frac{\mathbb{E}\,[\|\jmath_R^*X_{R,N}-\jmath_R^*X_R\|^n_{L_n^{-\kappa}(\mathbb{R}^2,\nu_L^{1/n})}\,\exp(-Y_{R,N})]}{\mathbb{E}\,[\exp(-Y_{R,N})\,]}\Big| \leq \frac{C}{N^\kappa}.\end{aligned}$$

Using Proposition 3.4.6 with $F=1$ combined with the preceding expression one obtains

$$\int \|\jmath_R^*\phi\|^n_{L_n^{-\kappa}(\mathbb{R}^2,v_L^{1/n})}\,\mu^g_R(\mathrm{d}\phi) = \lim_{N\to\infty}\int \|\jmath_R^*\phi\|^n_{L_n^{-\kappa}(\mathbb{R}^2,v_L^{1/n})}\,\mu^g_{R,N}(\mathrm{d}\phi) \leq C\,.$$

On account of Theorem 2.3.10 Item (C) the embedding $L_n^{-\kappa}(\mathbb{R}^2,v_L^{1/n})\to L_n^{-2\kappa}(\mathbb{R}^2,v_L^{2/n})$ is compact. As a result, by Lemma 2.5.14 the sequence of measures $(\jmath_R^*\sharp\mu^g_R)_{R\in\mathbb{N}_+}$ on $L_n^{-2\kappa}(\mathbb{R}^2,v_L^{2/n})$ is tight, hence, by the Prokhorov theorem it has a weakly convergent *subsequence, i.e., there exists some diverging sequence* $(R_l)_{l\in\mathbb{N}_+}$ *such that for every bounded and continuous functional* $F : L_n^{-2\kappa}(\mathbb{R}^2, v_L^{2/n}) \to \mathbb{R}$ *it holds*

$$\mu^g(F) = \lim_{l\to\infty} \jmath_{R_l}^* \sharp \mu_{R_l}^g(F) = \lim_{l\to\infty} \lim_{N\to\infty} \jmath_{R_l}^* \sharp \mu_{R_l,N}^g(F) = \lim_{l\to\infty} \lim_{N\to\infty} \mu_{R_l,N}^g(F \circ \jmath_{R_l}^*)\,.$$

Observe that the uniform bound stated in Prop. 5.2.1 does not depend on $g \in C^\infty(\mathbb{S}_R)$. *Hence, using the preceding proposition with* $g = 0$ *one infers the existence of a weak limit for the family of the measures* $(\jmath_R^* \sharp \mu_{R,N})_{R,N\in\mathbb{N}_+}$*, i.e.,* μ.

Definition 5.2.3. *Any weak limit of a subsequence of the sequence of measures* $(\jmath_R^* \sharp \mu_{R,N})_{R,N\in\mathbb{N}_+}$ *on* $\mathscr{S}'(\mathbb{R}^2)$ *is called a* $P(\Phi)_2$ *measure.*

Remark 5.2.4. *Observe that by Hölder's inequality one has*

$$\|\jmath_R^*\phi\|_{L_2^{-1}(\mathbb{R}^2, v_L^{1/2})} \le \|v_L^{1/2-1/n}\|_{L_{2n/(n-2)}(\mathbb{R}^2)} \|\jmath_R^*\phi\|_{L_n^{-1}(\mathbb{R}^2, v_L^{1/n})}.$$

This implies that there exists $C \in (0,\infty)$ *such that for all* $R \in \mathbb{N}_+$ *it holds*

$$\int \|\jmath_R^*\phi\|^n_{L_2^{-1}(\mathbb{R}^2, v_L^{1/2})}\, \mu_R(\mathrm{d}\phi) \le \int \|\jmath_R^*\phi\|^n_{L_n^{-1}(\mathbb{R}^2, v_L^{1/n})}\, \mu_R(\mathrm{d}\phi) < C\,.$$

It follows from Prop. 5.2.1 combined with the fact that the norm $\|v_L^{1/2-1/n}\|_{L_{2n/(n-2)}(\mathbb{R}^2)}$ *is bounded, since* $n \ge 4$ *and* $v_L \le 1$.

Integrability and Osterwalder-Schrader axioms

Let μ be any weak limit of the sequence of measures $(\jmath_R^*\sharp\mu_{R,N})_{R,N\in\mathbb{N}_+}$ as was determined in Def. 5.2.3. Let $B \subset \mathscr{S}(\mathbb{R}^2)$ be a ball w.r.t. some Schwarz semi-norm and $f \in B$ be a test function. In Prop. 6.1.1 we shall first verify that $(\phi(f))^n$ is exponentially integrable with respect to the measures μ, where $n \geq 4$ is the degree of the polynomial $P(\phi)$ fixed in Eq. (3.0.1). Note that exponential integrability of $(\phi(f))^n$ manifests the non-Gaussianity property of the measure μ. Next, we shall verify the Osterwalder-Schrader axioms, i.e., regularity, reflection positivity and Euclidean invariance properties, cf. Def 1.0.1. Once we verify the Osterwalder-Schrader axioms, the existence of a local relativistic QFT on Minkowski space-time satisfying the Wightman axioms, will follow from the reconstruction theorem [50, Ch's. 6, 19].

6.1 Integrability

Proposition 6.1.1. *There exists a ball $B \subset \mathscr{S}(\mathbb{R}^2)$ with respect to some Schwartz semi-norm centered at the origin such that for all $f \in B$ it holds that*

$$\int \exp(\phi(f)^n)\,\mu(\mathrm{d}\phi) \leq 2\,. \tag{6.1.1}$$

Proof. By Remark 2.1.8 for all $f \in \mathscr{S}(\mathbb{R}^2)$ and $R \in \mathbb{N}_+$ there exists $g_R \in C^\infty(\mathbb{S}_R)$ such that $w_R\jmath_R^* g_R = f$. Now let B be contained in the ball from the statement of Prop. 5.2.1 and suppose that $f \in B$. On account of Remark 2.1.8 for arbitrary $\phi \in \mathscr{D}'(\mathbb{S}_R)$ it holds

$$\phi(g_R) = (\jmath_R^*\phi)(w_R\jmath_R^* g_R) = (\jmath_R^*\phi)(f). \tag{6.1.2}$$

Then by Lemma 2.5.11 one has

$$\int \exp\left(\phi(g_R)^n/n\right)\,\mu_{R,N}(\mathrm{d}\phi) \leq \exp\left(\frac{1}{n}\int \phi(g_R)^n\,\mu_{R,N}^{g_R}(\mathrm{d}\phi)\right).$$

Note that the expression on the LHS is integrable by Lemma 3.3.4. The identity (6.1.2), Höder's inequality and Proposition 5.2.1 yield

$$\int \phi(g_R)^n\, \mu_{R,N}^{g_R}(\mathrm{d}\phi) \leq \hat{C}\, \|v_L^{-1/n} w_R \jmath_R^* g_R\|^n_{L^\kappa_{n/(n-1)}(\mathbb{R}^2)} \int \|\jmath_R^*\phi\|^n_{L_n^{-\kappa}(\mathbb{R}^2, v_L^{1/n})}\, \mu_R^{g_R}(\mathrm{d}\phi)$$
$$\leq C\, \|v_L^{-1/n} w_R \jmath_R^* g_R\|^n_{L^\kappa_{n/(n-1)}(\mathbb{R}^2)}$$

for some constants $\hat{C}, C \in (0,\infty)$ independent of R, N and g_R. Choosing the ball B so that $\|v_L^{-1/n} f\|^n_{L^\kappa_{n/(n-1)}(\mathbb{R}^2)} \leq n/2C$ for all $f \in B$ by the above inequalities and Proposition 3.4.6 we obtain

$$\int \exp\left((\jmath_R^*\phi)(f)^n/n\right)\, \mu_R(\mathrm{d}\phi) = \lim_{N\to\infty} \int \exp\left(\phi(g_R)^n/n\right)\, \mu_{R,N}(\mathrm{d}\phi) \leq 2\,.$$

This concludes the proof. □

Recall that $n \geq 4$ in Eq. (6.1.1). Prop. 6.1.1 implies that $\mu(\mathrm{d}\phi)$ is non-Gaussian, since Gaussian measures do not integrate functions growing so fast. Furthermore, one has, for $f \in \mathscr{S}(\mathbb{R}^2)$,

$$\int \exp(|\phi(f)|)\, \mu(\mathrm{d}\phi) < \infty \tag{6.1.3}$$

without the restriction $f \in B$. It follows from the fact that for all $\varepsilon > 0$, it holds $|\phi(f)| = \varepsilon^{-1}|\phi(\varepsilon f)| \leq \varepsilon^{-p} + |\phi(\varepsilon f)|^n$ for $1/p + 1/n = 1$, by the Young inequality. Now, for any given f, we have $\varepsilon f \in B$ for ε sufficiently small. Note that Eq. (6.1.3) implies that μ has moments of all orders, since the convergence of the integrals $\int \exp(\phi(f))\, \mu(\mathrm{d}\phi)$ implies that $\int |\phi(f)|^m \mu(\mathrm{d}\phi)$ is finite for all $m \in \mathbb{N}_+$ [50, Prop. 6.1.4].

6.2 Regularity

Proposition 6.2.1. *There holds the regularity axiom OS0: There is a Schwartz norm* $\|\bullet\|_{\mathrm{s}}$ *and* $K, \beta > 0$ *such that for all* $m \in \mathbb{N}$ *and* $f_1, \ldots, f_m \in \mathscr{S}(\mathbb{R}^2)$

$$\Big|\int \phi(f_1)\ldots\phi(f_m)\, \mu(\mathrm{d}\phi)\Big| \leq (m!)^\beta\, K^m \prod_{i=1}^m \|f_i\|_{\mathrm{s}}.$$

Proof. Since $\frac{x^m}{m!} \leq \mathrm{e}^x$, we can write for $f_0 \in B$:

$$\int \phi(f_0)^m\, \mu(\mathrm{d}\phi) \leq m! \int \mathrm{e}^{\phi(f_0)} \mu(\mathrm{d}\phi) \leq m!\,\mathrm{e} \int \mathrm{e}^{\phi(f_0)^n} \mu(\mathrm{d}\phi) \leq m!\,(2\,\mathrm{e}). \tag{6.2.1}$$

where we used $e^{\phi(f_0)} \leq e^{1+\phi(f_0)^n}$ and Eq. (6.1.1). Now, by the Young inequality, and (6.2.1), we write for $f_{1,0}, \ldots, f_{m,0} \in B$

$$\Big| \int \phi(f_{1,0}) \ldots \phi(f_{m,0})\, \mu(\mathrm{d}\phi) \Big| \leq \sum_{i=1}^{m} \int |\phi(f_{i,0})|^m \, \mu(\mathrm{d}\phi) \leq m!\, (2\,\mathrm{e})\, m.$$

Finally, we set $f_{i,0} = \varepsilon f_i / \|f_i\|_{\mathrm{s}}$ and obtain the bound from the statement of the proposition. □

6.3 Reflection positivity

In this section we verify that any accumulation point of the family of measures $(\jmath_R^* \sharp \mu_R)_{R \in \mathbb{N}_+}$ on $\mathscr{S}'(\mathbb{R}^2)$ is reflection positive. To this end, we shall utilize the fact that for all $R \in \mathbb{N}_+$ the Gaussian measure ν_R with covariance $G_R = (1 - \Delta_R)^{-1}$ is reflection positive. Then, we introduce a UV regularization, which preserves the reflection positivity property of the Gaussian measure ν_R. In general, to construct a nonlinear quantum field, it is more convenient to establish reflection positivity for some measures which occur as intermediate steps in the construction [50, p. 196]. That is why in the next step, we shall first show that for all $R \in \mathbb{N}_+$ the finite volume measure μ_R on $\mathscr{D}'(\mathbb{S}_R)$ is reflection positive. Then, in Lemma 6.3.20 the statement is verified for any weak limit of the family of measures $(\jmath_R^* \sharp \mu_R)_{R \in \mathbb{N}_+}$ on $\mathscr{S}'(\mathbb{R}^2)$. To this end, we take advantage of the fact that reflection positivity is preserved under limits.

Remark 6.3.1. *The significance of reflection positivity relies on the fact that it yields the existence of the Hilbert space H of quantum mechanical states. In particular, it guarantees the positivity of the inner product in the Hilbert space [50, Prop. 6.1.1]. Moreover, it gives rise to the existence of a positive-energy Hamiltonian, which enables us to perform an analytic continuation from Euclidean space to Minkowski space. Reflection positivity also establishes a correspondence between the unitary representations of the Euclidean group on the Euclidean side and the Poincaré group on the Minkowski side, cf. [73, Sec. 8.2].*

Recall that by Remark 2.4.9 the integral kernel of the covariance $G_R = (1-\Delta_R)^{-1}$ has a singularity at the coinciding points. To regularize G_R, we shall utilize an approximate Dirac delta written in terms of the geodesic distance. Similar regularization on $\mathbb{R}^2$ is used in [50, Sec. 8.5].

Definition 6.3.2. *Fix* $h \in C^\infty(\mathbb{R})$ *such that* $\operatorname{supp} h \subset (-1,1)$, $h = 1$ *on* $[-1/2, 1/2]$ *with* $2\pi \int_0^\infty \theta\, |h(\theta)|\, \mathrm{d}\theta = 1$. *For* $R, N \in \mathbb{N}_+$ *consider the measurable function* $\hat{K}_{R,N}$: $\mathbb{S}_R \times \mathbb{S}_R \to \mathbb{R}$ *defined by*

$$\hat{K}_{R,N}(\mathrm{x},\mathrm{y}) := N^2 h(N\, d_R(\mathrm{x},\mathrm{y})),$$

which is a smooth function with support in a ball of radius $1/N$. *For all* $R, N \in \mathbb{N}_+$ *one defines the bounded operator* $\hat{K}_{R,N} : L_2(\mathbb{S}_R) \to L_2(\mathbb{S}_R)$ *by*

$$(\hat{K}_{R,N} f)(\mathrm{x}) = \int_{\mathbb{S}_R} \hat{K}_{R,N}(\mathrm{x},\mathrm{y})\, f(\mathrm{y}) \rho_R(\mathrm{dy})\,.$$

Remark 6.3.3. *The restriction on the integral of the function* h *comes from the assumption of Lemma 2.4.2, which ensures the existence of the bounded operator* $\hat{K}_{R,N}$. *We demand that for some* $C \in (0,\infty)$ *it holds*

$$\|\hat{K}_{R,N}\| \le \sup_{\mathrm{x}\in\mathbb{S}_R} \int |\hat{K}(\mathrm{x},\mathrm{y})|\, \rho_R(\mathrm{dy}), \quad \|\hat{K}_{R,N}\| \le \sup_{\mathrm{y}\in\mathbb{S}_R} \int |\hat{K}(\mathrm{x},\mathrm{y})|\, \rho_R(\mathrm{dx}) \le C\,.$$

Explicity, one has

$$\|\hat{K}_{R,N}\| \le \sup_{\mathrm{x}\in\mathbb{S}_R} \int_{\mathbb{S}_R} N^2 |h(N d_R(\mathrm{x},\mathrm{y}))|\, \rho_R(\mathrm{dy}) \le 2\pi \int_0^\infty \theta\, |h(\theta)|\, \mathrm{d}\theta = 1\,.$$

It follows from a change of variable, i.e., $\theta' = N\, R\, \theta$, $\sin\theta \le \theta$ *and the support property of the function* h.

Remark 6.3.4. *By the Funk–Hecke Formula [9, Thm. 2.22] one gets*

$$(\hat{K}_{R,N} Y_{l,m})(\mathrm{x}) = \int_{\mathbb{S}_R} N^2 h(N\, R\, \theta(\mathrm{x},\mathrm{y}))\, Y_{l,m}(\mathrm{y})\, \rho_R(\mathrm{dy}) = h_l\, Y_{l,m}(\mathrm{x})\,,$$

where $h_l = \int_{\mathbb{S}_R} N^2 h(N\, R\, \theta(\mathrm{x},\mathrm{y}))\, P_l(\mathrm{x}\cdot\mathrm{y}/R^2)\, \rho_R(\mathrm{dy})$.

Definition 6.3.5. *For all* $R, N \in [1,\infty)$ *we set*

$$\mathbb{S}^{\pm}_{R,N} := \{(\mathrm{x}_1,\mathrm{x}_2,\mathrm{x}_3) \in \mathbb{S}_R \,|\, \pm\mathrm{x}_1 > 1/N\}, \quad \mathbb{S}_{R,N} := \mathbb{S}^{+}_{R,N} \cup \mathbb{S}^{-}_{R,N}, \quad \mathbb{S}^{\pm}_{R} := \cup_{N\in[1,\infty)} \mathbb{S}^{\pm}_{R,N}\,.$$

Definition 6.3.6. *Let* $R \in [1,\infty)$. *A functional* $F : \mathscr{D}'(\mathbb{S}_R) \to \mathbb{C}$ *is called cylindrical iff there exists* $k \in \mathbb{N}_+$, $G \in C^\infty_{\mathrm{b}}(\mathbb{R}^k)$ *and* $f_l \in C^\infty_{\mathrm{c}}(\mathbb{S}_R) := C^\infty(\mathbb{S}_R)$, $l \in \{1,\ldots,k\}$, *such that*

$$F(\phi) = G(\phi(f_1),\ldots,\phi(f_k)). \tag{6.3.1}$$

The algebra of cylindrical functions is denoted by $\mathcal{F}_R$. The subalgebras of $\mathcal{F}_R$ consisting of functionals of the form (6.3.1) with $\operatorname{supp} f_l \subset \mathbb{S}_R^\pm$, $l \in \{1,\ldots,k\}$, or such that $\operatorname{supp} f_l \subset \mathbb{S}_{R,N}^\pm$, $l \in \{1,\ldots,k\}$, are denoted by $\mathcal{F}_R^\pm$ and $\mathcal{F}_{R,N}^\pm$, respectively. The definitions of $\mathcal{F}$ and $\mathcal{F}^\pm$ are analogous to the definitions of $\mathcal{F}_R$ and $\mathcal{F}_R^\pm$ with $\mathbb{S}_R$ and $\mathbb{S}_R^\pm$ replaced by $\mathbb{R}^2$ and the half-plane $\{(x_1,x_2) \in \mathbb{R}^2 | \pm x_1 > 0\}$, respectively.

Definition 6.3.7. *Let $R \in [1,\infty)$, $\Theta_R \in O(3)$ with $\det \Theta_R = -1$. For $f \in C^\infty(\mathbb{S}_R)$ we define $\Theta_R f \in C^\infty(\mathbb{S}_R)$ by the formula $(\Theta_R f)(\mathrm{x}_1,\mathrm{x}_2,\mathrm{x}_3) := f(-\mathrm{x}_1,\mathrm{x}_2,\mathrm{x}_3)$. For $\phi \in \mathscr{D}'(\mathbb{S}_R)$ we define $\Theta_R\phi \in \mathscr{D}'(\mathbb{S}_R)$ by the formula $\langle \Theta_R\phi, f\rangle := \langle \phi, \Theta_R f\rangle$ for all $f \in C^\infty(\mathbb{S}_R)$. Observe that $\Theta_R^2 = \mathbb{1}$, i.e., it is an involution.*

Definition 6.3.8. *Let $f \in C^\infty(\mathbb{R}^2)$ and $\Theta \in O(2)$ with $\det \Theta = -1$ we define $\Theta f \in C^\infty(\mathbb{R}^2)$ by the formula $(\Theta f)(x_1,x_2) := f(-x_1,x_2)$. For $\phi \in \mathscr{S}'(\mathbb{R}^2)$ we define $\Theta\phi \in \mathscr{S}'(\mathbb{R}^2)$ by the formula $\langle \Theta\phi, f\rangle := \langle \phi, \Theta f\rangle$ for all $f \in \mathscr{S}(\mathbb{R}^2)$. Observe that $\Theta^2 = \mathbb{1}$, i.e., it is an involution.*

Remark 6.3.9. *It holds $\jmath_R^* \circ \Theta_R^* = \Theta^* \circ \jmath_R^*$, which implies that $\jmath_R^* \circ \Theta_R\,\phi = \Theta \circ \jmath_R^*\,\phi$ for all $\phi \in \mathscr{D}'(\mathbb{S}_R)$.*

Proposition 6.3.10. *Let μ be a weak limit of a subsequence of the sequence of measures $(\jmath_R^* \sharp \mu_R)_{R\in\mathbb{N}_+}$ on $\mathscr{S}'(\mathbb{R}^2)$. For all $F \in \mathcal{F}^+$ it holds $\int \overline{F(\Theta\phi)} F(\phi)\,\mu(\mathrm{d}\phi) \geq 0$.*

Remark 6.3.11. *We call this property the Glimm-Jaffe (GJ) reflection positivity, as it is similar to the variant stated in [50]. We will show how to obtain from it the OS reflection positivity (cf. Def. 1.0.1) at the end of this subsection.*

Proof. It is enough to prove that

$$
\begin{aligned}
\int \overline{F(\Theta\phi)} F(\phi)\,(\jmath_R^* \sharp \mu_R)(\mathrm{d}\phi) &= \int \overline{F(\Theta \jmath_R^* \phi)} F(\jmath_R^*\phi)\,\mu_R(\mathrm{d}\phi) \\
&= \int \overline{F(\jmath_R^* \Theta_R \phi)} F(\jmath_R^*\phi)\,\mu_R(\mathrm{d}\phi) \geq 0
\end{aligned}
$$

for all $R \in \mathbb{N}_+$ and $F \in \mathcal{F}^+$. By Def. 6.3.6 and Remark 2.1.8 for every $F \in \mathcal{F}^+$ it holds $F \circ \jmath_R^* \in \mathcal{F}_R^+$. Hence, the last bound above follows from the reflection positivity of the measure μ_R, which is given in Lemma 6.3.20 Item (D). □

We introduce a regularized free field $\hat{X}_{R,N}$ using the bounded operator $\hat{K}_{R,N}$ such that its law enjoys the reflection positivity, see Lemma 6.3.20 (B). Next, we show that the measure μ_R can be approximated using the corresponding regularized measure to $\hat{X}_{R,N}$, see Lemma 6.3.20 (C).

Definition 6.3.12. *Let* $\hat{c}_{R,N} := \mathrm{Tr}(\hat{K}_{R,N} G_R \hat{K}_{R,N})/4\pi R^2$*. By definition* $\hat{X}_{R,N} := \hat{K}_{R,N} X_R$*,*

$$\hat{X}^{:m:}_{R,N} := \sum_{k=0}^{\lfloor m/2 \rfloor} \frac{(-1)^k m!}{(m-2k)!k!2^k} (\hat{c}_{R,N})^k \hat{X}^{m-2k}_{R,N}, \qquad \hat{X}^{:m:}_{R,N}(h) := \int_{\mathbb{S}_R} \hat{X}^{:m:}_{R,N}(\mathrm{x})\, h(\mathrm{x})\, \rho_R(\mathrm{dx}),$$

$$\hat{Y}_{R,N} := \sum_{m=0}^{n} a_m \hat{X}^{:m:}_{R,N}(\mathbb{1}_{\mathbb{S}_R}), \qquad \tilde{Y}^{\pm}_{R,N} := \sum_{m=0}^{n} a_m \hat{X}^{:m:}_{R,N}(\mathbb{1}_{\mathbb{S}^{\pm}_{R,N}}), \qquad \tilde{Y}_{R,N} = \tilde{Y}^{+}_{R,N} + \tilde{Y}^{-}_{R,N},$$

where $h \in L_\infty(\mathbb{S}_R)$ *and* $\mathbb{1}_B$ *denotes the characteristic function of the set* B*.*

Remark 6.3.13. *Note that* $\hat{X}_{R,N}$ *introduced above and* $X_{R,N} = K_{R,N} X_R$ *introduced in Sec. 3.4 are free fields on* $\mathbb{S}_R$ *with different UV cutoffs. We use the same symbol* $N \in \mathbb{N}_+$ *to denote both cutoffs.*

Remark 6.3.14. *By Lemma B.1 Item (B) and Lemma 2.3.14 it holds* $\hat{X}_{R,N} \in L^1_2(\mathbb{S}_R) \subset L_n(\mathbb{S}_R)$ *almost surely. In particular,* $\hat{Y}_{R,N}$*,* $\tilde{Y}_{R,N}$ *are well-defined. The same follows using Remark 2.6.8 and Remark 6.3.17 combined with the fact that* $Y_{R,N}$ *given in Eq. (3.4.1) is well-defined. Moreover, there exists* $C \in (0,\infty)$ *such that for all* $N, R \in \mathbb{N}_+$ *it holds* $|\hat{c}_{R,N} - 1/2\pi \log N| \leq C$ *by the bound (3.3.2) and Remark 6.3.17.*

Remark 6.3.15. *Remark 6.3.4 implies that the operators* $\hat{K}_{R,N}$ *and* $(-\Delta_R)$ *both have the same set of eigenfunctions, i.e., they are diagonalizable simultaneously. Now using Remark 2.4.1 Item (A) and Item (B) with* $T = (-\Delta_R)$*,* $F(T) = (1 - \Delta_R/N^2)^{-1} = K_{R,N}$ *and* $S = \hat{K}_{R,N}$ *one infers that for all* $R, N \in \mathbb{N}_+$ *the operators* $\hat{K}_{R,N}$ *and* $K_{R,N}$ *commute, i.e., for all* $R, N \in \mathbb{N}_+$ *the bounded operator* $\hat{K}_{R,N}$ *commutes with every orthogonal projection* $(2l+1)\mathcal{P}_{R,l}$ *for all* $l \in \mathbb{N}_+$*, see Remark 2.4.1 Item (C).*

Remark 6.3.16. *Assume that* A, B *are self-adjoint i.e,* $A^* = A$*,* $B^* = B$ *and commuting. The spectral theorem implies that* A *and* B *behave like multiplication operators* a*,* b *on a common* L_2 *space. Therefore, one can apply the triangle inequality,* $|a + b| \leq |a| + |b|$ *pointwise to infer that* $|A + B| \leq |A| + |B|$*, where* $|A| = (A^*A)^{1/2}$*.*

Remark 6.3.17. *Recall that* $K_{R,N} = (1 - \Delta_R/N^2)^{-1}$*,* $G_R = (1 - \Delta_R)^{-1}$ *and the counterterms* $c_{R,N}$*,* $\hat{c}_{R,N}$ *were introduced in Eq. (3.3.1) and Def. 6.3.12, respectively. Note that by Remark 6.3.15 the operators* $G_R, K_{R,N}, \hat{K}_{R,N}$ *commute. Using Lemma B.9 and the fact that for commuting self-adjoint operators* A, B *it holds* $|A + B| \leq |A| + |B|$ *and* $|A - B| = |A - 1 - (B - 1)| \leq |A - 1| + |B - 1|$*, we obtain*

$$|\hat{K}^2_{R,N} - K^2_{R,N}| \leq |\hat{K}_{R,N} - K_{R,N}|\, |\hat{K}_{R,N} + K_{R,N}| \leq 2\, C\, (C+1)\, \frac{(1 - \Delta_R)/N^2}{(1 - \Delta_R/N^2)^2}\,,$$

where we used the fact that

$$|1-K_{R,N}|+|1-\hat{K}_{R,N}|\leq(C+1)\,\frac{(1-\Delta_R)/N^2}{1-\Delta_R/N^2}\,.$$

Observe that

$$\frac{(1-\Delta_R)/N^2}{(1-\Delta_R/N^2)^2}G_R=N^{-2}\frac{1}{(1-\Delta_R/N^2)^2}=\frac{(1-\Delta_R)}{N^2}\frac{1}{(1-\Delta_R/N^2)^2}\frac{1}{(1-\Delta_R)}$$

and decompose $(1-\Delta_R)/N^2=(1-\Delta_R/N^2)-1+1/N^2$. *Consequently, it holds*

$$\begin{aligned}|\hat{c}_{R,N}-c_{R,N}|&\leq\operatorname{Tr}(|\hat{K}^2_{R,N}-K^2_{R,N}|\,G_R)/4\pi R^2\\&\leq\frac{2C(C+1)}{4\pi R^2}\Big[\operatorname{Tr}((1-\Delta_R/N^2)^{-1}(1-\Delta_R)^{-1})-\operatorname{Tr}((1-\Delta_R/N^2)^{-2}(1-\Delta_R)^{-1})\\&\qquad+\operatorname{Tr}((1-\Delta_R/N^2)^{-2}(1-\Delta_R)^{-1})/N^2\Big].\end{aligned}$$

By Lemma 2.4.10 the RHS of the last inequality above is bounded by a constant independent of $R,N\in\mathbb{N}_+$.

Lemma 6.3.18. *The sequence* $(F(\hat{X}_{R,N})\exp(-\tilde{Y}_{R,N}))_{N\in\mathbb{N}_+}$ *converges in probability to* $F(X_R)\exp(-Y_R)$.

Proof. From Lemma B.2 for all $\kappa\in(0,\infty)$ and $\delta\in[0,2]$ one has

$$\mathbb{P}\Big(\|X_R-\hat{X}_{R,N}\|^2_{L_2^{-\kappa-\delta}(\mathbb{S}_R)}\geq\epsilon\Big)\leq\epsilon^{-1}\,\mathbb{E}\|X_R-\hat{X}_{R,N}\|^2_{L_2^{-\kappa-\delta}(\mathbb{S}_R)}\leq\epsilon^{-1}\,R^2\,C^2\,N^{-2\delta}$$

and by Lemma B.6 Item (C) and Item (D) one gets

$$\begin{aligned}\mathbb{P}\Big(|Y_R-\tilde{Y}_{R,N}|^2\geq\epsilon\Big)&\leq\mathbb{P}\Big(|Y_R-\hat{Y}_{R,N}|^2\geq\epsilon/2\Big)+\mathbb{P}\Big(|\hat{Y}_{R,N}-\tilde{Y}_{R,N}|^2\geq\epsilon/2\Big)\\&\leq\epsilon^{-1}\,C^2\,\Big(N^{-1/n}+N^{-1}\Big)\,.\end{aligned}$$

Using the above bounds one shows that the sequence $(\hat{X}_{R,N})_{N\in\mathbb{N}_+}$ and $(\tilde{Y}_{R,N})_{N\in\mathbb{N}_+}$ converge in probability as N approaches infinity to the random variables X_R and Y_R. One concludes the statement using the fact that $F:\mathscr{D}'(\mathbb{S}_R)\to\mathbb{R}$ and $t\mapsto\exp(t)$ are continuous. Hence, by Remark 2.5.3 the sequences $(F(X_{R,N}))_{N\in\mathbb{N}_+}$ and $(\exp(-\tilde{Y}_{R,N}))_{N\in\mathbb{N}_+}$ converge in probability to $F(X_R)$ and $\exp(-Y_R)$. This finishes the proof. □

Lemma 6.3.19. *For all $R \in \mathbb{N}_+$ and all bounded and continuous $F : \mathscr{D}'(\mathbb{S}_R) \to \mathbb{R}$ it holds*

$$\lim_{N\to\infty} \mathbb{E}F(\hat{X}_{R,N}) \exp(-\tilde{Y}_{R,N}) = \lim_{N\to\infty} \mathbb{E}F(X_{R,N}) \exp(-Y_{R,N}).$$

Proof. The proof follows from the same strategy implemented in the proof of the proposition 3.4.6. By Lemma 6.3.18 the sequences $(F(\hat{X}_{R,N}) \exp(-\tilde{Y}_{R,N}))_{N\in\mathbb{N}_+}$ and $(F(X_{R,N}) \exp(-Y_{R,N}))_{N\in\mathbb{N}_+}$ converge in probability to $F(X_R) \exp(-Y_R)$. To conclude we show that the aforementioned sequences are uniformly integrable by repeating verbatim the argument from the proof of Proposition 3.4.6. □

Lemma 6.3.20. *The following statements hold true for all $R, N \in \mathbb{N}_+$:*

(A) If $F \in \mathcal{F}^+_R$, then $\mathbb{E}\overline{F(\Theta_R X_R)}F(X_R) \geq 0$.

(B) If $F \in \mathcal{F}^+_{R,N}$, then $\mathbb{E}\overline{F(\Theta_R \hat{X}_{R,N})}F(\hat{X}_{R,N}) \geq 0$.

(C) If $F \in \mathcal{F}^+_{R,N}$, then $\mathbb{E}\overline{F(\Theta_R \hat{X}_{R,N})}F(\hat{X}_{R,N}) \exp(-\tilde{Y}_{R,N}) \geq 0$.

(D) For all $F \in \mathcal{F}^+_R$ it holds $\int \overline{F(\Theta_R \phi)}F(\phi)\, \mu_R(\mathrm{d}\phi) \geq 0$.

Proof. For the proof of Item (A) see [33, Theorem 2] or [3, Thm 8.3]. To prove Item (B), denote by x^- the reflection of $\mathrm{x} \in \mathbb{S}_R$ corresponding to the action of Θ_R. Observe that $d_R(\mathrm{x}^-, \mathrm{y}) = d_R(\mathrm{x}, \mathrm{y}^-)$, which implies that $\hat{K}_{R,N}(\mathrm{x}^-, \mathrm{y}) = \hat{K}_{R,N}(\mathrm{x}, \mathrm{y}^-)$. Hence, by Remark 2.4.1 one deduces that $[\Theta_R, \hat{K}_{R,N}] = 0$, which yields $\Theta_R \hat{K}_{R,N} X_R = \hat{K}_{R,N} \Theta_R X_R$. By assumption $F \in \mathcal{F}^+_{R,N}$. Observe that for all $f \in C^\infty(\mathbb{S}_R)$ it holds $(\hat{K}_{R,N} X_R)(f) = X_R(\hat{K}_{R,N} f)$, which implies that

$$F(\hat{K}_{R,N} X_R) = G(X_R(\hat{K}_{R,N} f_1), \ldots, X_R(\hat{K}_{R,N} f_k)).$$

Utilizing the fact that f is supported in $\mathbb{S}^+_{R,N}$ and the integral kernel of $K_{R,N}$ is supported in a $1/N$-neighbourhood of the diagonal, $\hat{K}_{R,N} f$ belongs to $\mathcal{F}^+_R$. Consequently, the statement follows from Item (A). To prove Item (C), note that

$$\mathbb{E}\overline{F(\Theta_R \hat{X}_{R,N})}F(\hat{X}_{R,N}) \exp(-\tilde{Y}_{R,N}) = \mathbb{E}\overline{F(\Theta_R \hat{X}_{R,N}) \exp(-\tilde{Y}^-_{R,N})}F(\hat{X}_{R,N}) \exp(-\tilde{Y}^+_{R,N})\,.$$

Denote $H(\hat{X}_{R,N}) := F(\hat{X}_{R,N}) \exp(-\tilde{Y}^+_{R,N})$. Using that $\tilde{Y}^-_{R,N}(X_{R,N}) = \tilde{Y}^+_{R,N}(\Theta_R X_{R,N})$ one infers

$$\mathbb{E}\overline{F(\Theta_R \hat{X}_{R,N})}F(\hat{X}_{R,N}) \exp(-\tilde{Y}_{R,N}) = \mathbb{E}\overline{H(\Theta_R \hat{X}_{R,N})}H(\hat{X}_{R,N})\,.$$

Let us now show that (C) follows from (A). Denote $H(X_R) := F(\hat{K}_{R,N}X_R)\exp(-\tilde{Y}^+_{R,N})$. We want to approximate H by $H_k \in \mathcal{F}^+_R$ in the topology of $L_2(\mathscr{D}'(\mathbb{S}_R), \nu_R)$. We recall the definition of $\tilde{Y}^+_{R,N}$:

$$\tilde{Y}^+_{R,N} = \int P((\hat{K}_{R,N}X_R(\mathrm{x})), \hat{c}_{R,N})\mathbb{1}_{\mathbb{S}^+_{R,N}}(\mathrm{x})\,\rho_R(\mathrm{dx}).$$

By continuity of $\hat{K}_{R,N}X_{R,N}$ a.e. in $\omega \in \Omega$, we can approximate the integral by a Riemann sum

$$(\tilde{Y}^+_{R,N})_k := \sum_{\mathrm{x}_j \in \mathrm{Part}_k} (\Delta\mathrm{x})_j\, P((\hat{K}_{R,N}X_R(\mathrm{x}_j)), \hat{c}_{R,N})\,\mathbb{1}_{\mathbb{S}^+_{R,N}}(\mathrm{x}_j),$$

which converges pointwise in ω. Partitions of the sphere can be obtained, e.g., in spherical coordinates, in which case $(\Delta x)_j = \sin(\theta_i)\Delta\theta_i\Delta\varphi_i$. Exploiting again the convergence of the Riemann sums, we can assume $\sum_{\mathrm{x}_j \in \mathrm{Part}_k}(\Delta\mathrm{x})_j \le 2\,\mathrm{vol}(\mathbb{S}_R)$, for sufficiently fine partitions. Then, by Lemma 3.3.2,

$$(\tilde{Y}^+_{R,N})_k \ge -\sum_{x_j \in \mathrm{Part}_k} (\Delta\mathrm{x})_j\, A\, c^{n/2}\mathbb{1}_{\mathbb{S}^+_{R,N}}(\mathrm{x}_j) \ge -A\,c^{n/2} 2\,\mathrm{vol}(\mathbb{S}_R).$$

Hence

$$\exp(-(\tilde{Y}^+_{R,N})_k) \le \exp(A\,c^{n/2}2\mathrm{Vol}(\mathbb{S}_R)). \tag{6.3.5}$$

We note that $H_{0,k}$ is cylindrical due to smearing of X_R with $\hat{K}_{R,N}$. Now we want to show that the cylindrical functions

$$H_k(X_R) := F(\hat{K}_{R,N}X_R)\exp(-(\tilde{Y}^+_{R,N})_k)$$

approximate H in $L_2(\mathscr{D}'(\mathbb{S}_R), \nu_R)$, i.e.,

$$\lim_{k\to\infty} \int |H(\phi) - H_k(\phi)|^2\,\nu_R(\mathrm{d}\phi) = 0. \tag{6.3.7}$$

We note that by Eq. (6.3.5)

$$|H_k(\phi)| := |F(\hat{K}_{R,N}\phi)|\exp(-H_{0,k}(\phi)) \le C,$$

uniformly in k, thus dominated convergence gives (6.3.7). This concludes the proof of (C). Let us turn to the proof of Item (D). First note that for any $F \in \mathcal{F}^+_R$ there exists $M \in \mathbb{N}_+$ such that $F \in \mathcal{F}^+_{R,M}$. Hence, it suffices to show that $\int \overline{F(\Theta_R\phi)}F(\phi)\,\mu_R(\mathrm{d}\phi) \ge$

0 for all $R, M \in \mathbb{N}_+$ and $F \in \mathcal{F}^+_{R,M}$. To establish this claim we note that by Lemma 6.3.19

$$\int \overline{F(\Theta_R\phi)} F(\phi)\, \mu_R(\mathrm{d}\phi) = \lim_{N\to\infty} \frac{\mathbb{E}\overline{F(\Theta_R \hat{X}_{R,N})} F(\hat{X}_{R,N}) \exp(-\tilde{Y}_{R,N})}{\mathbb{E}\exp(-\tilde{Y}_{R,N})}$$

and use Item (C) together with the fact that $\mathcal{F}^+_{R,M} \subset \mathcal{F}^+_{R,N}$ for all $N \geq M$. □

Recall that cylindrical functions $F \in \mathcal{F}$ have the form

$$F(\phi) = G(\phi(f_1), \dots, \phi(f_m)),$$

where $f_1, \dots, f_m \in \mathscr{S}(\mathbb{R}^2)$ and G is a smooth, bounded, complex-valued function. We have the following

$$\int \overline{F(\Theta\phi)} F(\phi)\, (\jmath^*_R \sharp \mu_R)(\mathrm{d}\phi) \geq 0.$$

We consider the Schwinger functions $S_m \in \mathscr{S}'(\mathbb{R}^{3m})$. We want to verify the OS reflection positivity which we recall here for convenience:

OS2: Let $\mathbb{R}^{dm}_+ := \{(x^{(1)}, \dots, x^{(m)}) \in \mathbb{R}^{dm} : x^{(j)}_0 > 0, j = 1, \dots, m\}$ and

$$\mathscr{S}(\mathbb{R}^{3n}_+; \mathbb{C}) := \{ f \in \mathscr{S}(\mathbb{R}^{dm}; \mathbb{C}) \,:\, \mathrm{supp}(f) \subset \mathbb{R}^{dm}_+ \}.$$

For all sequences $f^{(m)} \in \mathscr{S}(\mathbb{R}^{dm}_+; \mathbb{C})$, $m \in \mathbb{N}_0$, with finitely many non-zero elements

$$\sum_{\ell, m \in \mathbb{N}_0} S_{\ell+m}(\overline{\Theta f^{(\ell)}} \otimes f^{(m)}) \geq 0.$$

Lemma 6.3.21. *It suffices to show reflection positivity for $f^{(m)}_* \in \mathscr{S}(\mathbb{R}^{dm}_+; \mathbb{C})$ of the form*

$$f^{(m)}_* = \sum_{\alpha \in \mathcal{I}_m, |\alpha| = m} f^\alpha, \tag{6.3.9}$$

where $f^\alpha(x^{(1)}, \dots, x^{(m)}) = f_{\alpha_1}(x^{(1)}) \dots f_{\alpha_m}(x^{(m)})$, $f_{\alpha_i} \in \mathscr{S}(\mathbb{R}^d_+)$ and $\mathcal{I}_m$ is a finite set of multiindices.

Proof. By standard density arguments, given any $f^{(m)} \in \mathscr{S}(\mathbb{R}^{3m}_+; \mathbb{C})$ we can find a sequence of $f^{(m)}_{*,\epsilon}$, $\epsilon > 0$, of functions of the form (6.3.9), possibly without the support restriction, such that

$$f^{(m)} = \lim_{\epsilon \to 0} f^{(m)}_{*,\epsilon} \tag{6.3.10}$$

in the Schwartz topology. To ensure the support property of the functions on the RHS, we choose, for our given $f^{(m)}$, a bounded smooth function η supported in $\mathbb{R}^d_+$, such that

$$(\eta(x^{(1)})\dots\eta(x^{(m)}))f^{(m)}(x^{(1)},\dots,x^{(m)}) = f^{(m)}(x^{(1)},\dots,x^{(m)}).$$

By multiplying both sides of Eq. (6.3.10) with such products of functions η and making use of the fact that this operation is continuous in the Schwartz topology we ensure that the RHS of Eq. (6.3.10) consists of functions from $\mathscr{S}(\mathbb{R}^{dm}_+;\mathbb{C})$. Using that the Schwinger functions are tempered distributions and positivity is preserved in the limit, we obtain the claim. □

Now we define $\phi(f)^\alpha = \phi(f_{\alpha_1})\dots\phi(f_{\alpha_{|\alpha|}})$. We also set

$$\phi(f)^\alpha_\varepsilon := \phi(f_{\alpha_1})\mathrm{e}^{-\varepsilon\phi(f_{\alpha_1})^2}\dots\phi(f_{\alpha_{|\alpha|}})\mathrm{e}^{-\varepsilon\phi(f_{\alpha_m})^2}$$

and note that this is a cylindrical function. We write

$$\begin{aligned}
&\sum_{\ell,m\in\mathbb{N}_0} S_{\ell+m}(\overline{\Theta f^{(\ell)}_*}\otimes f^{(m)}_*)\\
&= \int \overline{\sum_\ell \sum_{\alpha\in\mathcal{I}_\ell,|\alpha|=\ell} \phi(\Theta f)^{\alpha_\ell}} \sum_m \sum_{\beta\in\mathcal{I}_m,|\beta|=m} \phi(f)^{\beta_m}\,\mu(\mathrm{d}\phi)\\
&= \lim_{\varepsilon\to 0}\int \overline{\sum_\ell \sum_{\alpha\in\mathcal{I}_\ell,|\alpha|=\ell} \phi(\Theta f)^{\alpha_\ell}_\varepsilon} \sum_m \sum_{\beta\in\mathcal{I}_m,|\beta|=m} \phi(f)^{\beta_m}_\varepsilon\,\mu(\mathrm{d}\phi) \geq 0,
\end{aligned}$$

where we made use of dominated convergence, the existence of the moments of the measure and the GJ-reflection positivity to conclude the OS reflection positivity.

6.4 Euclidean invariance

In this section we demonstrate that every accumulation point μ of a subsequence of the sequence of measures $(\jmath^*_R\sharp\mu_R)_{R\in\mathbb{N}_+}$ on $\mathscr{S}'(\mathbb{R}^2)$ is invariant under the action of the Euclidean group $T(2)\rtimes O(2)$. To this end, we appeal to the fact that the symmetry groups of the plane and the sphere have the same dimension, see Remark 3.1.2. Hence, one might hope to establish a one-to-one correspondence between the actions of these groups. To this end, we shall use the fact that for all $R\in\mathbb{N}_+$, the measure μ_R is invariant under the action of the orthogonal group $O(3)$, which is a consequence of the invariance property of the Gaussian measure ν_R with covariance $G_R = (1 -$

$\Delta_R)^{-1}$, see Lemma 6.4.20. Observe that every rotations around x_3 fixing the north pole $N = (0,0,R)$ stabilize the x_1x_2-plane and form a copy of $SO(2) \subset SO(3)$, see Remark 2.1.2. Consequently, one shows that the rotations of $\mathbb{S}_R$ around the x_3 axis are mapped under the stereographic projection to the rotations of the plane for any arbitrary angles. As a result, one easily proves the rotational invariance property for the measures $(\jmath_R^*\sharp\mu_R)_{R\in\mathbb{N}_+}$ for $R \in \mathbb{N}_+$ on $\mathscr{S}'(\mathbb{R}^2)$. However, the proof of the translational invariance just holds in a restricted sense, i.e., it holds true in the limit as $R \to \infty$ provided that one starts off by rotations around x_i, $i = 1, 2$ whose angles are inversely proportional to the radius of $\mathbb{S}_R$. Furthermore, the reflection invariance is deduced using Remark 2.1.1 with the help of the involutions Θ_R and Θ defined in Def. 6.3.7 and Def. 6.3.8, respectively.

Remark 6.4.1. *The proof of the Euclidean invariance is independent of the value of the coupling constant $\lambda \in (0,\infty)$ and holds true for any positive coupling constant.*

In the remaining part of this chapter, we first establish a one-to-one correspondence between the action of the group $SO(3)$ on $\mathbb{S}_R$ and the action of the group $T(2) \rtimes SO(2)$ on the plane, and we proceed to show that weak limit points of the sequence $(\jmath_R^*\sharp\mu_R)_{R,\in\mathbb{N}_+}$ on $\mathscr{S}'(\mathbb{R}^2)$ are invariant under the action of $T(2) \rtimes SO(2)$. Then, we demonstrate the reflection invariance property of weak limit points of the sequence $(\jmath_R^*\sharp\mu_R)_{R,\in\mathbb{N}_+}$ on $\mathscr{S}'(\mathbb{R}^2)$. One infers the Euclidean invariance of weak limit points of the sequence $(\jmath_R^*\sharp\mu_R)_{R,\in\mathbb{N}_+}$ by combining the preceding results.

Definition 6.4.2. *For $R \in \mathbb{N}_+$, $\alpha \in \mathbb{R}$ and $\mathrm{x} = (\mathrm{x}_1, \mathrm{x}_2, \mathrm{x}_3) \in \mathbb{S}_R$ we specify the actions of the generators of the group $SO(3)$ on $\mathbb{S}_R$ by the maps $\mathcal{R}_{R,\alpha}, \mathcal{T}_{R,\alpha}, \mathcal{T}'_{R,\alpha} : \mathbb{S}_R \to \mathbb{S}_R$ defined by*

$$\begin{aligned}\mathcal{R}_{R,\alpha}(\mathrm{x}) &= (\mathrm{x}_1\cos\alpha + \mathrm{x}_2\sin\alpha, \mathrm{x}_1\sin\alpha - \mathrm{x}_2\cos\alpha, \mathrm{x}_3),\\ \mathcal{T}_{R,\alpha}(\mathrm{x}) &= (\mathrm{x}_1\cos(\alpha/R) - \mathrm{x}_3\sin(\alpha/R), \mathrm{x}_2, \mathrm{x}_1\sin(\alpha/R) + \mathrm{x}_3\cos(\alpha/R)),\\ \mathcal{T}'_{R,\alpha}(\mathrm{x}) &= (\mathrm{x}_1, \mathrm{x}_2\cos(\alpha/R) - \mathrm{x}_3\sin(\alpha/R), \mathrm{x}_2\sin(\alpha/R) + \mathrm{x}_3\cos(\alpha/R)).\end{aligned}$$

Remark 6.4.3. *Any rotation $Q \in SO(3)$ can be represented using the generators of $SO(3)$ cf. [30, Lemma A.2.2].*

Remark 6.4.4. *Let $Q_{R,\alpha} \in \{\mathcal{R}_{R,\alpha}, \mathcal{T}_{R,\alpha}, \mathcal{T}'_{R,\alpha}\}$ defined by $Q_{R,\alpha} : \mathbb{S}_R \to \mathbb{S}_R$. Note that $Q_{R,\alpha}$ acts orthogonally, transitively and in an orientation preserving manner on $\mathbb{S}_R$, and give rise to a group of unitary operators on $L_2(\mathbb{S}_R)$. Let $Q^*_{R,\alpha} \in \{\mathcal{R}^*_{R,\alpha}, \mathcal{T}^*_{R,\alpha}, \mathcal{T'}^*_{R,\alpha}\}$ be maps $Q^*_{R,\alpha} : \mathscr{D}'(\mathbb{S}_R) \to \mathscr{D}'(\mathbb{S}_R)$ defined as follows: First, for all $f \in C^\infty(\mathbb{S}_R)$ we set $Q^*_{R,\alpha}f := f \circ Q_{R,\alpha} \in C^\infty(\mathbb{S}_R)$. Next, for $\phi \in \mathscr{D}'(\mathbb{S}_R)$ we set $Q^*_{R,\alpha}\phi := \phi \circ Q^*_{R,-\alpha} \in$*

$\mathscr{D}'(\mathbb{S}_R)$. *It holds* $\langle Q^*_{R,\alpha}\phi, f\rangle = \langle \phi, Q^*_{R,-\alpha} f\rangle {=} \langle \phi, f \circ Q_{R,-\alpha}\rangle$, *which is due to the rotational invariance of the Riemannian volume form* $\rho_R(\mathrm{dy})$.

Remark 6.4.5. *Let* $Q_{R,\alpha} \in \{\mathcal{R}_{R,\alpha}, \mathcal{T}_{R,\alpha}, \mathcal{T}'_{R,\alpha}\}$ *as in Remark 6.4.4. It holds* $\mathrm{T}_p\, Q_{R,\alpha} : \mathrm{T}_p\, \mathbb{S}_R \to \mathrm{T}_{Q_{R,\alpha}(p)}\, \mathbb{S}_R$ *for any* $p \in \mathbb{S}_R$. *Observe that* $\mathrm{T}_p\, \mathbb{S}_R = p^\perp$ *and* $\mathrm{T}_{Q_{R,\alpha}(p)}\, \mathbb{S}_R = (Q_{R,\alpha}(p))^\perp$ *both are of dimension two, i.e., they are isomorphic to* $\mathbb{R}^2$ *as vector spaces. This is due to the fact that* $Q_{R,\alpha}$ *is a diffeomorphism and it preserves the angles and lengths and its differential is an isomorphism so* $\det(\mathrm{T}_p\, Q_{R,\alpha}) \neq 0$ [1]. *In particular, since* $Q_{R,\alpha}$ *preserves orientation on* $\mathbb{S}_R$, *its differential map* $\mathrm{T}_p\, Q_{R,\alpha}$ *acts as orientation-preserving isometry of the corresponding tangent spaces such that* $\mathrm{T}_p\, Q_{R,\alpha} \in SO(2)$ *with* $\det(\mathrm{T}_p\, Q_{R,\alpha}) = 1$ *for all* $p \in \mathbb{S}_R$, *see Remark 2.1.3.*

Remark 6.4.6. *The algebra of cylindrical functionals* $\mathcal{F}$ *separates points in* $L_2^{-1}(\mathbb{R}^2, v_L^{1/2}) \subset \mathscr{D}'(\mathbb{R}^2)$. *Hence, if* μ_j, $j = 1, 2$, *are Borel probability measures on* $L_2^{-1}(\mathbb{R}^2, v_L^{1/2})$ *such that* $\mu_1(F) = \mu_2(F)$ *for all* $F \in \mathcal{F}$, *then* $\mu_1 = \mu_2$ *by [43, Theorem 4.5(a), Ch. 3]. The same is true if* μ_j, $j = 1, 2$ *are Borel probability measures on* $L_2^1(\mathbb{R}^2, v_L^{1/2}) \subset \mathscr{D}'(\mathbb{R}^2)$.

Lemma 6.4.7. *Let* $Q^*_{R,\alpha} \in \{\mathcal{R}^*_{R,\alpha}, \mathcal{T}^*_{R,\alpha}, \mathcal{T'}^*_{R,\alpha}\}$ *defined by* $Q^*_{R,\alpha} : \mathscr{D}'(\mathbb{S}_R) \to \mathscr{D}'(\mathbb{S}_R)$. *For all* $R, N \in [1, \infty)$

(A) The regularized Gaussian measure $\nu_{R,N}$ *has rotational invariance.*

(B) The measure μ_R *is rotationally invariant, i.e.,* $\mu_R(F) = \mu_R(F \circ Q^*_{R,\alpha})$ *for all bounded and measurable* $F : \mathscr{D}'(\mathbb{S}_R) \to \mathbb{R}$ *and all* $\alpha \in \mathbb{R}$.

Proof. To prove Item (A), it suffices to show that for all $R \in \mathbb{N}_+$ the covariance $G_{R,N}$ is rotationally invariant and refer to the Bochner-Minlos theorem (Thm. 2.6.2). Utilizing Remark 2.4.7 one infers that for all $R, N \in \mathbb{N}_+$ and for all $\mathrm{x}, \mathrm{y} \in \mathbb{S}_R$ it holds $G_{R,N}(\mathrm{x}, \mathrm{y}) = G_{R,N}(Q_{R,\alpha}\mathrm{x}, Q_{R,\alpha}\mathrm{y}) = \int_{\mathscr{D}'(\mathbb{S}_R)} \phi(\mathrm{x})\, \phi(\mathrm{y})\, \nu_{R,N}(\mathrm{d}\phi)$. Thus, from the preceding relation for all $f, g \in \mathscr{D}(\mathbb{S}_R)$ one gets

$$G_{R,N}(f, g) = \int_{\mathbb{S}_R \times \mathbb{S}_R} G_{R,N}(\mathrm{x}, \mathrm{y})\, f(\mathrm{x})\, g(\mathrm{y})\, \rho_R(\mathrm{dx})\, \rho_R(\mathrm{dy}) = G_{R,N}(Q^*_{R,-\alpha} f, Q^*_{R,-\alpha} g).$$

It follows from the facts that for all $R \in \mathbb{N}_+$, the Riemannian volume form ρ_R is rotationally invariant, $Q_{R,\alpha}$ acts transitively on $\mathbb{S}_R$ and Remark 6.4.4. This finishes the proof of Item (A). To prove Item (B) recall that for all $R, N \in [1, \infty)$ the measure $\nu_{R,N}$ is concentrated in $L_2^1(\mathbb{S}_R)$, see Remark 3.3.4. From Remark 6.4.6 it suffices to prove

[1]Taking derivative of $Q^{-1}_{Q(p)} \circ Q(p) = \mathrm{Id}_p$ one gets $T_{Q(p)}\, Q^{-1} \circ T_p\, Q = \mathrm{Id}_{\mathrm{T}_p\, \mathbb{S}_R}$. Hence, $\det(T_{Q(p)}\, Q^{-1})\, \det(T_p\, Q) = \det(\mathrm{Id}_{\mathrm{T}_p\, \mathbb{S}_R}) = 1$.

that for all $\alpha \in \mathbb{R}$ and all cylindrical functionals $F \in \mathcal{F}_{R,N}$ it holds $\mu_{R,N}(F \circ Q^*_{R,\alpha}) = \mu_{R,N}(F)$. It holds

$$\begin{aligned}\int_{L^1_2(\mathbb{S}_R)} F(Q^*_{R,\alpha}\phi)\, \mu_{R,N}(\mathrm{d}\phi) &= \frac{1}{\mathcal{Z}_{R,N}} \int_{L^1_2(\mathbb{S}_R)} F(Q^*_{R,\alpha}\phi)\, \mathcal{U}_{R,N}(Q^*_{R,\alpha}\phi)\, \nu_{R,N}(\mathrm{d}\phi) \\ &= \frac{1}{\mathcal{Z}_{R,N}} \int_{L^1_2(\mathbb{S}_R)} F(\phi)\, \mathcal{U}_{R,N}(\phi)\, (Q^*_{R,\alpha}\sharp\nu_{R,N})(\mathrm{d}\phi) \\ &= \frac{1}{\mathcal{Z}_{R,N}} \int_{L^1_2(\mathbb{S}_R)} F(\phi)\, \mathcal{U}_{R,N}(\phi)\, \nu_{R,N}(\mathrm{d}\phi) \\ &= \int_{L^1_2(\mathbb{S}_R)} F(\phi)\, \mu_{R,N}(\mathrm{d}\phi),\end{aligned}$$

where in the third line, we used Item (A), i.e., $(Q^*_{R,\alpha}\sharp\nu_{R,N})(A) = \nu_{R,N}(A)$ for all $A \in \mathrm{Borel}(L^1_2(\mathbb{S}_R))$. Now taking the limit $N \to \infty$ we conclude the lemma. □

Definition 6.4.8. *For $\alpha \in \mathbb{R}$ and all $x := (x_1, x_2) \in \mathbb{R}^2$ we determine the actions of the generators of the group $T(2) \rtimes SO(2)$ on $\mathbb{R}^2$ by the maps $\mathcal{R}_\alpha, \mathcal{T}_\alpha, \mathcal{T}'_\alpha : \mathbb{R}^2 \to \mathbb{R}^2$ defined by*

$$\mathcal{R}_\alpha(x_1, x_2) := (x_1 \cos\alpha + x_2 \sin\alpha, x_1 \sin\alpha - x_2 \cos\alpha),$$
$$\mathcal{T}_\alpha(x_1, x_2) := (x_1 + \alpha, x_2), \quad \mathcal{T}'_\alpha(x_1, x_2) := (x_1, x_2 + \alpha).$$

Remark 6.4.9. *Let $Q_\alpha \in \{\mathcal{R}_\alpha, \mathcal{T}_\alpha, \mathcal{T}'_\alpha\}$ and $Q^*_\alpha \in \{\mathcal{R}^*_\alpha, \mathcal{T}^*_\alpha, \mathcal{T}'^*_\alpha\}$ defined by $Q_\alpha : \mathbb{R}^2 \to \mathbb{R}^2$ and by $Q^*_\alpha : \mathscr{D}'(\mathbb{R}^2) \to \mathscr{D}'(\mathbb{R}^2)$, respectively. For $f \in C^\infty_{\mathrm{c}}(\mathbb{R})$ and $\phi \in \mathscr{D}'(\mathbb{R}^2)$ we set*

$$Q^*_\alpha f := f \circ Q_\alpha \in C^\infty_{\mathrm{c}}(\mathbb{R}^2), \quad Q^*_\alpha \phi := \phi \circ Q^*_{-\alpha} \in \mathscr{D}'(\mathbb{R}^2).$$

This implies that $\langle Q^*_\alpha \phi, f\rangle = \langle \phi, Q^*_{-\alpha} f\rangle = \langle \phi, f \circ Q_{-\alpha}\rangle$, which is due to the Euclidean invariance of the Lebesgue measure.

Definition 6.4.10. *Let $R \in \mathbb{N}_+$, $\alpha \in (-R, R)$ and consider the ball $B_R := \{x \in \mathbb{R}^2 \,|\, |x| < R\} \subset \mathbb{R}^2$. We define the map $\mathcal{S}_{R,\alpha} : B_R \to \mathbb{R}^2$ by*

$$\mathcal{S}_{R,\alpha}(x_1, x_2) := \frac{2(R \sin(\alpha/R)(1 - (x_1^2 + x_2^2)/4R^2) + x_1 \cos(\alpha/R), x_2)}{1 + \cos(\alpha/R) + (x_1^2 + x_2^2)/4R^2\,(1 - \cos(\alpha/R)) - (x_1/R)\,\sin(\alpha/R)}\,.$$

Note that for all $x = (x_1, x_2) \in B_R$ formally as $R \to \infty$ one recovers $\lim_{R\to\infty} \mathcal{S}_{R,\alpha}(x) = (x_1 + \alpha, x_2) = \mathcal{T}_\alpha(x_1, x_2)$. It follows from the L'Hopital rule

$$\lim_{R\to\infty} 2R\sin(\alpha/R)\Big(1 - \frac{(x_1^2 + x_2^2)}{4R^2}\Big) = \lim_{R\to\infty} 2R\sin(\alpha/R) = 2\alpha\,.$$
$$\lim_{R\to\infty} (1 + \cos(\alpha/R)) + (\frac{x_1^2 + x_2^2}{4R^2})(1 - \cos(\alpha/R)) - \frac{x_1}{R}\sin(\alpha/R) = 2\,.$$

Lemma 6.4.11. *(A) For all $R \in \mathbb{N}_+$ and $\alpha \in (-R, R)$ on the ball B_R one has $\mathcal{T}_{R,\alpha} \circ \jmath_R = \jmath_R \circ \mathcal{S}_{R,\alpha}$.*

(B) For all $R \in \mathbb{N}_+$ and $\alpha \in \mathbb{R}$ it holds $\mathcal{R}_{R,\alpha} \circ \jmath_R = \jmath_R \circ \mathcal{R}_\alpha$.

Proof. Both items can be verified using direct calculations and are similar. Let us prove Item (A). For all $x = (x_1, x_2) \in B_R \subset \mathbb{R}^2$ it holds $\mathcal{T}_{R,\alpha} \circ \jmath_R(x_1, x_2) = (\mathrm{x}'_1, \mathrm{x}'_2, \mathrm{x}'_3)$, where

$$\begin{aligned}
\mathrm{x}'_1 &:= \frac{4R^2 x_1}{4R^2 + x_1^2 + x_2^2} \cos(\alpha/R) - \frac{R(x_1^2 + x_2^2 - 4R^2)}{4R^2 + x_1^2 + x_2^2} \sin(\alpha/R)\,, \\
\mathrm{x}'_2 &:= \frac{4R^2\, x_2}{4R^2 + x_1^2 + x_2^2}\,, \\
\mathrm{x}'_3 &:= \frac{4R^2 x_1}{4R^2 + x_1^2 + x_2^2} \sin(\alpha/R) + \frac{R(x_1^2 + x_2^2 - 4R^2)}{4R^2 + x_1^2 + x_2^2} \cos(\alpha/R)\,.
\end{aligned}$$

Using the map $\jmath_R^{-1}(\mathrm{x}_1, \mathrm{x}_2, \mathrm{x}_3) = 2R(\mathrm{x}_1, \mathrm{x}_2)/(R - \mathrm{x}_3)$ given in Def. 2.1.7 one has

$$\begin{aligned}
R - \mathrm{x}'_3 &= 4R^2\, \frac{R}{4R^2 + x_1^2 + x_2^2} \Big(1 + \cos(\alpha/R) + (\frac{x_1^2 + x_2^2}{4R^2})(1 - \cos(\alpha/R)) - \frac{x_1}{R} \sin(\alpha/R)\Big) \\
2R\,\mathrm{x}'_1 &= 2R\, \frac{4R^2}{4R^2 + x_1^2 + x_2^2} \left(x_1 \cos(\alpha/R) + R \sin(\alpha/R) \Big(1 - \frac{(x_1^2 + x_2^2)}{4R^2}\Big) \right).
\end{aligned}$$

Combining the obtained results one verifies the identity $\mathcal{S}_{R,\alpha}(x) = \jmath_R^{-1} \circ \mathcal{T}_{R,\alpha} \circ \jmath_R(x)$ for all $x \in B_R$. This finishes the proof of item (A). In a similar manner, one could prove Item (B). □

Remark 6.4.12. *The preceding Remark implies for all $R \in \mathbb{N}_+$ and $\alpha \in \mathbb{R}$ it holds $\mathcal{S}^*_{R,\alpha} \circ \jmath^*_R = \jmath^*_R \circ \mathcal{T}^*_{R,\alpha}$ and $\mathcal{R}^*_\alpha \circ \jmath^*_R = \jmath^*_R \circ \mathcal{R}^*_{R,\alpha}$.*

Remark 6.4.13. *Let U, V be two non-empty open sets in $\mathbb{R}^2$ and $F : U \to V$ be a $C^\infty-$ diffeomorphism and $\varphi \in \mathscr{D}'(V)$. We define $\varphi \circ F \in \mathscr{D}'(U)$ by*

$$\langle \varphi \circ F, f \rangle := \int_U \varphi(F(x))\, f(x)\, \mathrm{d}x = \int_V \varphi(y)\, f(F^{-1}(y))\, |\det(\mathrm{T}(F^{-1}))|\, \mathrm{d}y\,,$$

where $\det(\mathrm{T}F^{-1})$ denote the determinant of the Jacobian of the map F^{-1}. The informal expression $\varphi(F(x))$ in the first step serves only to motivate the second step. The expression on the RHS is well defined provided $f(F^{-1}(\bullet))\, |\det(\mathrm{T}(F^{-1}))|$ is a test-function supported in V.

Remark 6.4.14. *As $R \to \infty$ by the property of the determinant function combined with Remark 6.4.5 we expect that $\det(\mathrm{T}\mathcal{S}_{R,\alpha}) = \det(\mathrm{T}(\jmath_R^{-1} \circ \mathcal{T}_{R,\alpha} \circ \jmath_R)) \to 1$. Let $R \in$*

$\mathbb{N}_+$, $\alpha \in (-R, R)$. *For* $f \in C_c^\infty(\mathbb{R}^2)$ *and all* $\phi \in \mathscr{D}'(\mathbb{R}^2)$, $\operatorname{supp}\phi \subset B_R$, *we set* $\mathcal{S}^*_{R,\alpha} f := f \circ \mathcal{S}_{R,\alpha} \in C^\infty(B_R)$ *and* $\mathcal{S}^*_{R,\alpha}\phi = \phi \circ \mathcal{S}^*_{R,-\alpha} \in \mathscr{D}'(\mathbb{R}^2)$. *Utilizing Remark 6.4.13 one has* $\langle \mathcal{S}^*_{R,\alpha}\phi, f\rangle := \langle \phi, \det(\mathrm{T}\mathcal{S}_{R,-\alpha})\, \mathcal{S}^*_{R,-\alpha} f\rangle$ *for all* $f \in C_c^\infty(\mathbb{R}^2)$.

Remark 6.4.15. *Let* $\alpha \in \mathbb{R}$, $a \in \mathbb{N}_0^2$ *and* $M \in (0, \infty)$. *By direct calculations one verifies that there exists* $C \in (0, \infty)$ *such that for all* $x \in B_M$ *and* $R \in (|\alpha| \vee M, \infty)$ *it holds*

$$\mathcal{T}_{R,\alpha} \circ \jmath_R(x) = \jmath_R \circ \mathcal{S}_{R,\alpha}(x) \qquad \text{and} \qquad |\partial^a \mathcal{S}_{R,\alpha}(x) - \partial^a \mathcal{T}_\alpha(x)| \le C/R\,.$$

A succinct proof for the above bound is given in Appendix E.

Remark 6.4.16. *Let* Q *be* $n \times n$, *positive matrix such that* $Q = I + E$, *where* E *is a small perturbation matrix, i.e.,* $\|E\| < 1$. *One has*

$$\mathrm{Tr}\,(\log(I+E)) = \mathrm{Tr}\,(E - \frac{1}{2}E^2 + O(E^3)) = \mathrm{Tr}\,(E) - \frac{1}{2}\mathrm{Tr}\,(E^2) + O\,(\|E\|^3)\,.$$

Using the fact that $\det(Q) = \exp(\mathrm{Tr}\,(\log(Q)))$, *which holds true for any invertible matrix, one has* $\det(Q) = 1 + \mathrm{Tr}(E) + O(\|E\|^2)$, *where we used the fact that for sufficiently small* $x \in (0,1)$ *it holds* $\exp(x) = 1 + x + O(x^2)$. *As a result, one finds* $|\det(Q) - 1| \le |\mathrm{Tr}(E)| + O(\|E\|^2) \le C_n\,\|E\|$. *Generally, it holds* $\det(I + \varepsilon E) = \big(1 + \varepsilon\,\mathrm{Tr}\,(E) + O(\varepsilon^2)\big)$, *where* $\varepsilon > 0$.

Remark 6.4.17. *By definition one has* $\mathcal{S}^*_{R,-\alpha} f(x) - \mathcal{T}^*_{-\alpha} f(x) = f(\mathcal{S}_{R,-\alpha}(x)) - f(\mathcal{T}_{-\alpha}(x))$. *Moreover, from the Taylor expansion one obtains*

$$f(\mathcal{S}_{R,-\alpha}(x)) = f(\mathcal{T}_{-\alpha}(x)) + \vec{\nabla} f(\mathcal{T}_{-\alpha}(x))^{\mathrm{T}} \times (\mathcal{S}_{R,-\alpha}(x) - \mathcal{T}_{-\alpha}(x)) + R(x), \tag{6.4.1}$$

where $R(x)$ *is the higher order remainder, i.e.,*

$$R(x) = \frac{1}{2}(\mathcal{S}_{R,-\alpha}(x) - \mathcal{T}_{-\alpha}(x))^{\mathrm{T}}\, H_f\, (\mathcal{S}_{R,-\alpha}(x) - \mathcal{T}_{-\alpha}(x)),$$

where $(H_f)_{i,j} = \partial^2 f/\partial x_i \partial x_j$ *is the Hessian matrix of* f *evaluated at some point between* $\mathcal{S}_{R,-\alpha}(x)$ *and* $\mathcal{T}_{-\alpha}(x)$. *By Remark 6.4.15, it holds*

$$|R(x)| \le C^2/2R^2 \sup_{x \in B_M} |H_f(x)|,$$
$$|\vec{\nabla} f(\mathcal{T}_{-\alpha}(x))^{\mathrm{T}}\, (\mathcal{S}_{R,-\alpha}(x) - \mathcal{T}_{-\alpha}(x))\,| \le C/R \sup_{x \in B_M} |\vec{\nabla} f(x)|\,.$$

Combining the above results one rewrites Eq. (6.4.1)

$$|f(\mathcal{S}_{R,-\alpha}(x)) - f(\mathcal{T}_{-\alpha}(x))| \le C/R \sup_{x \in B_M} \Big(|\vec{\nabla} f(x)| + C/2R\,|H_f(x)|\Big)\,.$$

Noting that $\det(\mathrm{T}\,\mathcal{T}_{-\alpha}) = 1$*, now from Remark 6.4.16 and the second bound in Remark 6.4.15 with* $a = 1$ *one has* $|\det(\mathrm{T}\mathcal{S}_{R,-\alpha}) - 1)| \le C/R$*. To see this, we use* $\mathrm{T}T_{-\alpha} = 1$ *to write*

$$\det(\mathrm{T}\mathcal{S}_{R,-\alpha}) = \det(1 + \mathrm{T}(\mathcal{S}_{R,-\alpha} - T_{-\alpha}))$$

and note that the tangent map T *amounts to first derivatives of* $\mathcal{S}_{R,-\alpha} - T_{-\alpha}$*. Hence, for all* $\alpha \in \mathbb{R}$ *and* $f \in C_c^\infty(\mathbb{R}^2)$ *one infers that there exists* $C \in (0,\infty)$ *such that for all sufficiently large* $R \in \mathbb{N}_+$ *it holds*

$$\begin{aligned}
&\|\det(\mathrm{T}\mathcal{S}_{R,-\alpha})\mathcal{S}^*_{R,-\alpha} f - \mathcal{T}^*_{-\alpha} f\|_{L^1_2(\mathbb{R}^2, v_L^{-1/2})} \\
&\quad \le \|(\det(\mathrm{T}\mathcal{S}_{R,-\alpha}) - 1) f \circ \mathcal{S}_{R,-\alpha}\|_{L^1_2(\mathbb{R}^2, v_L^{-1/2})} + \|f \circ \mathcal{S}_{R,-\alpha} - f \circ \mathcal{T}_{-\alpha}\|_{L^1_2(\mathbb{R}^2, v_L^{-1/2})} \\
&\le C/R \left(\|f \circ \mathcal{S}_{R,-\alpha}\|_{L^1_2(\mathbb{R}^2, v_L^{-1/2})} + \|\vec{\nabla} f\|\|_{L^1_2(\mathbb{R}^2, v_L^{-1/2})} + (C/2R)\, \|H_f\|_{L^1_2(\mathbb{R}^2, v_L^{-1/2})} \right).
\end{aligned}$$

Hence, for all $\phi \in \mathscr{D}'(\mathbb{R}^2)$*,* $\operatorname{supp}\phi \subset B_R$ *and all* $f \in C_c^\infty(\mathbb{R}^2)$ *one has*

$$\begin{aligned}
|\langle \mathcal{S}^*_{R,\alpha}\phi, f\rangle - \langle \mathcal{T}^*_\alpha \phi, f\rangle| &\le \|\det(\mathrm{T}\mathcal{S}_{R,-\alpha})\mathcal{S}^*_{R,-\alpha} f - \mathcal{T}^*_{-\alpha} f\|_{L^1_2(\mathbb{R}^2, v_L^{-1/2})} \|\phi\|_{L^{-1}_2(\mathbb{R}^2, v_L^{1/2})} \\
&\le C/R\, \|\phi\|_{L^{-1}_2(\mathbb{R}^2, v_L^{1/2})}\,.
\end{aligned}$$

Proposition 6.4.18. *Let* μ *be a weak limit of a subsequence of the sequence of measures* $(\jmath^*_R \sharp \mu_R)_{R\in\mathbb{N}_+}$ *on* $\mathscr{S}'(\mathbb{R}^2)$*. It holds* $\mu(F) = \mu(F \circ \mathcal{R}^*_\alpha)$ *and* $\mu(F) = \mu(F \circ \mathcal{T}^*_\alpha)$ *for all bounded and measurable* $F : \mathscr{S}'(\mathbb{R}^2) \to \mathbb{R}$ *and all* $\alpha \in \mathbb{R}$*.*

Proof. Suppose that the sequence of measures $\jmath^*_R \sharp \mu_R$ on $\mathscr{S}'(\mathbb{R}^2)$ converges to μ along the subsequence $(R_M)_{M\in\mathbb{N}_+}$. By Remarks 5.2.2 and 5.2.4 the measure μ is concentrated on $L^{-1}_2(\mathbb{R}^2, v_L^{1/2})$. For every $F \in \mathcal{F}$ we obtain

$$\begin{aligned}
\mu(F \circ \mathcal{R}^*_\alpha - F) &= \lim_{M\to\infty} \jmath^*_{R_M} \sharp \mu_{R_M}(F \circ \mathcal{R}^*_\alpha - F) = \lim_{M\to\infty} \mu_{R_M}(F \circ \mathcal{R}^*_\alpha \circ \jmath^*_{R_M} - F \circ \jmath^*_{R_M}) \\
&= \lim_{M\to\infty} \mu_{R_M}(F \circ \jmath^*_{R_M} \circ \mathcal{R}^*_{R_M,\alpha} - F \circ \jmath^*_{R_M}) = \lim_{M\to\infty} (\jmath^*_{R_M} \sharp \mu_{R_M} - \mu)(F) = 0.
\end{aligned}$$

It follows from Lemma 6.4.7 and Remark 6.4.12. On account of Remark 6.4.6 one shall infer the statement for an arbitrary functional. This finishes the proof of the rotational invariance.

Let us turn to the proof of the translational invariance. Note that by Remark 6.4.12 for every $F \in \mathcal{F}$ and all sufficiently large $R \in \mathbb{N}_+$ it holds

$$\jmath^*_R \sharp \mu_R(F) = \mu_R(F \circ \jmath^*_R) = \mu_R(F \circ \jmath^*_R \circ \mathcal{T}^*_{R,\alpha}) = \mu_R(F \circ \mathcal{S}^*_{R,\alpha} \circ \jmath^*_R) = \jmath^*_R \sharp \mu_R(F \circ \mathcal{S}^*_{R,\alpha}).$$

Using Remark 6.4.17 one deduces that for every $\alpha \in \mathbb{R}$ and $F \in \mathcal{F}$ there exists $C \in (0,\infty)$ such that for all $\psi \in L_2^{-1}(\mathbb{R}^2, v_L^{1/2})$ and all sufficiently large $R \in \mathbb{N}_+$ it holds

$$|F(\mathcal{S}^*_{R,\alpha}\psi) - F(\mathcal{T}^*_\alpha\psi)| \leq (C/R)\, \|\psi\|_{L_2^{-1}(\mathbb{R}^2, v_L^{1/2})}.$$

By Remark 5.2.4 for all $\alpha \in \mathbb{R}$ and $F \in \mathcal{F}$ one has

$$\lim_{R\to\infty} \jmath^*_R \sharp \mu_R (F \circ \mathcal{S}^*_{R,\alpha} - F \circ \mathcal{T}^*_\alpha) = 0.$$

Consequently,

$$\begin{aligned}\mu(F \circ \mathcal{T}^*_\alpha - F) &= \lim_{M\to\infty} \jmath^*_{R_M} \sharp \mu_{R_M} (F \circ \mathcal{T}^*_\alpha - F) \\ &= \lim_{M\to\infty} \jmath^*_{R_M} \sharp \mu_{R_M} (F \circ \mathcal{S}^*_{R_M,\alpha} - F) = \lim_{M\to\infty} (\jmath^*_{R_M} \sharp \mu_{R_M} - \mu)(F) = 0.\end{aligned}$$

This finishes the proof. □

Lemma 6.4.19. *Let $\Theta_R : \mathscr{D}'(\mathbb{S}_R) \to \mathscr{D}'(\mathbb{S}_R)$, which was defined in Def. 6.3.7. For all $R, N \in [1,\infty)$*

(A) The regularized Gaussian measure $\nu_{R,N}$ is Θ_R- invariant.

(B) The measure μ_R is Θ_R-invariant, i.e., $\mu_R(F) = \mu_R(F \circ \Theta_R)$ for all bounded and measurable $F : \mathscr{D}'(\mathbb{S}_R) \to \mathbb{R}$.

Proof. The proof follows from the same strategy implemented in the proof of the lemma. 6.4.7. To prove Item (A), it suffices to show that for all $R, N \in \mathbb{N}_+$ the covariance $G_{R,N}$ is Θ_R-invariant and refer to the Bochner-Minlos theorem (Thm. 2.6.2). Utilizing Remark 2.4.7 one infers that for all $R, N \in \mathbb{N}_+$ and for all $\mathrm{x}, \mathrm{y} \in \mathbb{S}_R$ it holds $G_{R,N}(\mathrm{x},\mathrm{y}) = G_{R,N}(\Theta_R\mathrm{x}, \Theta_R\mathrm{y})$. Thus, from the preceding relation for all $f, g \in \mathscr{D}(\mathbb{S}_R)$ one gets

$$G_{R,N}(f,g) = \int_{\mathbb{S}_R\times\mathbb{S}_R} G_{R,N}(\mathrm{x},\mathrm{y})\, f(\mathrm{x})\, g(\mathrm{y})\, \rho_R(\mathrm{dx})\, \rho_R(\mathrm{dy}) = G_{R,N}(\Theta_R f, \Theta_R g).$$

It follows from the fact that $\Theta_R^2 = \mathbb{1}$ combined with

$$(\Theta_R \times \Theta_R)(\rho_R(\mathrm{dx}) \otimes \rho_R(\mathrm{dy})) = ((-1)\rho_R(\mathrm{dx})) \otimes ((-1)\rho_R(\mathrm{dy})) = (\rho_R(\mathrm{dx})) \otimes (\rho_R(\mathrm{dy})),$$

where the (-1) reflects the change of the orientation of the integral. This finishes the proof of Item (A). To prove Item (B) recall that for all $R, N \in [1,\infty)$ the measure

$\nu_{R,N}$ is concentrated in $L^1_2(\mathbb{S}_R)$. Thus, by Remark 6.4.6, it suffices to prove that for all cylindrical functionals $F \in \mathcal{F}_{R,N}$ it holds $\mu_{R,N}(F \circ \Theta_R) = \mu_{R,N}(F)$. One obtains

$$\int_{L^1_2(\mathbb{S}_R)} F(\Theta_R \phi)\, \mu_{R,N}(\mathrm{d}\phi) = \frac{1}{\mathcal{Z}_{R,N}} \int_{L^1_2(\mathbb{S}_R)} F(\phi)\, \mathcal{U}_{R,N}(\phi)\, (\Theta_R \sharp \nu_{R,N})(\mathrm{d}\phi)$$
$$= \int_{L^1_2(\mathbb{S}_R)} F(\phi)\, \mu_{R,N}(\mathrm{d}\phi),$$

where we used the fact that $\mathcal{U}_{R,N}(\Theta_R\, \phi) = \mathcal{U}_{R,N}(\phi)$ combined with Item (A), i.e., $(\Theta_R \sharp \nu_{R,N})(A) = \nu_{R,N}(A)$ for all $A \in \mathrm{Borel}(L^1_2(\mathbb{S}_R))$. Now taking the limit $N \to \infty$ we conclude the proof. □

Remark 6.4.20. *Combining Lemma 6.4.7 with Lemma 6.4.19 gives rise to the invariance of the measure μ_R on $\mathscr{D}'(\mathbb{S}_R)$ under the action of the orthogonal group $O(3)$. This follows from a similar justification as the one made below the proof of Prop. 6.4.21, see Remark 2.1.1.*

Proposition 6.4.21. *Let μ be a weak limit of a subsequence of the sequence of measures $(\jmath^*_R \sharp \mu_R)_{R \in \mathbb{N}_+}$ on $\mathscr{S}'(\mathbb{R}^2)$. It holds $\mu(F) = \mu(F \circ \Theta)$ and for all bounded and measurable $F : \mathscr{S}'(\mathbb{R}^2) \to \mathbb{R}$.*

Proof. Suppose that the sequence of measures $\jmath^*_R \sharp \mu_R$ on $\mathscr{S}'(\mathbb{R}^2)$ converges to μ along the subsequence $(R_M)_{M \in \mathbb{N}_+}$. By Remarks 5.2.2 and 5.2.4 the measure μ is concentrated on $L^{-1}_2(\mathbb{R}^2, v^{1/2}_L)$. For every $F \in \mathcal{F}$ we obtain

$$\mu(F \circ \Theta - F) = \lim_{M \to \infty} \jmath^*_{R_M} \sharp \mu_{R_M}(F \circ \Theta - F) = \lim_{M \to \infty} \mu_{R_M}(F \circ \Theta \circ \jmath^*_{R_M} - F \circ \jmath^*_{R_M})$$
$$= \lim_{M \to \infty} \mu_{R_M}(F \circ \jmath^*_{R_M} \circ \Theta_R - F \circ \jmath^*_{R_M}) = \lim_{M \to \infty} (\jmath^*_{R_M} \sharp \mu_{R_M} - \mu)(F) = 0,$$

where in the second line we first used Remark 6.3.9, then Lemma 6.4.19. On account of Remark 6.4.6 one infers the statement for an arbitrary functional. This finishes the proof of the reflection invariance. □

Let us now deduce the invariance of any weak limit point of the sequence of measures $(\jmath^*_R \sharp \mu_R)_{R \in \mathbb{N}_+}$ on $\mathscr{S}'(\mathbb{R}^2)$ under the action of an arbitrary element of $E(2) = T(2) \rtimes O(2)$: On account of Remark 2.1.1 any element $g \in O(2)$ either belongs to $SO(2)$ or satisfies $\det(g) = -1$. In the former case the invariance follows directly from Prop. 6.4.18. In the latter case we write $g = \Theta(\Theta g)$, so that $\Theta g \in SO(2)$. Then, the invariance follows by combining Prop. 6.4.18 with Prop. 6.4.21. The invariance under reflections is a special property of scalar polynomial theories.

Euclidean invariance of the Schwinger functions, as required by the axiom OS3, can now be checked by similar arguments as in the last part of Sec. 6.3.

Appendices

A Bessel potential

The modified Bessel function satisfies [8, Eq. (14.97)]

$$\rho^2 \frac{\mathrm{d}^2}{\mathrm{d}\rho^2} P_m(k\rho) + \rho \frac{\mathrm{d}}{\mathrm{d}\rho} P_m(k\rho) - (k^2\rho^2 + m^2) P_m(k\rho) = 0 \,.$$

Let $k\rho = z$ one gets

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2} + \frac{1}{z} \frac{\mathrm{d}}{\mathrm{d}z} - \frac{m^2}{z^2} - 1 \right) P_m(z) = 0 \,.$$

Indeed, the modified Bessel equation is the eigenequation of $\left(-\frac{\mathrm{d}^2}{\mathrm{d}z^2} - \frac{1}{z}\frac{\mathrm{d}}{\mathrm{d}z} + \frac{m^2}{z^2} \right)$ with eigenvalue -1. Let us restrict ourselves to the case where $\mathrm{Re}(z) > 0$ and $m \in \mathbb{Z}$. The preceding equation has two solutions: first in terms of power series [8, Eq. (14.99)]

$$I_m(z) = \sum_{n=0}^{\infty} \frac{(\frac{z}{2})^{2n+m}}{n!\, \Gamma(m+n+1)}$$

such that $I_m = I_{-m}$ for $m \in \mathbb{Z}$ [8, Eq. (14.101)]. The second solution is given in the integral form, which is absolutely convergent and is known as the second modified Bessel function

$$K_m(z) := \frac{1}{2} \int_0^\infty \exp\left(-\frac{z}{2}(s + s^{-1}) \right) s^{m-1} \, \mathrm{d}s \,.$$

Alternatively, for $\mathrm{Re}(z) > 0$ one has

$$K_m(z) = \frac{1}{2} \left(\frac{z}{2}\right)^m \int_0^\infty s^{-m-1} \exp\left(-s - \frac{z^2}{4s}\right) \mathrm{d}s \,. \tag{A.1}$$

Starting from Eq. (A.1) and using a change of variable $s = \sqrt{\frac{a}{b}}\, t$, one gets

$$\int_0^\infty s^{m-1} \exp\left(-\frac{a}{s} - bs\right) \mathrm{d}s = 2\left(\frac{a}{b}\right)^{\frac{m}{2}} K_m(2\sqrt{ab}) . \tag{A.2}$$

Note that $K_m = K_{-m}$ for $m \in \mathbb{Z}$. It holds [8, Eq. (14.107)]

$$K_m(z) = K_{-m}(z) = \frac{\pi}{2\,\sin(m\pi)}\,(I_{-m}(z) - I_m(z)) .$$

Limiting behaviour

In what follows we write $f(z) \sim g(z)$ as $z \to a$ if $\lim_{z\to a} \frac{f(z)}{g(z)} = 1$. Using the little-o notation one rewrites the preceding expression as $f(z) = g(z)\,(1 + o(1))$. Using this convention, for small z and for $m \neq -1, -2, \cdots$ the limiting behaviour of I_m will be of the form $I_m(z) = z^m\, 2^{-m}\, [\Gamma(m+1)]^{-1}(1 + o(1))$ [8, Eq. (14.100)] and as $z \to \infty$ one has $I_m(z) = \frac{1}{\sqrt{2\pi z}}\, \mathrm{e}^z(1 + o(1))$. Furthermore, one also determines the lowest-order terms of $K_m(z)$ as $z \to 0$ via $K_0(z) = -\log(z) - \gamma + \log(2) + \cdots$ and $K_m(z) = 2^{\nu-1}\Gamma(\nu)\, z^{-m} + \cdots$ [8, Eqs. (14.110–111)]. Using the introduced convention one rewrites the above limiting behaviour

$$K_0(z) = -\log(z)(1 + o(1)), \qquad K_m(z) = 2^{m-1}\Gamma(z)\, z^{-m}(1 + o(1)). \tag{A.3}$$

This indicates that $K_0(z)$ is irregular at $z = 0$. Moreover, one shows that $K_m(z)$ is exponentially decaying, i.e., $K_m(z) = \sqrt{\frac{\pi}{2z}}\, \mathrm{e}^{-z}(1 + o(1))$ [8, Eq. (14.127)]. More precisely, for all $z \in \mathbb{C}$ such that $|\mathrm{arg} z| < \pi - \epsilon$ it holds

$$\lim_{|z|\to\infty} \frac{K_m(z)}{\mathrm{e}^{-z}\sqrt{\frac{\pi}{2z}}} = 1 .$$

Bessel potential using the Fourier transform

Let $(1 - \Delta)^{\alpha/2}\, G(x) = \delta(x)$, where Δ is the Laplacian on $\mathbb{R}^d$. Let $p^2 = \vec{p} \cdot \vec{p}$. Using the Fourier transform one gets

$$G(x) = \mathcal{F}^{-1}\left[\frac{1}{(1 + p^2)^{\alpha/2}}\right](x) = \int \frac{\mathrm{e}^{i\vec{p}\cdot\vec{x}}}{(1 + p^2)^{\alpha/2}}\, \frac{\mathrm{d}p}{(2\pi)^d} .$$

Remark A.1. *It holds that*

$$A^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty \mathrm{e}^{-sA}\, s^{\alpha-1}\, \mathrm{d}s, \qquad \int \mathrm{e}^{-sp^2}\, \mathrm{e}^{i\vec{p}\cdot\vec{x}}\, \frac{\mathrm{d}p}{(2\pi)^d} = \frac{1}{(4\pi\, s)^{d/2}}\, \mathrm{e}^{-\frac{x^2}{4s}} .$$

Lemma A.2. *Let $|x|$ denote the Euclidean norm on $\mathbb{R}^d$ and* $\operatorname{Re}(\alpha) > d/2$*. One has*

$$\int \frac{e^{i\vec{p}\cdot\vec{x}}}{(1+p^2)^{\alpha/2}} \frac{dp}{(2\pi)^d} = \frac{1}{\Gamma(\frac{\alpha}{2})\,\pi^{d/2}\,2^{\frac{d+\alpha}{2}-1}} |x|^{(\alpha-d)/2} K_{(d-\alpha)/2}(|x|)\,.$$

Proof. One combines Remark A.1 with Eqs. (A.2) and (A.1) to conclude. See also [31, Thm. 2.1]. ☐

Remark A.3. *Let G denote a fundamental solution for the positive self-adjoint operator $(1-\Delta)$, where Δ is the Laplacian on $\mathbb{R}^2$, i.e., $(1-\Delta)G(x,x') = \delta(x-x')$. Evoking lemma. A.2 with $d=2$, and $\alpha=2$ and taking into account Eq. (A.3) for small $\|x-x'\|$ one deduces*

$$G(x,x') = -\frac{1}{2\pi}\log\|x-x'\|\,(1+o(1)).$$

Moreover, using the fact that $K_m(z)$ is exponentially decaying as $\|x-x'\| \to \infty$ one has

$$G(x,x') = \frac{e^{-\|x-x'\|}}{\sqrt{8\,\pi\,\|x-x'\|}}\,(1+o(1)).$$

B Stochastic estimates

In this appendix, we work with the probability space $(\mathscr{D}'(\mathbb{S}_R), \mathrm{Borel}(\mathscr{D}'(\mathbb{S}_R), \nu_R)$ and $\mathbb{E}$ stands for the integration with respect to the Gaussian probability measure ν_R with covariance $G_R = (1-\Delta_R)^{-1}$. Moreover, for all $R, N \in \mathbb{N}_+$ we let the stochastic processes $X_R, X_{R,N} : \mathscr{D}'(\mathbb{S}_R) \to L^{-1}_2(\mathbb{S}_R)$.

Lemma B.1. *For every $\kappa \in (0,\infty)$, $N \in \mathbb{N}_+$ there exists $C \in (0,\infty)$ such that for all $R \in \mathbb{N}_+$ it holds*

(A) $\mathbb{E}\|X_{R,N}\|^2_{L^{2-\kappa}_2(\mathbb{S}_R)} \leq R^2\,C^2.$

(B) $\mathbb{E}\|\hat{X}_{R,N}\|^2_{L^{2-\kappa}_2(\mathbb{S}_R)} \leq R^2\,C^2.$

Proof. It suffices to just prove Item (A). By Lemma B.9 the proof of Item (B) is identical to the proof of Item (A). One starts off by writing

$$\begin{aligned}\mathbb{E}\|X_{R,N}\|^2_{L^{2-\kappa}_2(\mathbb{S}_R)} &= \mathbb{E}\|(1-\Delta_R)^{(2-\kappa)/2}X_{R,N}\|^2_{L_2(\mathbb{S}_R)} \\ &= \mathbb{E}\int_{\mathbb{S}_R} ((1-\Delta_R)^{(2-\kappa)/2}X_{R,N})(\mathrm{x})\,\overline{((1-\Delta_R)^{(2-\kappa)/2}X_{R,N})(\mathrm{x})}\,\rho_R(\mathrm{dx})\,.\end{aligned}$$

Utilizing Remark 2.2.4 with the notations introduced in Remark 2.4.11 one gets

$$\mathbb{E}\|X_{R,N}\|^2_{L_2^{2-\kappa}(\mathbb{S}_R)} = \mathbb{E}\sum_{l=0}^{\infty}\sum_{m=-l}^{l} \ll l \gg^{(2-\kappa)} (X_{R,N})_{l,m}\,\overline{(X_{R,N})_{l,m}}\,.$$

From the fact that $X_{R,N}$ is real-valued, the Fubini theorem and Lemma 2.6.7 one has

$$\begin{aligned}\mathbb{E}(X_{R,N})_{l,m}\,\overline{(X_{R,N})_{l,m}} &= \int_{\mathbb{S}_R}\int_{\mathbb{S}_R} \mathbb{E}(X_{R,N}(\mathrm{y})X_{R,N}(\mathrm{z}))\overline{Y_{l,m}}(\mathrm{y})Y_{l,m}(\mathrm{z})\,\rho_R(\mathrm{dy})\rho_R(\mathrm{dz})\\ &= \int_{\mathbb{S}_R}\int_{\mathbb{S}_R} G_{R,N}(\mathrm{y},\mathrm{z})\,\overline{Y_{l,m}}(\mathrm{y})\,Y_{l,m}(\mathrm{z})\,\rho_R(\mathrm{dy})\,\rho_R(\mathrm{dz})\,.\end{aligned}$$

Applying Remarks 2.4.7 and 2.2.3 to the preceding expression and using the notations introduced in Remark 2.4.11 culminate in

$$\begin{aligned}\mathbb{E}\|X_{R,N}\|^2_{L_2^{2-\kappa}(\mathbb{S}_R)} = \frac{1}{(4\pi R^2)^2}\sum_{l=0}^{\infty} \ll l \gg^{(2-\kappa)} (2l+1)\sum_{l_1=0}^{\infty}\langle l_1\rangle^{-1}(2l_1+1)\\ \times\int_{\mathbb{S}_R}\int_{\mathbb{S}_R} P_{l_1}(\frac{\mathrm{y}\cdot\mathrm{z}}{R^2})\,P_l(\frac{\mathrm{y}\cdot\mathrm{z}}{R^2})\,\rho_R(\mathrm{dy})\,\rho_R(\mathrm{dz})\,.\end{aligned}$$

Let $t := (\mathrm{y}\cdot\mathrm{z})/R^2 = \cos\theta$, $\mathrm{d}t = \sin\theta\mathrm{d}\theta$ and recall that $\rho_R(\mathrm{dy}) = 2\pi R^2\sin\theta\,\mathrm{d}\theta$. From rotation invariance property of P_l and Remark 2.2.5 one obtains that

$$\begin{aligned}\int_{\mathbb{S}_R}\int_{\mathbb{S}_R} P_{l_1}(\frac{\mathrm{y}\cdot\mathrm{z}}{R^2})\,P_l(\frac{\mathrm{y}\cdot\mathrm{z}}{R^2})\,\rho_R(\mathrm{dy})\,\rho_R(\mathrm{dz}) &= (4\pi R^2\times 2\pi R^2)\int_{-1}^{1} P_{l_1}(t)P_l(t)\,\mathrm{d}t\\ &= (4\pi R^2\times 2\pi R^2)\frac{2}{2l_1+1}\,\delta_{l_1,l}.\end{aligned}$$

Substituting this to the previous equation and referring to Lemma 2.4.10 for the R^2-dependence of the constants, we conclude the proof. ☐

Lemma B.2. *For every $\kappa\in(0,\infty)$, $\delta\in[0,2]$ there exists $C\in(0,\infty)$ such that for all $R,N\in\mathbb{N}_+$ it holds*

(A) $\mathbb{E}\|X_R\|^2_{L_2^{-\kappa}(\mathbb{S}_R)} \le R^2\,C^2$,

(B) $\mathbb{E}\|X_R - X_{R,N}\|^2_{L_2^{-\kappa-\delta}(\mathbb{S}_R)} \le R^2\,C^2\,N^{-2\delta}$,

(C) $\mathbb{E}\|X_R - \hat{X}_{R,N}\|^2_{L_2^{-\kappa-\delta}(\mathbb{S}_R)} \le R^2\,C^2\,N^{-2\delta}$.

Proof. By definition it holds

$$\begin{aligned}\mathbb{E}\|X_{R,N}\|^2_{L_2^{-\kappa}(\mathbb{S}_R)} &= \sum_{l=0}^{\infty}(1+\frac{l(l+1)}{R^2})^{-\kappa}\,(2l+1)\,(1+\frac{l(l+1)}{R^2})^{-1}(1+\frac{(l(l+1)}{N^2R^2})^{-2}\\ &= \sum_{l=0}^{\infty}(2l+1)\left(1+\frac{l(l+1)}{R^2}\right)^{-1-\kappa}\left(1+\frac{l(l+1)}{R^2N^2}\right)^{-2}\\ &\le \sum_{l=0}^{\infty}(2l+1)\left(1+\frac{l(l+1)}{R^2}\right)^{-1}\left(1+\frac{l(l+1)}{R^2N^2}\right)^{-2}.\end{aligned}$$

Now, Item (A) follows from Lemma 2.4.10. To prove Item (B) we start off by writing

$$\mathbb{E}\|X_R - X_{R,N}\|^2_{L_2^{-\kappa-\delta}(\mathbb{S}_R)} = \mathbb{E}\|(1-\Delta_R)^{\frac{-\kappa-\delta}{2}}(X_R - X_{R,N})\|^2_{L_2(\mathbb{S}_R)}\,.$$

The above expression can be expanded by

$$\mathrm{Tr}\left((1-\Delta_R)^{-1-\kappa-\delta}\right) - 2\,\mathrm{Tr}\left((1-\Delta_R)^{-1-\kappa-\delta}(1-\frac{\Delta_R}{N^2})^{-1}\right) + \mathrm{Tr}\left(1-\Delta_R)^{-1-\kappa-\delta}(1-\frac{\Delta_R}{N^2})^{-2}\right),$$

which is equal to

$$\begin{aligned}&\mathrm{Tr}\Big((1-\Delta_R)^{-1-\kappa-\delta}(1-(1-\frac{\Delta_R}{N^2})^{-1})^2\Big)\\ &\quad= \sum_{l=0}^{\infty}(2l+1)\,\left(1+\frac{l(l+1)}{R^2}\right)^{-\kappa-\delta-1}\left(1-(1+\frac{l(l+1)}{R^2N^2})^{-1}\right)^2.\end{aligned}$$

It follows from the fact that $\mathrm{Tr}(A)+\mathrm{Tr}(B)=\mathrm{Tr}(A+B)$. Observe that for all $\delta\in[0,2]$ it holds

$$\left(1-\left(1+\frac{l(l+1)}{R^2N^2}\right)^{-1}\right)^2 \le N^{-2\delta}\left(\frac{l(l+1)}{R^2}\right)^{\delta}.$$

Hence,

$$\begin{aligned}\mathbb{E}\|X_R - X_{R,N}\|^2_{L_2^{-\kappa-\delta}(\mathbb{S}_R)} &= \mathrm{Tr}\big((1-\Delta_R)^{-1-\kappa-\delta}(1-(1-\Delta_R/N^2)^{-1})^2\big)\\ &\le N^{-2\delta}\,\mathrm{Tr}\big((1-\Delta_R)^{-1-\kappa}\big) = \sum_{l=0}^{\infty}\frac{N^{-2\delta}\,(2l+1)}{(1+l(l+1)/R^2)^{1+\kappa}}.\end{aligned}$$

Now, Item (B) follows from Lemma 2.4.10. Thanks to Lemma B.9 the proof of Item (C) is the same as the proof of Item (B). □

Lemma B.3. *Let $(X_n)_{n\in\mathbb{N}_+}$ be a sequence of Gaussian random variables with values in a Hilbert space H with distributions $\mathcal{N}(m_n, Q_n)$. Assume that $X_n \to X$ in $L_2(\Omega, \mathcal{F}, \mu : H)$. Then X is a Gaussian random variable with mean $m = \lim_{n\to\infty} m_n$ and covariance operator $\langle Qh, h\rangle := \lim_{n\to\infty}\langle Q_n h, h\rangle$.*

Proof. Convergence in L_2 implies convergence in law, hence pointwise convergence of the characteristic functions. Considering that

$$\mathbb{E}\, \mathrm{e}^{i\langle X_n(\omega), f\rangle} = \mathrm{e}^{i\langle m_n, f\rangle - \frac{1}{2}\langle Q_n f, f\rangle}.$$

We observe that $\langle m_n, f\rangle$ and $\langle Q_n f, f\rangle$ must converge for any fixed f and thus determine the mean m and covariance Q of the limiting characteristic function. From its form we see that X is Gaussian. $\square$

Remark B.4. *Combing Lemma B.3 and Lemma B.2 one infers that X_R is a Gaussian random variable with mean zero and covariance operator $\langle G_R h, h\rangle := \lim_{N\to\infty}\langle G_{R,N} h, h\rangle$.*

Remark B.5. *For all $a, b \in \mathbb{R}$ and $p \geq 1$ integer, it holds that*

$$|a^p - b^p| \leq \frac{p}{2}\,|a-b|\,(|a|^{p-1} + |b|^{p-1}).$$

To prove this, we use

$$a^p - b^p = (a-b)\sum_{k=0}^{p-1} a^{p-1-k} b^k$$

and apply the Young inequality for products $|a|^{p-1-k}|b|^k \leq \frac{1}{\tilde{p}}|a|^{(p-1-k)\tilde{p}} + \frac{1}{\tilde{q}}|b|^{k\tilde{q}}$ for $\tilde{p} := \frac{p-1}{p-1-k}$, $\tilde{q} = \frac{p-1}{p-1-k}$. It follows that

$$\sum_{l=1}^{n-1} |a^l - b^l| \leq C\,|a-b|\left[2 + \sum_{l=1}^{n-2}(|a|^l + |b|^l)\right].$$

Lemma B.6. *Let $R \in \mathbb{N}_+$. There exists a real-valued random variable Y_R and $C \in (0,\infty)$ such that for all $N \in \mathbb{N}_+$ it holds*

(A) $\mathbb{E}Y_R^2 \leq C^2$,

(B) $\mathbb{E}(Y_R - Y_{R,N})^2 \leq C^2\, N^{-1/n}$,

(C) $\mathbb{E}(Y_R - \hat{Y}_{R,N})^2 \leq C^2\, N^{-1/n}$,

(D) $\mathbb{E}(\hat{Y}_{R,N} - \tilde{Y}_{R,N})^2 \leq C^2\, N^{-1}$.

Remark B.7. *Recall that $n \in 2\mathbb{N}_+$, $n \geq 4$, is the degree of the polynomial P given in Eq.* (3.0.1) *and the random variables $Y_{R,N}$ and $\hat{Y}_{R,N}, \tilde{Y}_{R,N}$ were introduced in Eq.* (3.4.1) *and Def. 6.3.12, respectively.*

Proof. To prove Items (A) and (B) it is enough to show that for every $m \in \{1, \dots, n\}$ there exists $C \in (0, \infty)$ such that for all $N, M \in \mathbb{N}_+$ it holds

$$\mathbb{E} X_{R,N}^{:m:}(\mathbb{1}_{\mathbb{S}_R})(X_{R,N}^{:m:} - X_{R,M}^{:m:})(\mathbb{1}_{\mathbb{S}_R}) \leq C^2\,(N \wedge M)^{-1}.$$

Let $G_{R,N,M} := K_{R,N} G_R K_{R,M}$. By Lemma 2.6.7

$$\begin{aligned}\mathbb{E} X_{R,N}^{:m:}&(1_{\mathbb{S}_R})(X_{R,N}^{:m:} - X_{R,M}^{:m:})(\mathbb{1}_{\mathbb{S}_R}) \\ &= m! \int_{\mathbb{S}_R} \int_{\mathbb{S}_R} (G_{R,N,N}^m(\mathrm{x},\mathrm{y}) - G_{R,N,M}^m(\mathrm{x},\mathrm{y}))\, \rho_R(\mathrm{dx})\rho_R(\mathrm{dy}).\end{aligned}$$

On account of Remark B.5 there exists $C \in (0, \infty)$ such that it holds

$$|G_{R,N,N}^m - G_{R,N,M}^m| \leq C\, |G_{R,N,N} - G_{R,N,M}|\, \big| |G_{R,N,N}|^{m-1} + |G_{R,N,M}|^{m-1} \big|.$$

For every $m \in \{1, \dots, n\}$ from Hölder's inequality with $1/m + (m-1)/m = 1$, we obtain that there exists $C \in (0, \infty)$ such that for all $N, M \in \mathbb{N}_+$ and all $\mathrm{x} \in \mathbb{S}_R$ it holds

$$\begin{aligned}&|\mathbb{E} X_{R,N}^{:m:}(\mathbb{1}_{\mathbb{S}_R})(X_{R,N}^{:m:} - X_{R,M}^{:m:})(\mathbb{1}_{\mathbb{S}_R})| \\ &\leq C\, \|(G_{R,N,N} - G_{R,N,M})(\bullet,\bullet)\|_{L_m(\mathbb{S}_R^2)} \big(\|G_{R,N,N}(\bullet,\bullet)\|_{L_m(\mathbb{S}_R^2)}^{m-1} + \|G_{R,N,M}(\bullet,\bullet)\|_{L_m(\mathbb{S}_R^2)}^{m-1} \big) \\ &\leq \hat{C}\, \|(G_{R,N,N} - G_{R,N,M})(\bullet,\bullet)\|_{L_n(\mathbb{S}_R^2)} \big(\|G_{R,N,N}(\bullet,\bullet)\|_{L_n(\mathbb{S}_R^2)}^{m-1} + \|G_{R,N,M}(\bullet,\bullet)\|_{L_n(\mathbb{S}_R^2)}^{m-1} \big) \\ &= \check{C}\, \|(G_{R,N,N} - G_{R,N,M})(\mathrm{x},\bullet)\|_{L_n(\mathbb{S}_R)} \big(\|G_{R,N,N}(\mathrm{x},\bullet)\|_{L_n(\mathbb{S}_R)}^{m-1} + \|G_{R,N,M}(\mathrm{x},\bullet)\|_{L_n(\mathbb{S}_R)}^{m-1} \big),\end{aligned}$$

where in the last step above we used the rotational invariance property of $G_{R,N,N}$ and $G_{R,N,M}$. Note that $\hat{C} = (4\pi R^2)^{(2-2m/n)} C$, $\check{C} = (4\pi R^2)^{m/n} \hat{C}$. Evoking the rotational invariance property combined with Lemma 2.3.14 there exist $\hat{C}, C \in (0, \infty)$ such that for all $N \in \mathbb{N}_+$ it holds

$$\begin{aligned}\|(G_{R,N,N} - G_{R,N,M})(\mathrm{x},\bullet)\|_{L_n(\mathbb{S}_R)}^2 &\leq \hat{C}\, \|(G_{R,N,N} - G_{R,N,M})(\mathrm{x},\bullet)\|_{L_2^{(n-2)/n}(\mathbb{S}_R)}^2 \\ &= (4\pi R^2)^{-1}\, \hat{C}\, \big(\mathrm{Tr}\big[G_R^{(n+2)/n} K_{R,N}^2 (K_{R,N} - K_{R,M})^2 \big] \big) \leq C\,(N \wedge M)^{-2/n}.\end{aligned}$$

The last bound follows from the fact that

$$\begin{aligned}
&\operatorname{Tr}\left(G_R^{(n+2)/n}\,\left(K_{R,N}^2\,(K_{R,N}-K_{R,M})^2\right)\right)\\
&\quad\le \sum_{l=0}^\infty \frac{(2l+1)}{(1+l(l+1)/R^2)^{1+2/n}}\left(\frac{1}{1+l(l+1)/(RN)^2}-\frac{1}{1+l(l+1)/(RM)^2}\right)^2\\
&\quad\le \sum_{l=0}^\infty \frac{(2l+1)}{(1+l(l+1)/R^2)^{1+2/n}}\left((1+l(l+1)/R^2)^{1/n}\left(M^{-2/n}+N^{-2/n}\right)\right)\\
&\quad\le 2\,(N\wedge M)^{-2/n}\sum_{l=0}^\infty \frac{(2l+1)}{(1+l(l+1)/R^2)^{1+1/n}}\,.
\end{aligned}$$

The last sum is bounded as $(1+(l(l+1))/(RN)^2)\ge 1$, $n\ge 4$. By an analogous reasoning we obtain

$$\begin{aligned}
\|G_{R,N,N}(\mathrm{x},\bullet)\|^2_{L_n(\mathbb{S}_R)} &\le \hat C\,\|G_{R,N,N}(\mathrm{x},\bullet)\|^2_{L_2^{(n-2)/n}(\mathbb{S}_R)}\\
&= (4\pi R^2)^{-1}\,\hat C\,\big(\operatorname{Tr}\big[G_R^{(n+2)/n}K_{R,N}^4\big]\big).
\end{aligned}$$

Observe that

$$\operatorname{Tr}\left[G_R^{(n+2)/n}K_{R,N}^4\right] \;=\; \sum_{l=0}^\infty \frac{(2l+1)}{(1+l(l+1)/R^2)^{1+2/n}}\left(\frac{1}{1+l(l+1)/(RN)^2}\right)^4 \;\le\; C$$

for some constants $C,\hat C$ independent of N and m. This proves (A) and (B). Thanks to Lemma B.9 the above estimates are also valid when $X_{R,N}$ is replaced with $\hat X_{R,N}$ and $G_{R,N,M}$ is replaced with $\hat G_{R,N,M} := \hat K_{R,N} G_R \hat K_{R,M}$. Hence, (C) follows. To prove Item (D) note that for every $m\in\{1,\ldots,n\}$ there exists $C\in(0,\infty)$ such that for all $N\in\mathbb{N}_+$ and $\mathrm{x}\in\mathbb{S}_R$ it holds

$$\begin{aligned}
\mathbb{E}\hat X^{:m:}_{R,N}(1_{\mathbb{S}_R\setminus\mathbb{S}_{R,N}})\hat X^{:m:}_{R,N}(1_{\mathbb{S}_R\setminus\mathbb{S}_{R,N}}) &\le \|\hat G_{R,N,N}(\bullet,\bullet)\|^m_{L_m(\mathbb{S}_R\setminus\mathbb{S}_{R,N}\times\mathbb{S}_R\setminus\mathbb{S}_{R,N})}\\
&\le \|\hat G_{R,N,N}(\bullet,\bullet)\|^m_{L_m(\mathbb{S}_R\setminus\mathbb{S}_{R,N}\times\mathbb{S}_R)} \le C/N\,\|\hat G_{R,N,N}(\mathrm{x},\bullet)\|^m_{L_m(\mathbb{S}_R)},
\end{aligned}$$

where in the last step we used the rotational invariance of $\hat G_{R,N,N}$ and the fact that the volume of $\mathbb{S}_R\setminus\mathbb{S}_{R,N}$ is bounded by C/N. It holds

$$\begin{aligned}
\|\hat G_{R,N,N}(\mathrm{x},\bullet)\|^2_{L_m(\mathbb{S}_R)} \le \check C\,\|\hat G_{R,N,N}(\mathrm{x},\bullet)\|^2_{L_n(\mathbb{S}_R)} &\le \hat C\,\|\hat G_{R,N,N}(\mathrm{x},\bullet)\|^2_{L_2^{(n-2)/n}(\mathbb{S}_R)}\\
&= (4\pi R^2)^{-1}\,\hat C\,\big(\operatorname{Tr}\big[\hat G_R^{1+2/n}\hat K_{R,N}^4\big]\big) \le C.
\end{aligned}$$

This finishes the proof. □

Lemma B.8. *Let $m \in \mathbb{N}_+$, $p \in [1,\infty)$, $\kappa \in (0,\infty)$ and $L \in [1,\infty)$. There exists $C \in (0,\infty)$ such that for all $R,N \in \mathbb{N}_+$, $R \geq L$, it holds*

(A) $\mathbb{E}\|\jmath_R^* X_{R,N}^{:m:}\|^p_{L_p^{-\kappa}(\mathbb{R}^2,v_L^{1/p})} \leq C$,

(B) $\mathbb{E}\,\|\jmath_R^*(X_R - X_{R,N})\|^p_{L_p^{-\kappa}(\mathbb{R}^2,v_L^{1/p})} \leq C N^{-\kappa}$.

Proof. By Jensen's inequality it suffices to prove the statement for $p \in 2\mathbb{N}_+$. Let $q = (4/\kappa) \vee 4$. There exists $C \in (0,\infty)$ depending on p and κ such that for all $R,N \in \mathbb{N}_+$ it holds

$$\begin{aligned}\mathbb{E}\|\jmath_R^* X_{R,N}^{:m:}\|^p_{L_p^{-\kappa}(\mathbb{R}^2,v_L^{1/p})} &\leq \|v_L w_L^{-1/q}\|_{L_1(\mathbb{R}^2)}\,\|\mathbb{E}((1-\Delta)^{-\kappa/2}\jmath_R^* X_{R,N}^{:m:}(\bullet))^p\|_{L_\infty(\mathbb{R}^2,w_L^{1/q})}\\ &\leq C\,\|\mathbb{E}((1-\Delta)^{-\kappa/2}\jmath_R^* X_{R,N}^{:m:}(\bullet))^2\|^{p/2}_{L_\infty(\mathbb{R}^2,w_L^{1/q})},\end{aligned}$$

where the last bound is a consequence of Lemma 2.6.9. Recall that $\mathbb{E}X_{R,N} \otimes X_{R,N} = G_{R,N}(\bullet,\bullet)$, where $G_{R,N} = K_{R,N} G_R K_{R,N}$. By Lemma 2.6.7

$$\mathbb{E}\jmath_R^* X_{R,N}^{:m:} \otimes \jmath_R^* X_{R,N}^{:m:} = m!\,\tilde{G}_{R,N}^m, \qquad \tilde{G}_{R,N} := (\jmath_R^* \otimes \jmath_R^*) G_{R,N}(\bullet,\bullet)\,.$$

Hence, by Fubini's theorem and explicit formula for the kernel in terms of spherical harmonics

$$\begin{aligned}\mathbb{E}(1-\Delta)^{-\kappa/2}\jmath_R^* X_{R,N}^{:m:} \otimes\ &(1-\Delta)^{-\kappa/2}\jmath_R^* X_{R,N}^{:m:}\\ &= m!\,\big((1-\Delta)^{-\kappa/2} \otimes (1-\Delta)^{-\kappa/2}\big)\,\tilde{G}_{R,N}^m \in C(\mathbb{R}^2 \times \mathbb{R}^2).\end{aligned}$$

Since for $F \in C(\mathbb{R}^2 \times \mathbb{R}^2)$ it holds $\sup_{x\in\mathbb{R}^2} F(x,x) \leq \sup_{y\in\mathbb{R}^2}\sup_{x\in\mathbb{R}^2} F(x,y)$ we obtain

$$\begin{aligned}&\|\mathbb{E}((1-\Delta)^{-\kappa/2}\jmath_R^* X_{R,N}^{:m:}(\bullet))^2\|_{L_\infty(\mathbb{R}^2,w_L^{1/q})}\\ &\quad\leq m!\,\sup_y w_L^{1/q}(y)\,\|((1-\Delta)^{-\kappa/2}\otimes 1)\big(1\otimes(1-\Delta)^{-\kappa/2})\tilde{G}_{R,N}^m\big)(\bullet,y)\|_{L_\infty(\mathbb{R}^2)}\\ &\quad= m!\,\sup_y w_L^{1/q}(y)\,\|\big(1\otimes(1-\Delta)^{-\kappa/2})\tilde{G}_{R,N}^m\big)(\bullet,y)\|_{L_\infty^{-\kappa}(\mathbb{R}^2)}.\end{aligned}$$

By Theorem 2.3.10 (B) there exists $C \in (0,\infty)$ such that for all $R,N \in \mathbb{N}_+$ the above expression is bounded by

$$\begin{aligned}&\sup_{y\in\mathbb{R}^2} w_L^{1/q}(y)\,\|\big(1\otimes(1-\Delta)^{-\kappa/2})\tilde{G}_{R,N}^m\big)(\bullet,y)\|_{L_\infty(\mathbb{R}^2)}\\ &\quad= \sup_{x\in\mathbb{R}^2}\|\big(1\otimes(1-\Delta)^{-\kappa/2})\tilde{G}_{R,N}^m\big)(x,\bullet)\|_{L_\infty(\mathbb{R}^2,w_L^{1/q})}\\ &\quad= \sup_{x\in\mathbb{R}^2}\|\tilde{G}_{R,N}^m(x,\bullet)\|_{L_\infty^{-\kappa}(\mathbb{R}^2,w_L^{1/q})}\end{aligned}$$

up to a multiplicative constant C, which depends on m. The first equality above follows from the fact that for $F \in C(\mathbb{R}^2 \times \mathbb{R}^2)$ it holds $\sup_{x\in\mathbb{R}^2}\sup_{y\in\mathbb{R}^2} F(x,y) = \sup_{y\in\mathbb{R}^2}\sup_{x\in\mathbb{R}^2} F(x,y)$. By Theorem 2.3.10 (B), since $q > 2/\kappa$, the above expression is bounded by

$$\begin{aligned}
\sup_{x\in\mathbb{R}^2} \|\tilde{G}^m_{R,N}(x,\bullet)\|_{L_q(\mathbb{R}^2,w_L^{1/q})} &\leq \sup_{x\in\mathbb{R}^2} \|\tilde{G}^m_{R,N}(x,\bullet)\|_{L_q(\mathbb{R}^2,w_R^{1/q})} = \sup_{\mathrm{x}\in\mathbb{S}_R} \|G_{R,N}(\mathrm{x},\bullet)^m\|_{L_q(\mathbb{S}_R)} \\
&= \sup_{\mathrm{x}\in\mathbb{S}_R} \|G_{R,N}(\mathrm{x},\bullet)\|^m_{L_{mq}(\mathbb{S}_R)} \leq C \sup_{\mathrm{x}\in\mathbb{S}_R} \|G_{R,N}(\mathrm{x},\bullet)\|^m_{L_2^{(mq-2)/mq}(\mathbb{S}_R)} \\
&= C(4\pi R^2)^{-m/2}\,[\mathrm{Tr}(G_{R,N}(1-\Delta_R)^{(mq-2)/mq}G_{R,N})]^{m/2}.
\end{aligned}$$

The first bound above is true because $R \geq L$. The second bound is a consequence of the Sobolev embedding stated in Lemma 2.3.14. It holds

$$\mathrm{Tr}(G_{R,N}(1-\Delta_R)^{(mq-2)/mq}G_{R,N}) = \mathrm{Tr}\left(\frac{1}{(1-\Delta_R)^{1+2/mq}(1-(\Delta_R/N^2))^4}\right).$$

Now, Item (A) follows from Lemma 2.4.10 applied with $N' = 1$ and $\kappa' = 2/mq$. To prove Item (B) we use exactly the same strategy as above with $m = 1$. There exists $C \in (0,\infty)$ depending on p and κ such that for all $R, N \in \mathbb{N}_+$ it holds

$$\begin{aligned}
\mathbb{E}\|\jmath^*_R(X_R - X_{R,N})\|^p_{L_p^{-\kappa}(\mathbb{R}^2,v_L^{1/p})} &= \mathbb{E}\|\jmath^*_R(1-K_{R,N})X_R\|^p_{L_p^{-\kappa}(\mathbb{R}^2,v_L^{1/p})} \\
\leq \|v_L w_L^{-1/q}\|_{L_1(\mathbb{R}^2)}\, &\|\mathbb{E}((1-\Delta)^{-\kappa/2}\jmath^*_R(1-K_{R,N})X_R(\bullet))^p\|_{L_\infty(\mathbb{R}^2,w_L^{1/q})} \\
&\leq C\,\|\mathbb{E}\left((1-\Delta)^{-\kappa/2}\jmath^*_R(1-K_{R,N})X_R(\bullet)\right)^2\|^{p/2}_{L_\infty(\mathbb{R}^2,w_L^{1/q})}.
\end{aligned}$$

where the last bound is a consequence of Lemma 2.6.9. It holds

$$\mathbb{E}\,(\jmath^*_R(1-K_{R,N})X_R \otimes \jmath^*_R(1-K_{R,N})X_R) = (\jmath^*_R \otimes \jmath^*_R)\,\tilde{G}_{R,N}(\bullet,\bullet)\,.$$

Utilizing the result obtained in the previous part of the proof with $m = 1$ and replacing $G_{R,N} \to \tilde{G}_{R,N}$ yields

$$\begin{aligned}
\mathrm{Tr}\left(\tilde{G}_{R,N}(1-\Delta_R)^{1-2/q}\tilde{G}_{R,N}\right) &= \mathrm{Tr}\left(\frac{1}{(1-\Delta_R)^{1+2/q}}\left(\frac{-(\Delta_R/N^2)}{1-(\Delta_R/N^2)}\right)^4\right) \\
&\leq N^{-\kappa}\,\mathrm{Tr}\left(\frac{1}{(1-\Delta_R)^{1+2/q-\kappa/2}}\right).
\end{aligned}$$

The above expression is converging provided that $2/q > \kappa/2$, which amount to the restriction $q < 4/\kappa$. This concludes the proof. □

Lemma B.9. *There exists $C \in (0,\infty)$ such that for all $R, N \in \mathbb{N}_+$ it holds*

$$|\hat{K}_{R,N}| \leq C\,\frac{1}{1-\Delta_R/N^2}, \qquad |1-\hat{K}_{R,N}| \leq C\,\frac{(1-\Delta_R)/N^2}{1-\Delta_R/N^2}\,.$$

Remark B.10. *Observe that for all $R, N \in \mathbb{N}_+$ it holds $1-\Delta_R/N^2 \geq 1$. Given the first bound in the statement of the lemma, for both cases when $\Delta_R/N^2 \leq 1$ and $\Delta_R/N^2 \geq 1$, there is $C \in (0,\infty)$ such that the second bound holds.*

Proof. Using Remark 2.4.6 one has $\hat{K}_{R,N} = \sum_{l=0}^{\infty}(2l+1)\mathrm{Tr}(\hat{K}_{R,N}\mathcal{P}_{R,l})\,\mathcal{P}_{R,l}$. Consequently, by the triangle inequality for the commuting self-adjoint operators it is enough to show that there exists $C \in (0,\infty)$ such that for all $R, N \in \mathbb{N}_+$ and $l \in \mathbb{N}_0$ it holds

$$\begin{gathered}(1+l(l+1)/R^2N^2)\,|\mathrm{Tr}(\hat{K}_{R,N}\mathcal{P}_{R,l})| \leq C,\\ |\mathrm{Tr}((1-\hat{K}_{R,N})\mathcal{P}_{R,l})| \leq C\,(1+l(l+1))/R^2N^2\,,\end{gathered} \tag{B.1}$$

where we used Remark 2.4.7. Using Remark 6.3.4 one has

$$\begin{aligned}\mathrm{Tr}(\hat{K}_{R,N}\mathcal{P}_{R,l}) &= \frac{1}{4\pi R^2}\int_{\mathbb{S}_R}\int_{\mathbb{S}_R} N^2\,h(NR\,\theta(\mathrm{x},\mathrm{y}))\,P_l(\mathrm{x}\cdot\mathrm{y}/R^2)\,\rho_R(\mathrm{d}\mathrm{x})\,\rho_R(\mathrm{d}\mathrm{y})\\ &= 2\pi\int_0^{\pi}(NR)^2\,h(RN\theta)\,P_l(\cos\theta)\,\sin\theta\,\mathrm{d}\theta\\ &= 2\pi\int_0^{1}h(\theta)\,P_l(\cos(\theta/RN))\,RN\,\sin(\theta/RN)\,\mathrm{d}\theta\,,\end{aligned}$$

where we used a change of variable $\theta' = NR\,\theta$, the support property of the function h and the rotation invariance property of the whole expression. Observe that $RN\sin(\theta/RN) \leq \theta$ and from Eq. (2.2.1) one has $|P_l(\cos(\theta/RN))| \leq 1$. This easily verifies that $|\mathrm{Tr}(\hat{K}_{R,N}\mathcal{P}_{R,l})| \leq C$. From integration by parts and the support property of the function h, one infers that there exists $C' \in (0,\infty)$ such that it holds

$$\int_0^{\pi} h(RN\theta)\Big|\frac{\mathrm{d}}{\mathrm{d}\theta}\,(\cos\theta\,P_l(\cos\theta)) - \frac{\mathrm{d}^2}{\mathrm{d}\theta^2}\,(\sin\theta\,P_l(\cos\theta))\,\Big|\,\mathrm{d}\theta < C'\,.$$

The preceding bound combined with Eq. (2.2.2) implies that

$$(2\pi)\int_0^{\pi}(NR)^2\,h(RN\theta)\,P_l(\cos\theta)\,\sin\theta\,\mathrm{d}\theta \leq C\,\frac{(2\pi)(NR)^2}{l(l+1)}\,.$$

This concludes the proof of the first bound in Eq. (B.1). Next, using that $2\pi\int\theta\,|h(\theta)|\mathrm{d}\theta = 1$, we obtain that

$$|\mathrm{Tr}((1-\hat{K}_{R,N})\mathcal{P}_{R,l})| = 2\pi\int_0^{1}\Big(\theta - RN\,\sin(\theta/RN)\,P_l(\cos(\theta/RN))\Big)\,h(\theta)\,\mathrm{d}\theta\,.$$

One rewrites the RHS of the above expression

$$2\pi \int_0^1 \Big(1 - \; RN/\theta \, \sin(\theta/RN)\big[-1 + 1 - P_l(\cos(\theta/RN))\big]\Big)\, \theta \, h(\theta) \, \mathrm{d}\theta \, .$$

Using Remark 2.2.2 one obtains

$$0 \le 1 - P_l(\cos(\theta/NR)) \le l(l+1)\,(1-\cos(\theta/NR))/2 \le l(l+1)\,(\theta/NR)^2/4.$$

Combining the above bound with the elementary estimate $0 \le 1 - \sin(\theta/NR)/(\theta/NR) \le (\theta/NR)^2/6$ and the support property of the function h, one verifies the second bound in (B.1) and finishes the proof. □

C Proof of Lemma 4.3.9

From Thm 4.3.5 a mild $L^1_2(\mathbb{S}_R)$−valued solution of Eq. (4.3.3) on $[0,T]$ is a function $\varPsi^g_{R,N} \in C([0,T], L^1_2(\mathbb{S}_R))$ such that almost surely it holds

$$\varPsi^g_{R,N}(t,\bullet) = \mathrm{e}^{-t\,Q_{R,N}}\,\psi^g_{R,N} + \int_0^t \mathrm{e}^{-(t-s)\,Q_{R,N}}\,F(\varPsi^g_{R,N}(s,\bullet))\,\mathrm{d}s \quad \text{for all} \quad 0 \le t \le T\,,$$

where $\psi^g_{R,N} \in L^1_2(\mathbb{S}_R)$ and $F(\varPsi^g_{R,N}(s,\bullet)) = P'((\varPsi^g_{R,N} + Z_{R,N})(s,\bullet), c_{R,N}) + ((\varPsi^g_{R,N} + Z_{R,N})(s,g))^{n-1} g(\bullet)$ for all $R, N \in \mathbb{N}_+$. Note that $F : L^1_2(\mathbb{S}_R) \to L_2(\mathbb{S}_R)$ and $F : L^3_2(\mathbb{S}_R) \to L_2(\mathbb{S}_R)$ are locally Lipschitz. The latter fact follows from calculations similar to those in the discussion of Eqs. (4.3.8)–(4.3.10) with the help of Remark 2.4.15. Moreover, Theorem 4.3.8 implies that $\varPsi^g_{R,N} \in C([0,\infty), L^1_2(\mathbb{S}_R))$ is the global mild solution to Eq. (4.3.3). To verify the statement, we shall first show that $\varPsi^g_{R,N} \in C((0,\infty), L^3_2(\mathbb{S}_R))$, which implies $\varPsi^g_{R,N} \in C((0,\infty), L^{-3}_2(\mathbb{S}_R))$. Observe that at $t = 0$ we cannot have a better regularity than the one of the initial condition. Using the fact that for all $N \in \mathbb{N}_+$ there are some $\tilde{c}, c \in (0,\infty)$ such that $c\,Q^{1/3}_{R,N} \le (1-\Delta_R) \le \tilde{c}\,Q^{1/3}_{R,N}$ one gets

$$\|\mathrm{e}^{-tQ_{R,N}}\,\psi^g_{R,N}\|_{L^3_2(\mathbb{S}_R)} \le \tilde{c}\,\|Q^{1/2}_{R,N}\,\mathrm{e}^{-tQ_{R,N}}\,\psi^g_{R,N}\|_{L^1_2(\mathbb{S}_R)} \le C\,\|\psi^g_{R,N}\|_{L^1_2(\mathbb{S}_R)}\,,$$

where $C \in (0,\infty)$. The last bound follows from the fact that for all $t > 0$, the map $x \mapsto x^s\,\mathrm{e}^{-tx}$ is uniformly bounded for all $x > 0$ by C'/t^s for some $C' \in (0,\infty)$. Note that for all $N \in \mathbb{N}_+$ there exists some $C \in (0,\infty)$ such that it holds

$$\|\int_0^t \mathrm{e}^{-(t-s)Q_{R,N}}\,F(s,\bullet)\,\mathrm{d}s\|_{L^3_2(\mathbb{S}_R)} \le C\,\|Q^{1/2}_{R,N}\int_0^t \mathrm{e}^{-(t-s)Q_{R,N}}\,F(s,\bullet)\,\mathrm{d}s\|_{L_2(\mathbb{S}_R)},$$

where we wrote $F(s,\bullet) := F(\Psi^g_{R,N}(s,\bullet))$ for brevity. Testing with $\phi \in \mathrm{Dom}(Q^{1/2}_{R,N})$ and using the Cauchy-Schwarz inequality one infers that the right hand side of the above expression can be bounded by

$$\begin{aligned}
&\|Q^{1/2}_{R,N}\int_0^t \mathrm{e}^{-(t-s)\,Q_{R,N}}\,F(s,\bullet)\,\mathrm{d}s\|_{L_2(\mathbb{S}_R)}\\
&\qquad= \sup_{\|\phi\|_{L_2(\mathbb{S}_R)}\le 1} |\,\langle \phi, Q^{1/2}_{R,N}\int_0^t \mathrm{e}^{-(t-s)\,Q_{R,N}}\,F(s,\bullet)\,\mathrm{d}s\rangle_{L_2(\mathbb{S}_R)}\,|\\
&\qquad= \sup_{\|\phi\|_{L_2(\mathbb{S}_R)}\le 1} |\int_0^t \langle\, Q^{1/2}_{R,N}\,\mathrm{e}^{-(t-s)\,Q_{R,N}}\phi,\ F(s,\bullet)\,\rangle_{L_2(\mathbb{S}_R)}\,\mathrm{d}s|\\
&\quad= \sup_{\|\phi\|_{L_2(\mathbb{S}_R)}\le 1} \Big[\int_0^t \|Q^{1/2}_{R,N}\,\mathrm{e}^{-(t-s)\,Q_{R,N}}\phi\,\|_{L_2(\mathbb{S}_R)}\,\mathrm{d}s\Big]\sup_{r\in[0,t]}\|F(r,\bullet)\|_{L_2(\mathbb{S}_R)}\\
&\qquad\qquad\le C'\sup_{\|\phi\|_{L_2(\mathbb{S}_R)}\le 1}\Big[\langle Q^{1/2}_{R,N}\,\phi, \frac{\mathrm{e}^{-t\,Q_{R,N}}-1}{Q_{R,N}}\,Q^{1/2}_{R,N}\,\phi\rangle\Big]^{1/2}\le C''\,,
\end{aligned}$$

where we used the Fubini theorem in the third line and $C'' \in (0,\infty)$ is some constant. Observe that $\sup_{r\in[0,t]}\|F(r,\bullet)\|_{L_2(\mathbb{S}_R)}$ is bounded. This verifies that for all $t > 0$, the solution $\Psi^g_{R,N}(t,\bullet)$ takes values in $L^3_2(\mathbb{S}_R)$. To proceed, we show that $\Psi^g_{R,N} \in C((0,\infty), L^3_2(\mathbb{S}_R))$. To this end, fix $t > 0$ and observe that the limit

$$\lim_{\Delta t\to 0}\|\mathrm{e}^{-t\,Q_{R,N}}(\mathrm{e}^{-\Delta t\,Q_{R,N}}-1)\,\psi^g_{R,N}\|_{L^3_2(\mathbb{S}_R)} = 0\,.$$

It follows from the semigroup property, the strong continuity of the semigroup $\mathrm{e}^{-t\,Q_{R,N}}$ and the fact that $\|Q^{1/2}_{R,N}\,\mathrm{e}^{-tQ_{R,N}}\|_{L_2(\mathbb{S}_R)\to L_2(\mathbb{S}_R)}$ is bounded. Now we aim to show that

$$\begin{aligned}
\lim_{\Delta t\to 0}\|\int_0^t \mathrm{e}^{-(t-s)\,Q_{R,N}}\,(\mathrm{e}^{-\Delta t\,Q_{R,N}}-1)\,F(s,\bullet)\,\mathrm{d}s&\\
+\int_t^{t+\Delta t}\mathrm{e}^{-(t+\Delta t-s)\,Q_{R,N}}\,F(s,\bullet)\,\mathrm{d}s\|_{L^3_2(\mathbb{S}_R)}& \qquad \text{(C.1)}
\end{aligned}$$

is zero. To verify this claim observe that for $s \neq t$

$$\|Q^{1/2}_{R,N}\,\mathrm{e}^{-(t-s)\,Q_{R,N}}\,(\mathrm{e}^{-\Delta t\,Q_{R,N}}-1)\,F(s,\bullet)\|_{L_2(\mathbb{S}_R)} \le \frac{C}{(t-s)^{1/2}} \qquad \text{(C.2)}$$

is integrable, where $C \in (0,\infty)$. Moreover, for any fixed $s \neq t$ the following limit

$$\lim_{\Delta t\to 0}\|Q^{1/2}_{R,N}\,\mathrm{e}^{-(t-s)\,Q_{R,N}}\,(\mathrm{e}^{-\Delta t\,Q_{R,N}}-1)\,F(s,\bullet)\|_{L_2(\mathbb{S}_R)} \qquad \text{(C.3)}$$

is zero, which gives the pointwise convergence of the integral. Combining Eqs. (C.2) and (C.3) from the dominated convergence theorem one shows that the first integral in Eq. (C.1) tends to zero. Now consider the second integral in Eq. (C.1). One has

$$\| \int_t^{t+\Delta t} \mathrm{e}^{-(t+\Delta t-s)\,Q_{R,N}}\, F(s,\bullet)\,\mathrm{d}s\|_{L_2^3(\mathbb{S}_R)}$$
$$= \| \int_0^\infty \chi_{[0,\Delta t]}(s)\,\mathrm{e}^{-s\,Q_{R,N}}\, F(t+\Delta t-s,\bullet)\,\mathrm{d}s\|_{L_2^3(\mathbb{S}_R)}\,.$$

Note that

$$\|\chi_{[0,\Delta t]}(s)\,Q_{R,N}^{1/2}\,\mathrm{e}^{-s\,Q_{R,N}}\,F(t+\Delta t-s,\bullet)\|_{L_2(\mathbb{S}_R)} \le C\,\chi_{[0,1]}(s)\,s^{-1/2}. \tag{C.4}$$

Furthermore, one has

$$\lim_{\Delta t\to 0} \|\chi_{[0,\Delta t]}(s)\,Q_{R,N}^{1/2}\,\mathrm{e}^{-s\,Q_{R,N}}\,F(t+\Delta t-s,\bullet)\|_{L_2(\mathbb{S}_R)} = 0\,. \tag{C.5}$$

Combining Eqs. (C.4) and (C.5) from the dominated convergence theorem one shows that the second integral in Eq. (C.1) is zero. This finishes the proof that $\Psi^g_{R,N} \in C((0,\infty), L_2^3(\mathbb{S}_R)$. To proceed, we shall show that $\Psi^g_{R,N}(t,\bullet) \in C^1((0,\infty), L_2^{-3}(\mathbb{S}_R))$. Using the fact that the desired derivative of $\mathrm{e}^{-t\,Q_{R,N}}\,\psi^g_{R,N}$ is $-Q_{R,N}\,\mathrm{e}^{-t\,Q_{R,N}}\,\psi^g_{R,N}$, we aim to show that

$$\lim_{\Delta t\to 0} \|\Big(\frac{(\mathrm{e}^{-\Delta t\,Q_{R,N}}-1)}{\Delta t} + Q_{R,N}\Big)\,\mathrm{e}^{-t\,Q_{R,N}}\,\psi^g_{R,N}\|_{L_2^{-3}(\mathbb{S}_R)} = 0.$$

One rewrites the above expression

$$(\Delta t) \int_0^1 \|Q_{R,N}^{-1/2}\,Q_{R,N}^2\,\mathrm{e}^{-t\,Q_{R,N}}\,\mathrm{e}^{-s\,\Delta t}(1-s)\,\psi^g_{R,N}\,\mathrm{d}s\|_{L_2(\mathbb{S}_R)}\,, \tag{C.6}$$

where we used Remark 2.4.5 with $n=1$ to have

$$\mathrm{e}^{-\Delta t\,Q_{R,N}} = 1 - \Delta t Q_{R,N} + Q_{R,N}^2(\Delta t)^2 \int_0^1 \mathrm{e}^{-s\,\Delta t\,Q_{R,N}}(1-s)\,\mathrm{d}s\,.$$

Observe that due to the positivity of the spectrum of the elliptic operator $Q_{R,N}$ there is $C\in(0,\infty)$ independent of Δt such that

$$\| \int_0^1 \mathrm{e}^{-s\,\Delta t Q_{R,N}}(1-s)\,\mathrm{d}s\|_{L_2(\mathbb{S}_R)\to L_2(\mathbb{S}_R)} \le |\int_0^1 (1-s)\,\mathrm{d}s| \le C\,. \tag{C.7}$$

Note that $\|Q_{R,N}^{-1/2}\, Q_{R,N}^2\, \mathrm{e}^{-tQ_{R,N}}\|_{L_2(\mathbb{S}_R)\to L_2(\mathbb{S}_R)}$ is bounded. Thus, on account of Eq. (C.7) and the dominated convergence one concludes that Eq. (C.6) tends to zero as $\Delta t \to 0$, i.e., $\psi_{R,N}^g \in C^1((0,\infty), L_2^{-3}(\mathbb{S}_R))$. Now we need to verify that

$$\lim_{\Delta t\to 0} \|\frac{1}{\Delta t} \int_0^t \mathrm{e}^{-(t-s)\,Q_{R,N}} \left(\mathrm{e}^{-\Delta\, t\, Q_{R,N}} - 1\right) F(s,\bullet)\,\mathrm{d}s + Q_{R,N} \int_0^t \mathrm{e}^{-(t-s)\,Q_{R,N}} F(s,\bullet)\,\mathrm{d}s\|_{L_2^{-3}(\mathbb{S}_R)}$$

$$\lim_{\Delta t\to 0} \|\frac{1}{\Delta t} \int_t^{t+\Delta t} \mathrm{e}^{-(t+\Delta t-s)\,Q_{R,N}}\, F(s,\bullet)\,\mathrm{d}s - F(t,\bullet)\|_{L_2^{-3}(\mathbb{S}_R)} = 0\,. \tag{C.8}$$

Consider the first integral in Eq. (C.8) and note that $\|Q_{R,N}^{-1/2}\, Q_{R,N}\, \mathrm{e}^{-(t-s)\,Q_{R,N}}\|_{L_2(\mathbb{S}_R)\to L_2(\mathbb{S}_R)}$ and $\|\int_0^1 \mathrm{e}^{-s\,\Delta t Q_{R,N}}\,\mathrm{d}s\|_{L_2(\mathbb{S}_R)\to L_2(\mathbb{S}_R)}$ are bounded. One shows that point-wise

$$\|\, Q_{R,N}^{-1/2}\, Q_{R,N} \int_0^t \mathrm{e}^{-(t-s)\,Q_{R,N}} \Big(1 - \int_0^1 \mathrm{e}^{-s\,\Delta\, t\, Q_{R,N}}\,\mathrm{d}s\Big)\, F(s,\bullet)\|_{L_2(\mathbb{S}_R)}\,,$$

converges zero as $\Delta t \to 0$. It follows from Remark 2.4.5 with $n = 0$

$$\mathrm{e}^{-\Delta t\, Q_{R,N}} - 1 = -Q_{R,N}\Delta t \int_0^1 \mathrm{e}^{-s\Delta t\, Q_{R,N}}\,\mathrm{d}s\,.$$

Hence, the first integral in Eq. (C.8) tends to zero as $\Delta t \to 0$ by the dominated convergence theorem. Now consider the second integral in Eq. (C.8). One has

$$\begin{aligned}
&\|\frac{1}{\Delta t} \int_t^{t+\Delta t} \mathrm{e}^{-(t+\Delta t-s)\,Q_{R,N}}\, F(s,\bullet)\mathrm{d}s - F(t,\bullet)\|_{L_2^{-3}(\mathbb{S}_R)} \\
&\qquad = \|\frac{1}{\Delta t} \int_0^\infty \chi_{[0,\Delta t]}(s)\, \mathrm{e}^{-s\, Q_{R,N}}\, F(t+\Delta t - s,\bullet)\,\mathrm{d}s - F(t,\bullet)\|_{L_2^{-3}(\mathbb{S}_R)}.
\end{aligned}$$

Applying the dominated convergence theorem as we did above and using continuity of $t \mapsto F(t,\bullet)$ one shows that the above limit is zero. This verifies that $\Psi_{R,N}^g \in C^1((0,\infty), L_2^{-3}(\mathbb{S}_R))$ and concludes the proof.

D Proof of Lemma 5.1.3

Notice that in what follows from line to line C might be different. There exists $C \in (0,\infty)$ such that $|\vec{\nabla}(w_R^{-1})| \le (C/L)\, w_R^{-1}, |\vec{\nabla}(v_L)| \le (C/L)\, v_L$. Using the product rule one gets $|\vec{\nabla}(w_R^{-1} v_L)| \le (C/L)\,|w_R^{-1} v_L|$. Similarly, $|\Delta(w_R^{-1} v_L)| = |(\Delta w_R^{-1})\, v_L + w_R^{-1}\,(\Delta v_L) + 2(\vec{\nabla} w_R^{-1})\cdot(\vec{\nabla} v_L)| \le (C/L)^2\, |w_R^{-1} v_L|$. Consider Item (A) of Lemma 5.1.3.

Using the integration by parts one gets

$$
\begin{aligned}
-\langle \Psi, v_L(w_R^{-1}\Delta)\Psi\rangle_{L_2(\mathbb{R}^2)} &= \langle \vec{\nabla}\Psi(v_L w^{-1}), \nabla\Psi\rangle + \langle \Psi\vec{\nabla}(v_L w^{-1}), \vec{\nabla}\Psi\rangle \\
&\geq \langle \vec{\nabla}\Psi, (v_L w^{-1})\nabla\Psi\rangle - |\langle \Psi, \vec{\nabla}(v_L w^{-1})\vec{\nabla}\Psi\rangle| \\
&\geq \|\vec{\nabla}\Psi\|^2_{L_2(w_R^{-1/2}v_L^{1/2})} - \frac{C}{L}|\langle \Psi, (v_L w^{-1})\vec{\nabla}\Psi\rangle| \,.
\end{aligned}
$$

Applying the Hölder and the Young inequalities yields

$$
-\langle \Psi, v_L(w_R^{-1}\Delta)\Psi\rangle_{L_2(\mathbb{R}^2)} \;\geq\; (1-\frac{C}{2L})\,\|\vec{\nabla}\Psi\|^2_{L_2(w_R^{-1/2}v_L^{1/2})} - \frac{C}{2L}\|\Psi\|^2_{L_2(w_R^{-1/2}v_L^{1/2})} \,.
$$

This finishes the proof of Item (A). To prove Item (B) note that

$$
\begin{aligned}
&\langle \Psi, v_L(-w_R^{-1}\Delta)^2\Psi\rangle = \langle \Psi, (v_L w_R^{-1})\Delta w_R^{-1}\Delta\Psi\rangle \\
&= \langle \Delta\Psi(v_L w_R^{-1}), w^{-1}\Delta\Psi\rangle + \langle \Psi, \Delta(v_L w_R^{-1}), w^{-1}\Delta\Psi\rangle + 2\langle \vec{\nabla}\Psi\cdot\vec{\nabla}(v_L w_R^{-1}), w^{-1}\Delta\Psi\rangle \\
&\geq \langle \Delta\Psi(v_L w_R^{-1}), w^{-1}\Delta\Psi\rangle - |\langle \Psi, \Delta(v_L w_R^{-1}), w^{-1}\Delta\Psi\rangle| - 2|\,\langle \vec{\nabla}\Psi\cdot\vec{\nabla}(v_L w_R^{-1}), w^{-1}\Delta\Psi\rangle| \\
&\qquad \geq \|\Delta\Psi\|^2_{L_2(w_R^{-1}v_L^{1/2})} - \frac{C^2}{2L^2}\|\Psi\|^2_{L_2(w_R^{-1}v_L^{1/2})} - \frac{C^2}{2L^2}\|\Delta\Psi\|_{L_2(w_R^{-1}v_L^{1/2})} \\
&\qquad\qquad - \frac{C}{L}\|\vec{\nabla}\Psi\|^2_{L_2(w_R^{-1}v_L^{1/2})} - \frac{C}{L}\|\Delta\Psi\|^2_{L_2(w_R^{-1}v_L^{1/2})} \,.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\langle \Psi, v_L(-w_R^{-1}\Delta)^2\Psi\rangle &\geq (1-\frac{C}{L}-\frac{C^2}{2L^2})\|\Delta\Psi\|^2_{L_2(w_R^{-1}v_L^{1/2})} \\
&\qquad - \frac{C}{L}\,\|\vec{\nabla}\Psi\|^2_{L_2(w_R^{-1}v_L^{1/2})} - \frac{C^2}{2L^2}\,\|\Psi\|^2_{L_2(w_R^{-1}v_L^{1/2})} \,. \qquad \text{(D.1)}
\end{aligned}
$$

Observe that from the Hölder and the Young inequalities one has

$$
\begin{aligned}
&\|\vec{\nabla}\Psi\|^2_{L_2(w_R^{-1}v_L^{1/2})} = \|w_R^{-1}v_L^{1/2}\vec{\nabla}\Psi\|^2_{L_2} = \sum_{j=1,2}\langle \partial_j\Psi, w_R^{-2}v_L\partial_j\Psi\rangle \\
&\qquad \leq |\sum_{j=1,2}\langle \Psi, (-\partial_j)(w_R^{-2}v_L)\,\partial_j\Psi\rangle| + |\langle \Psi, (w_R^{-2}v_L)(-\Delta)\Psi\rangle| \\
&\quad \leq \frac{C}{L}\|\Psi\|_{L_2(w_R^{-1}v_L^{1/2})}\|\vec{\nabla}\Psi\|_{L_2(w_R^{-1}v_L^{1/2})} + \|\Psi\|_{L_2(w_R^{-1}v_L^{1/2})}\|\Delta\Psi\|_{L_2(w_R^{-1}v_L^{1/2})} \\
&\leq \frac{C}{2L}\Big[\|\Psi\|^2_{L_2(w_R^{-1}v_L^{1/2})} + \|\vec{\nabla}\Psi\|^2_{L_2(w_R^{-1}v_L^{1/2})}\Big] + \frac{1}{2}\Big[\|\Psi\|^2_{L_2(w_R^{-1}v_L^{1/2})} + \|\Delta\Psi\|^2_{L_2(w_R^{-1}v_L^{1/2})}\Big] \,.
\end{aligned}
$$

In particular, it holds

$$(1-\frac{C}{2L})\|\vec{\nabla}\Psi\|^2_{L_2(w_R^{-1}v_L^{1/2})} \le (\frac{1}{2}+\frac{C}{2L})\,\|\Psi\|^2_{L_2(w_R^{-1}v_L^{1/2})} + 1/2\,\|\Delta\Psi\|^2_{L_2(w_R^{-1}v_L^{1/2})}\,.$$

Consequently,

$$\|\vec{\nabla}\Psi\|^2_{L_2(w_R^{-1}v_L^{1/2})} \le \frac{(\frac{1}{2}+\frac{C}{2L})}{(1-\frac{C}{2L})}\,\|\Psi\|^2_{L_2(w_R^{-1}v_L^{1/2})} + \frac{1/2}{(1-\frac{C}{2L})}\,\|\Delta\Psi\|^2_{L_2(w_R^{-1}v_L^{1/2})}$$

provided that $(1-\frac{C}{2L})>0$. Therefore, one can rewrite Eq. (D.1) as follows

$$\begin{aligned}\langle\Psi, v_L(-w_R^{-1}\Delta)^2\Psi\rangle \ge (1-\frac{C}{L}-\frac{C^2}{2L^2}-\frac{C}{L}\,\frac{1/2}{(1-\frac{C}{2L})})\|\Delta\Psi\|^2_{L_2(w_R^{-1}v_L^{1/2})}\\ -(\frac{C}{L}\,\frac{(\frac{1}{2}+\frac{C}{2L})}{(1-\frac{C}{2L})}+\frac{C^2}{2L^2})\,\|\Psi\|^2_{L_2(w_R^{-1}v_L^{1/2})}\,.\end{aligned}$$

This finishes the proof of Item (B). To prove Item (C) consider

$$\begin{aligned}&\langle\Psi, v_L(-w_R^{-1}\Delta)^3\Psi\rangle = -\langle\Delta\Psi(v_Lw_R^{-1}), w_R^{-1}(\Delta w_R^{-1})\Delta\Psi\rangle\\ &\quad -\langle\Psi\Delta(v_Lw_R^{-1}), w_R^{-1}(\Delta w_R^{-1})\Delta\Psi\rangle - 2\langle\vec{\nabla}\Psi\cdot\vec{\nabla}(v_Lw_R^{-1}), w_R^{-1}(\Delta w_R^{-1})\Delta\Psi\rangle\,.\end{aligned}\tag{D.2}$$

Consider the first term in the RHS of Eq. (D.2). Using integration by parts one has

$$\begin{aligned}-\langle\Delta\Psi(v_Lw_R^{-1})w_R^{-1}, \Delta w_R^{-1}\,\Delta\Psi\rangle &= \langle\vec{\nabla}\Delta\Psi(v_Lw_R^{-1})w_R^{-1}, \vec{\nabla}w_R^{-1}\Delta\Psi\rangle\\ &+\langle\Delta\Psi\vec{\nabla}(v_Lw_R^{-1})w_R^{-1}, \vec{\nabla}w_R^{-1}\Delta\Psi\rangle\\ &+\langle\Delta\Psi(v_Lw_R^{-1})\vec{\nabla}w_R^{-1}, \vec{\nabla}w_R^{-1}\Delta\Psi\rangle.\end{aligned}$$

Hence,

$$\begin{aligned}-\langle\Delta\Psi(v_Lw_R^{-1})w_R^{-1}, \Delta w_R^{-1}\,\Delta\Psi\rangle &\ge \langle\vec{\nabla}\Delta\Psi(v_Lw_R^{-1})w_R^{-1}, \vec{\nabla}[w_R^{-1}\Delta\Psi]\rangle\\ &-|\langle\Delta\Psi\vec{\nabla}(v_Lw_R^{-1})w_R^{-1}, \vec{\nabla}[w_R^{-1}\Delta\Psi]\rangle|\\ &-|\langle\Delta\Psi(v_Lw_R^{-1})\vec{\nabla}w_R^{-1}, \vec{\nabla}[w_R^{-1}\Delta\Psi]\rangle|\,.\end{aligned}\tag{D.3}$$

Consider the first term in the RHS of Eq. (D.3). Utilizing the Hölder and the Young inequalities one writes

$$\begin{aligned}&\langle\vec{\nabla}\Delta\Psi(v_Lw_R^{-1})w_R^{-1}, \vec{\nabla}[w_R^{-1}\Delta\Psi]\rangle\\ &\qquad\ge \langle\vec{\nabla}\Delta\Psi(v_Lw_R^{-3}), \vec{\nabla}\Delta\Psi\rangle - |\langle\nabla\Delta\Psi(v_Lw_R^{-1})w^{-1}, (\nabla w^{-1})\Delta\Psi\rangle|\\ &\qquad\ge \|\vec{\nabla}\Delta\Psi\|^2_{L_2(w_R^{-3/2}v_L^{1/2})} - \frac{C}{2L}\|\nabla\Delta\Psi\|^2_{L_2(w_R^{-3/2}v_L^{1/2})} - \frac{C}{2L}\|\Delta\Psi\|^2_{L_2(w_R^{-3/2}v_L^{1/2})}.\end{aligned}$$

One treats the second term in the RHS of Eq. (D.3) similarly. It holds that

$$
\begin{aligned}
&|\langle \Delta\Psi \vec{\nabla}(v_L w_R^{-1}) w_R^{-1}, \vec{\nabla}[w_R^{-1}\Delta\Psi]\rangle| \\
&\quad = |\langle \Delta\Psi \vec{\nabla}(v_L w_R^{-1}) w_R^{-1}, (\vec{\nabla} w_R^{-1})\Delta\Psi]\rangle| + |\langle \Delta\Psi \vec{\nabla}(v_L w_R^{-1}) w_R^{-1}, w_R^{-1}\vec{\nabla}\Delta\Psi]\rangle| \\
&\quad \leq \frac{C^2}{L^2}\|\Delta\Psi\|^2_{L_2(w_R^{-3/2}v_L^{1/2})} + \frac{C}{2L}\|\vec{\nabla}\Delta\Psi\|^2_{L_2(w_R^{-3/2}v_L^{1/2})} + \frac{C}{2L}\|\Delta\Psi\|^2_{L_2(w_R^{-3/2}v_L^{1/2})}.
\end{aligned}
$$

Moreover, one bounds the third term in the RHS of Eq. (D.3) as follows

$$
\begin{aligned}
&|\langle \Delta\Psi (v_L w_R^{-1}) \vec{\nabla} w_R^{-1}, \vec{\nabla}[w_R^{-1}\Delta\Psi]\rangle| \\
&\quad \leq \frac{C^2}{L^2}\|\Delta\Psi\|^2_{L_2(w_R^{-3/2}v_L^{1/2})} + \frac{C}{2L}\|\vec{\nabla}\Delta\Psi\|^2_{L_2(w_R^{-3/2}v_L^{1/2})} + \frac{C}{2L}\|\Delta\Psi\|^2_{L_2(w_R^{-3/2}v_L^{1/2})}.
\end{aligned}
$$

Hence, combining the above results one bounds Eq. (D.3)

$$
\begin{aligned}
&-\langle \Delta\Psi (v_L w_R^{-1}) w^{-1}, \Delta w_R^{-1}\,\Delta\Psi\rangle \\
&\qquad \geq (1 - \frac{3\,C}{2L})\|\vec{\nabla}\Delta\Psi\|^2_{L_2(w_R^{-3/2}v_L^{1/2})} - (\frac{2C^2}{L^2} + \frac{3C}{2L})\|\Delta\Psi\|^2_{L_2(w_R^{-3/2}v_L^{1/2})}. \qquad \text{(D.4)}
\end{aligned}
$$

Consider the second in the RHS of Eq. (D.2). Using the fact that

$$
\Delta(fgh) = (\Delta f)gh + f(\Delta g)h + fg(\Delta h) + 2(\vec{\nabla} f)\cdot(\vec{\nabla} g)h + 2(\vec{\nabla} f)\cdot g(\vec{\nabla} h) + 2f(\vec{\nabla} g)\cdot(\vec{\nabla} h)
$$

one has

$$
\begin{aligned}
&\langle \Psi\Delta(v_L w_R^{-1}) w_R^{-1}, (\Delta w_R^{-1})\Delta\Psi\rangle \\
&\qquad = \langle \Delta\Psi\Delta(v_L w_R^{-1}) w_R^{-1}, w_R^{-1}\Delta\Psi\rangle + \langle \Psi(\Delta)^2(v_L w_R^{-1}) w_R^{-1}, w_R^{-1}\Delta\Psi\rangle \\
&\quad + \langle \Psi\Delta(v_L w_R^{-1})(\Delta w_R^{-1}), w_R^{-1}\Delta\Psi\rangle + 2\langle \vec{\nabla}\Psi\cdot\vec{\nabla}\Delta(v_L w_R^{-1})\, w_R^{-1}, w_R^{-1}\Delta\Psi\rangle \\
&\quad + 2\langle \vec{\nabla}\Psi\cdot\Delta(v_L w_R^{-1})\,\vec{\nabla} w_R^{-1}, w_R^{-1}\Delta\Psi\rangle + 2\langle \Psi\vec{\nabla}\Delta(v_L w_R^{-1})\cdot\vec{\nabla}\, w_R^{-1}, w_R^{-1}\Delta\Psi\rangle.
\end{aligned}
$$

Using the Hölder and the Young inequalities one obtains

$$
\begin{aligned}
&|\langle \Psi\Delta(v_L w_R^{-1}) w_R^{-1}, (\Delta w_R^{-1})\Delta\Psi\rangle| \\
&\quad \leq \frac{C^2}{L^2}\|\Delta\Psi\|_{L_2(w_R^{-3/2}v_L^{1/2})} + \frac{2C^4}{L^4}\|\Psi\|^2_{L_2(w_R^{-3/2}v_L^{1/2})} + \frac{2C^4}{L^4}\|\Delta\Psi\|^2_{L_2(w_R^{-3/2}v_L^{1/2})} \\
&\qquad\qquad + \frac{2C^3}{L^3}\|\vec{\nabla}\Psi\|^2_{L_2(w_R^{-3/2}v_L^{1/2})} + \frac{2C^3}{L^3}\|\Delta\Psi\|^2_{L_2(w_R^{-3/2}v_L^{1/2})}.
\end{aligned}
$$

Combining the above results one bounds the second in the RHS of Eq. (D.2)

$$-|\langle \Psi\Delta(v_L w_R^{-1})w^{-1}, (\Delta w^{-1})\Delta\Psi\rangle| \geq -\frac{2C^4}{L^4}\|\Psi\|^2_{L_2(w_R^{-3/2}v_L^{1/2})} - \frac{2C^3}{L^3}\|\vec{\nabla}\Psi\|^2_{L_2(w_R^{-3/2}v_L^{1/2})} - (\frac{2C^4}{L^4} + \frac{2C^3}{L^3} + \frac{C^2}{L^2})\|\Delta\Psi\|^2_{L_2(w_R^{-3/2}v_L^{1/2})}. \quad \text{(D.5)}$$

Consider the third term in the RHS of Eq. (D.2). It holds,

$$\begin{aligned}
&\langle \vec{\nabla}\Psi \ \cdot \vec{\nabla}(v_L w_R^{-1})w^{-1}, (\Delta[w_R^{-1}\Delta\Psi])\rangle \\
&= \langle \vec{\nabla}\Delta\Psi \ \cdot \vec{\nabla}(v_L w_R^{-1})w^{-1}, w^{-1}\Delta\Psi\rangle + \langle \vec{\nabla}\Psi \ \cdot \vec{\nabla}\Delta(v_L w_R^{-1})w^{-1}, w^{-1}\Delta\Psi\rangle \\
&+ \langle \vec{\nabla}\Psi \ \cdot \vec{\nabla}(v_L w_R^{-1})(\Delta w^{-1}), w^{-1}\Delta\Psi\rangle + 2\langle \Delta\Psi \ \Delta(v_L w_R^{-1})w^{-1}, w^{-1}\Delta\Psi\rangle \\
&+ 2\langle \vec{\nabla}\Psi \ \Delta(v_L w_R^{-1}) \cdot \vec{\nabla}w_R^{-1}, w_R^{-1}\Delta\Psi\rangle + 2\langle \Delta\Psi \ \vec{\nabla}(v_L w_R^{-1}) \cdot \vec{\nabla}w_R^{-1}, w_R^{-1}\Delta\Psi\rangle.
\end{aligned}$$

Hence,

$$|\langle \vec{\nabla}\Psi \ \cdot \vec{\nabla}(v_L w_R^{-1})w^{-1}, (\Delta w_R^{-1})\Delta\Psi\rangle| \leq \frac{C}{2L}\|\vec{\nabla}\Delta\Psi\|^2_{L_2(w_R^{-3/2}v_L^{1/2})} + \frac{2C^3}{L^3}\|\vec{\nabla}\Psi\|^2_{L_2(w_R^{-3/2}v_L^{1/2})} + (\frac{2C^3}{L^3} + \frac{C}{2L} + \frac{4C^2}{L^2})\|\Delta\Psi\|^2_{L_2(w_R^{-3/2}v_L^{1/2})}.$$

Therefore, one bounds the third term in the RHS of Eq. (D.2) as follows

$$-2\langle \vec{\nabla}\Psi \cdot \vec{\nabla}(v_L w_R^{-1})w^{-1}, (\Delta w^{-1})\Delta\Psi\rangle \geq -\frac{C}{L}\|\vec{\nabla}\Delta\Psi\|^2_{L_2(w_R^{-3/2}v_L^{1/2})} - \frac{4C^3}{L^3}\|\vec{\nabla}\Psi\|^2_{L_2(w_R^{-3/2}v_L^{1/2})} - (\frac{4C^3}{L^3} + \frac{C}{L} + \frac{8C^2}{L^2})\|\Delta\Psi\|^2_{L_2(w_R^{-3/2}v_L^{1/2})}. \quad \text{(D.6)}$$

Putting Eqs. (D.4), (D.5) and (D.6) in Eq. (D.2) culminates in

$$\begin{aligned}
&\langle \Psi, v_L(-w_R^{-1}\Delta)^3\Psi\rangle \\
&\geq (1 - \frac{5C}{2L})\|\vec{\nabla}\Delta\Psi\|^2_{L_2(w_R^{-3/2}v_L^{1/2})} - \frac{6C^3}{L^3}\|\vec{\nabla}\Psi\|^2_{L_2(w_R^{-3/2}v_L^{1/2})} \\
&- \frac{2C^4}{L^4}\|\Psi\|^2_{L_2(w_R^{-3/2}v_L^{1/2})} - (\frac{5C}{2L} + \frac{11C^2}{L^2} + \frac{6C^3}{L^3} + \frac{2C^4}{L^4})\|\Delta\Psi\|^2_{L_2(w_R^{-3/2}v_L^{1/2})}. \quad \text{(D.7)}
\end{aligned}$$

Now we aim to write $\|\vec{\nabla}\Psi\|^2_{L_2(w_R^{-3/2}v_L^{1/2})}$ and $\|\Delta\Psi\|^2_{L_2(w_R^{-3/2}v_L^{1/2})}$ in terms of $\|\vec{\nabla}\Delta\Psi\|^2_{L_2(w_R^{-3/2}v_L^{1/2})}$ and $\|\Psi\|^2_{L_2(w_R^{-3/2}v_L^{1/2})}$. Consider

$$\begin{aligned}
\|\Delta\Psi\|^2_{L_2(w_R^{-3/2}v_L^{1/2})} &= \|w_R^{-3/2}v_L^{1/2}\,\Delta\Psi\|^2_{L_2} = \langle \Delta\Psi, (w_R^{-3}v_L)\Delta\Psi\rangle \\
&\leq |\langle \vec{\nabla}\Psi, [(-\vec{\nabla})(w_R^{-3}v_L)]\Delta\Psi\rangle| + |\langle \vec{\nabla}\Psi, (w_R^{-3}v_L)(-\vec{\nabla}\,\Delta\Psi)\rangle| \\
\leq \frac{C}{L}\,\|\vec{\nabla}\Psi\|_{L_2(w_R^{-3/2}v_L^{1/2})}\,&\|\Delta\Psi\|_{L_2(w_R^{-3/2}v_L^{1/2})} + \|\vec{\nabla}\Psi\|_{L_2(w_R^{-3/2}v_L^{1/2})}\,\|\vec{\nabla}\Delta\Psi\|_{L_2(w_R^{-3/2}v_L^{1/2})} \\
\leq \frac{C}{2L}\,\|\vec{\nabla}\Psi\|^2_{L_2(w_R^{-3/2}v_L^{1/2})} &+ \frac{C}{2L}\,\|\Delta\Psi\|^2_{L_2(w_R^{-3/2}v_L^{1/2})} + \frac{1}{2}\|\vec{\nabla}\Psi\|^2_{L_2(w_R^{-3/2}v_L^{1/2})} + \frac{1}{2}\|\vec{\nabla}\Delta\Psi\|^2_{L_2(w_R^{-3/2}v_L^{1/2})}.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\|\Delta\Psi\|^2_{L_2(w_R^{-3/2}v_L^{1/2})} \\
&\qquad\leq \frac{(1/2+\frac{C}{2L})}{(1-\frac{C}{2L})}\,\|\vec{\nabla}\Psi\|^2_{L_2(w_R^{-3/2}v_L^{1/2})} + \frac{\frac{1}{2}}{(1-\frac{C}{2L})}\|\vec{\nabla}\Delta\Psi\|^2_{L_2(w_R^{-3/2}v_L^{1/2})}. \qquad \text{(D.8)}
\end{aligned}$$

Observe that

$$\begin{aligned}
\|\vec{\nabla}\Psi\|^2_{L_2(w_R^{-3/2}v_L^{1/2})} &= \|w_R^{-3/2}v_L^{1/2}\,\vec{\nabla}\Psi\|^2_{L_2} = \sum_{j=1,2}\langle \partial_j\Psi, w_R^{-3}v_L\partial_j\Psi\rangle \\
&\leq \sum_{j=1,2}\langle |\Psi|, |(-\partial_j)(w_R^{-3}v_L)|\,|\partial_j\Psi|\rangle + |\langle \Psi, (w_R^{-3}v_L)(-\Delta)\Psi\rangle| \\
\leq \frac{C}{L}\|\Psi\|_{L_2(w_R^{-3/2}v_L^{1/2})}&\|\vec{\nabla}\Psi\|_{L_2(w_R^{-3/2}v_L^{1/2})} + \|\Psi\|_{L_2(w_R^{-3/2}v_L^{1/2})}\|\Delta\Psi\|_{L_2(w_R^{-3/2}v_L^{1/2})} \\
\leq \frac{C}{2L}\Big[\|\Psi\|^2_{L_2(w_R^{-3/2}v_L^{1/2})} + \|\vec{\nabla}\Psi\|^2_{L_2(w_R^{-3/2}v_L^{1/2})}\Big] &+ \frac{1}{2}\Big[\|\Psi\|^2_{L_2(w_R^{-3/2}v_L^{1/2})} + \|\Delta\Psi\|^2_{L_2(w_R^{-3/2}v_L^{1/2})}\Big].
\end{aligned}$$

One obtains

$$\begin{aligned}
&\|\vec{\nabla}\Psi\|^2_{L_2(w_R^{-3/2}v_L^{1/2})} \\
&\qquad\leq \frac{(1/2+\frac{C}{2L})}{(1-\frac{C}{2L})}\|\Psi\|^2_{L_2(w_R^{-3/2}v_L^{1/2})} + \frac{1/2}{(1-\frac{C}{2L})}\|\Delta\Psi\|^2_{L_2(w_R^{-3/2}v_L^{1/2})}. \qquad \text{(D.9)}
\end{aligned}$$

Observe that the preceding expression resembles Eq. (D.8) with the same coefficients. Combining Eqs. (D.8) and (D.9) one infers

$$(1-\frac{(1/2+\frac{C}{2L})}{(1-\frac{C}{2L})})\|\nabla\Psi\|^2_{L_2(w_R^{-3/2}v_L^{1/2})}+(1-\frac{1/2}{(1-\frac{C}{2L})})\,\|\Delta\Psi\|^2_{L_2(w_R^{-3/2}v_L^{1/2})}$$
$$\leq\frac{(1/2+\frac{C}{2L})}{(1-\frac{C}{2L})}\|\Psi\|^2_{L_2(w_R^{-3/2}v_L^{1/2})}+\frac{1/2}{(1-\frac{C}{2L})}\,\|\nabla\Delta\Psi\|^2_{L_2(w_R^{-3/2}v_L^{1/2})}\,.$$

Now substitute Eqs. (D.8) and (D.9) in Eq. (D.7). This finishes the proof.

E Proof of Remark 6.4.15

Recall the formula

$$\mathcal{S}_{R,\alpha}(x_1,x_2):=\frac{2(R\sin(\alpha/R)(1-(x_1^2+x_2^2)/4R^2)+x_1\cos(\alpha/R),x_2)}{1+\cos(\alpha/R)+(x_1^2+x_2^2)/4R^2\,(1-\cos(\alpha/R))-(x_1/R)\,\sin(\alpha/R)}\,.$$

It is known that rotations on the sphere correspond to Möbius transformations on the plane via the stereographic mapping. Specifically, let us write $w=x_1+\mathrm{i}x_2$ and consider the transformation

$$w'=\frac{aw+b}{cw+d}=\frac{\cos(\alpha/(2R))w+2R\sin(\alpha/(2R))}{-\frac{\sin(\alpha/(2R))}{2R}w+\cos(\alpha/(2R))},$$

where $a=\cos(\alpha/2R)$, $b=2R\,\sin(\alpha/2R)$, $c=-\sin(\alpha/2R)/2R$ and $d=\cos(\alpha/2R)$ such that $ad-cb=1$. Set

$$N(w):=a(w)+b=(ax_1+b)+\mathrm{i}ax_2,\quad D(w):=cw+d=(cx_1+d)+\mathrm{i}cx_2\,.$$

This implies that

$$w'=\frac{N(w)}{D(w)}=\frac{N(w)\overline{D(w)}}{D(w)^2}=x_1'+\mathrm{i}x_2'\,,$$

where

$$x_1'=\frac{(ax_1+b)(cx_1+d)+acx_2^2}{c^2(x_1^2+x_2^2)+2cdx_1+d^2},\quad x_2'=\frac{x_2}{c^2(x_1^2+x_2^2)+2cdx_1+d^2}\,.$$

Using the facts that $2\sin(\alpha/2R)\cos(\alpha/2R)=\sin(\alpha/R)$ and $\cos(\alpha/2R)^2-\sin(\alpha/2R)^2=\cos(\alpha/R)$ one gets

$$(ax_1+b)(cx_1+d)+acx_2^2=R\,\sin(\alpha/R)(1-\frac{1}{4R^2}\,(x_1^2+x_2^2))+x_1\cos(\alpha/R),$$
$$c^2(x_1^2+x_2^2)+2cdx_1+d^2=\frac{1}{4R^2}\sin(\alpha/2R)^2(x_1^2+x_2^2)+\cos(\alpha/2R)^2-\frac{x_1}{2R}\sin(\alpha/R).$$

From the fact that $\cos(\alpha/2R)^2 = \frac{1}{2}(1+\cos(\alpha/R))$ and $\sin(\alpha/2R)^2 = \frac{1}{2}(1-\cos(\alpha/R))$ one rewrites the second relation

$$\frac{1}{2}\left((1-\cos(\alpha/R)\,\frac{1}{4R^2}(x_1^2+x_2^2)+(1+\cos(\alpha/R))-\frac{x_1}{R}\sin(\alpha/R)\right).$$

This verifies that $(x_1', x_2') = \mathcal{S}_{R,\alpha}(x_1, x_2)$. We write $N(w) =: c_1 w + c_\alpha$, $D(w) =: c_0 w + c_1$, where the indices point to the limits to which the coefficients tend as $R \to \infty$. Clearly, we have

$$c_1 = 1 + O(1/R^2), \quad c_\alpha = \alpha + O(1/R^2), \quad c_0 = O(1/R^2).$$

Then, under the assumptions on the ranges of the parameter, stated in Remark 6.4.15,

$$\frac{N(w)}{D(w)} = \frac{w+\alpha+O(1/R^2)}{1+O(1/R^2)} = w+\alpha+O(1/R^2),$$

which covers the case $a = 0$ from this remark. Let us move on to the derivatives. Consider the expression

$$F(w) := \frac{N(w)}{D(w)} - w - \alpha.$$

Let ∂ denote the derivative w.r.t. x_1 or $\mathrm{i}x_2$. We have

$$\begin{aligned}\partial F(w) &= \frac{\partial N(w)}{D(w)} - \frac{(\partial D(w))N(w)}{D(w)^2} - 1\\ &= \frac{c_1}{c_0 w + c_1} - \frac{c_0(c_1 w + c_\alpha)}{(c_0 w + c_1)^2} - 1 = \frac{c_0 w}{c_0 w + c_1} - \frac{c_0(c_1 w + c_\alpha)}{(c_0 w + c_1)^2} = O(1/R^2).\end{aligned}$$

due to the factor c_0. We note that if we apply a second derivative to F, we will obtain two types of terms: First, those involving $\partial D(w)$, hence $O(1/R^2)$. Second, those involving second derivatives of $N(w)$ or $D(w)$, hence zero. This reasoning extends to higher derivatives.

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