REPORT ON THE PH.D. THESIS "TOPICS ON TOPOLOGICAL ROBOTICS: ON TOPOLOGICAL COMPLEXITY OF $K(G,1)$-SPACES AND EFFECTIVE TOPOLOGICAL COMPLEXITY" BY ARTURO ESPINOSA BARO

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Let X be a topological space. The topological complexity of X, denoted $TC(X)$, is a numerical invariant defined by M. Farber in [5]. It is a more complicated cousin of the Lusternik-Schnirelman category. The original definition was motivated by the fact, that $TC(X)$ equals the minimal number of continuous rules in a motion planning algorithm in X. But more importantly, at least from the point of view of theoretical mathematics, topological complexity is a homotopy invariant. Hence $TC(X)$ can be studied using various tools from algebraic topology. Moreover, if G is a discrete group, one can define $TC(G) = TC(K(G,1))$. Today topological complexity is a well established subfield of topology with leading mathematicians working in the field.

To proceed, we need to introduce one technical notion. Let $f\colon X \to Y$ be a continuous map between topological spaces. Suppose

$$Y = \bigcup_{i=0}^{n} U_i$$

is an open cover of Y by sets U_i that can be send back to X by continuous maps $s_i\colon U_i \to X$ such that $f \circ s_i$ is homotopic to the inclusion $U_i \hookrightarrow Y$ for $0 \leq i \leq n$. The sectional category of f, denoted $\mathrm{secat}(f)$, is the minimal number of sets in an open cover of Y satisfying the above condition. One of equivalent definitions of $TC(X)$ says that

$$TC(X) = \mathrm{secat}(\Delta_X \colon X \to X \times X),$$

where Δ_X is the diagonal map.

Since sectional category is a homotopy invariant, one can define $\operatorname{secat}(f)$ for homomorphisms of discrete groups: if $f\colon H \to G$ is a homomorphism, then

$$\operatorname{secat}(f) = \operatorname{secat}(f^*\colon K(H,1) \to K(G,1)),$$

where f^* realizes f on the fundamental groups. In this language one can say that $TC(G)$ is the sectional category of the diagonal inclusion $\Delta_G\colon G \hookrightarrow G \times G$.

The thesis is mostly devoted to executing the following project: what happens, if instead of the sectional category of the inclusion $\Delta_G\colon G \hookrightarrow G \times G$, one looks at $\operatorname{secat}(i\colon H \hookrightarrow G)$, for arbitrary inclusion i of arbitrary H? In particular, what known results on $TC(G)$ generalize to the new situation?

This project is consistently implemented in Chapters 4, 5, and 6 of the thesis. The main result of Chapter 4 is Theorem 4.1.4. It gives an equivalent definition of $\operatorname{secat}(i\colon H \hookrightarrow G)$ in terms of classifying spaces for actions with stabilizers conjugated to subgroups in H. It is a straightforward generalization of a fact proven for $TC(G)$ in [6] with the help of Bredon cohomology. The author of the thesis and his collaborators suggest a slightly different approach to this topic: instead of using Bredon cohomology, one can look at the seemingly easier Adamson cohomology theory. They develop and describe a few supplementing results and gadgets, some related to the thesis, and some of general interest. For example, they define a relative Berstein-Schwartz class, a related spectral sequence, and show (rather straightforward) relation between Adams and Bredon cohomology.

Chapter 5 focuses on inequalities between the cohomological dimension and sectional category. Again, the inequalities generalize known inequalities for $TC(G)$. In this chapter, there are two main results. First is Theorem 5.1.7 where the case when $i(H)$ is normal in G is analyzed. I like that the proof uses the relative Berstein-Schwartz class defined in Chapter 4. The other result is Theorem 5.3.3, which gives a refined lower bound on $\operatorname{secat}(i\colon H \hookrightarrow G)$ if G is geometrically finite (but $i(H)$ is arbitrary). The proof is quite involved and uses spectral sequences, however, it is heavily based on the strategy already used in [7].

The main result of Chapter 6 defines $\mathrm{secat}(i\colon H \hookrightarrow G)$ in terms of so called $\mathcal{A}$-genus. There are (I think new) inequalities that relate cohomological dimension and the sequential topological complexity (which is a more general variant of topological complexity).

Chapter 7 is not related to $\mathrm{secat}(i\colon H \hookrightarrow G)$. Here the author studies properties of effective topological complexity $TC^{G,\infty}(X)$, where G is a group acting on X. It is a notion defined in [4] to incorporate symmetries of X to the definition in such a way, that $TC^{G,\infty}(X) \leq TC(X)$ (note that one cannot simply define $TC^{G,\infty}(X) = TC(X/G)$). A novel part of Chapter 7 is the definition of the k-effective Lusternik-Schnirelman category and Theorem 7.4.2 which relates effective topological complexity to this effective category. The proofs are relatively straightforward, but the difficulty lies in introducing the right definitions. Then the author applies this theorem to particular cases when the orbit map $X \to X/G$ has a strict section or is a fibration. In these situations, some concrete computations can be done.

The thesis under review is quite broad. The part on $\mathrm{secat}(i\colon H \hookrightarrow G)$ touches several important aspects of the subject. Chapter 7 is a nice addition. The author had to master an important and complicated toolkit of algebraic topology, like different versions of cohomology theory, classifying spaces and spectral sequences. I am sure that this gives him a good base for future research.

The weak side of this work is that many results closely follow results and techniques already developed by other mathematicians or collaborators of the author. Most interesting applications of obtained results are given to (sequential) topological complexity. There are few (and rather artificial) examples where one uses embeddings $i\colon H \hookrightarrow G$ other than the diagonal embedding. Interesting examples of inclusions for which sectional category is computable would make the results of this dissertation much stronger.

The thesis is well organized. I appreciate a detailed introduction to the subject and preliminaries in Part I. Chapters 4, 5, and 7 consist of papers [3, 1, 2], with some additional bits. [3] is already published and the other papers are published on the ArXiv. The results of Chapter 6, according to the author, are about to be published. The "mathematical" parts of the thesis are well written and I had no problems understanding the arguments. However, "narrative" parts, like introductions, have flaws. For example, the author uses the expression "as such" in many random and inappropriate places. There are several mistakes and inaccuracies throughout the whole dissertation, but all are minor.

Chapters 4, 5, and 7 are based on joint work with more senior mathematicians. Chapter 6 is a solo work of the candidate. Chapter 6 and the quality of the solo parts (introduction and some subsections of Chapters 4, 5, and 7) convince me that the candidate did and can conduct independent research.

Overall, the thesis is a solid step in understanding topological complexity and its generalizations. In my opinion, the dissertation matches all the formal and usual requirements for awarding Arturo Espinosa Baro the doctoral degree in mathematics.

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