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**Review of the doctoral dissertation “Stochastic quantization and Osterwalder-Schrader axioms for quantum field theory models” by Azam Jahandideh**

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The main aim of the doctoral dissertation by Azam Jahandideh is the construction of the Gibbs measures associated with Euclidean  $P(\varphi)_2$  quantum field theories and the proof that these measures satisfy the Osterwalder–Schrader axioms. This proof is based on stochastic quantization techniques.

The construction and the proof of the basic properties of Gibbs measures related to Euclidean quantum field theory constitute a classical research field, starting from the pioneering results of Edward Nelson, who related many models of (bosonic) quantum fields to probability measures on Banach spaces of distributions. Thanks to these first insights and to the contributions of many authors, it has been possible to construct several different quantum field models in two and three dimensions. More recently, a different approach for handling this problem has been proposed and developed, known as stochastic quantization. Stochastic quantization is based on an idea introduced by Parisi and Wu in the 1980s, and consists in realizing the quantum field Gibbs measure as the invariant measure of a suitable stochastic partial differential equation (SPDE). Thanks to the progress made in the study of singular SPDEs over the last fifteen years, the stochastic quantization program has become a valid and powerful alternative for the construction and analysis of several quantum field models.

The main new result of the thesis is the proof that a Gibbs measure of a scalar bosonic quantum field with a polynomial interaction on  $\mathbb{R}^2$  satisfies the Osterwalder–Schrader axioms using only stochastic quantization methods. Some previous works, within the same research program of stochastic quantization, succeeded in proving the Osterwalder–Schrader axioms for other two-dimensional quantum field models, such as  $\exp(\varphi)_2$ ,  $\cosh(\varphi)_2$ , and  $\cos(\varphi)_2$  (in the latter case using the variational formulation of stochastic quantization and only for small coupling constants). In all these cases, the proof was essentially based on the uniqueness of the Gibbs measure associated with the quantum model. In the case of the polynomial models considered in the thesis, this property does not hold: for a generic polynomial of degree  $2n$ , one may expect up to  $n$  different pure ergodic Gibbs measures. Some partial results in the direction of proving the Osterwalder–Schrader axioms for  $P(\varphi)_2$  quantum field models using only stochastic quantization methods were obtained, in both two and three dimensions, by proving exponential integrability of the measure, reflection positivity of the associated process, invariance with respect to space-time translations, and, separately, invariance with respect to rotations. Howe-

ver, all these results, prior to the present work (and also subsequent works in the case of non-uniqueness of the Gibbs measures), fail to establish all these properties simultaneously. Consequently, in the absence of uniqueness of the invariant measure of the stochastic quantization SPDE, the previous results do not prove the existence of a Gibbs measure satisfying all the Osterwalder–Schrader axioms.

The thesis under review solves this problem by exploiting a very elegant and original idea, namely the definition of a stochastic quantization SPDE on the two-dimensional sphere  $\mathbb{S}_R$  of radius  $R$ . The invariant measures of these SPDEs enjoy invariance with respect to the (four-dimensional) group  $O(3)$  and satisfy reflection positivity (with respect to reflection across a fixed axis). These invariant measures are then mapped onto  $\mathbb{R}^2$  via stereographic projection, and finally the limit as the radius  $R \rightarrow \infty$  is taken. The resulting measure is both reflection positive and invariant under translations and rotations, since the projected  $O(3)$  symmetries on  $\mathbb{S}_R$  converge to the group of roto-translations on  $\mathbb{R}^2$ . Since all measures before taking the limit satisfy a uniform exponential integrability bound, the Osterwalder–Schrader axioms hold (with the possible exception of ergodicity with respect to time translations, which in general does not hold unless additional assumptions on the model are imposed).

The thesis is structured into six chapters and several appendices. The first chapter is an introduction to the work, presenting the main problems addressed in the thesis. It contains, in particular, a complete and precise review of the literature on stochastic quantization SPDEs of polynomial-type models in two and three dimensions.

Chapter 2 includes several preliminary technical results needed throughout the thesis. This chapter is well written and explains in a clear and detailed manner the technical tools used in Euclidean quantum field theory and stochastic quantization.

Chapter 3 contains the construction of the  $P(\varphi)_2$  measure with base space given by the sphere  $\mathbb{S}_R$ . The construction follows the classical argument first proposed by Edward Nelson for the construction of a similar measures on a compact flat space. The novelty of this chapter lies in the use of new stochastic estimates involving the free field on the sphere, proved in Appendix B.

Chapter 4 contains the proof of existence and uniqueness of global-in-time solutions to the regularized stochastic quantization SPDE on the sphere  $\mathbb{S}_R$ . Differently from the standard approach, where the regularization is obtained by mollifying the noise, here the regularization results from a modification of the linear part of the SPDE, in which the usual Laplacian on the sphere is replaced by a sixth-order operator. This choice is particularly suitable, as it preserves the form of the invariant measure while avoiding the use of non-local operators. The proof is based on the analysis of the regularity of the solution to the linear equation, namely the Ornstein–Uhlenbeck process on the sphere, and exploits properties of spherical harmonics. The argument then proceeds via the classical Da Prato–Debusche method and the derivation of suitable a priori estimates. The results in this chapter are original, due to both the spherical setting and the non-standard regularization employed.

Chapter 5 presents the first main result of the thesis, namely the proof of uniform a priori estimates for the field obtained by mapping on  $\mathbb{R}^2$  the invariant measure of the stochastic quantization SPDE on  $\mathbb{S}_R$  through the stereographic projection. Compared with the usual estimates for the stochastic quantization SPDE on  $\mathbb{R}^2$  and those obtained on  $\mathbb{S}_R$  in the previous chapter, these estimates involve an additional

weight  $w_R$  (due to the stereographic projection) and an additional  $H^2$ -type norm, which arises from the particular regularization chosen. By a careful and clever use of the properties of the weight  $w_R$ , it is shown that the invariant measure on the sphere, once projected onto the plane, converges to a limiting measure.

Finally, Chapter 6 is devoted to the proof that the Schwinger functions associated with the limiting measure constructed in the previous chapter satisfy the Osterwalder–Schrader axioms. The two most delicate axioms to establish are reflection positivity and invariance with respect to the Euclidean group on  $\mathbb{R}^2$ . Reflection positivity is proved by showing that a further regularized measure (obtained by applying an operator with a compactly supported integral kernel) converges to the invariant measure of the singular stochastic quantization SPDE on  $\mathbb{S}_R$ , projected onto  $\mathbb{R}^2$ , which is itself reflection positive with respect to reflections of the sphere  $\mathbb{S}_R$ . The invariance of the Gibbs measure under the Euclidean group on  $\mathbb{R}^2$  is then obtained by proving that the action of the group  $O(3)$  on  $\mathbb{S}_R$ , mapped onto  $\mathbb{R}^2$  via stereographic projection, converges to the action of the Euclidean group as  $R \rightarrow +\infty$ .

As a general comment, the thesis is very well written and provides a clear, self-contained, and valuable introduction to stochastic quantization and its relation with Euclidean quantum field theory, particularly with regard to the Osterwalder–Schrader axioms. This demonstrates the candidate’s deep knowledge of constructive quantum field theory and of the related mathematical tools and techniques. The thesis contains many new results in Chapters 3, 4, 5, and 6 within the literature on stochastic quantization, as discussed above. Of particular importance are the contributions and new ideas introduced in Chapters 5 and 6, which have been the subject of a scientific article published in the journal *Annales Henri Poincaré*. To the best of my knowledge, this work currently provides the only complete proof, using solely stochastic quantization and stochastic analysis techniques, that  $P(\varphi)_2$  models satisfy the Osterwalder–Schrader axioms when the associated Gibbs measure is not unique.

For all these reasons, I believe that the doctoral dissertation under review meets all the requirements stated in Article 187 of the Act of July 20, 2018, *Law on Higher Education and Science*. The dissertation fully justifies the awarding of a doctoral degree in mathematics to Azam Jahandideh.

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