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Referee's Report for Awarding the Degree of Habilitated Doctor to Dr. Sebastian Król

Brief Biography of Dr. Sebastian Król:

Dr. Sebastian Król obtained his Master degree at the University of Wrocław (Poland) in 2004 under supervision of Prof. A. Raczyński. He got PhD degree in 2010 at the Nicolaus Copernicus University in Toruń (Poland) under supervision of Prof. dr hab. Yuri Tomilov: Title of the thesis *Perturbation and spectral properties of generators of C_0-semigroups*. Then he intensively was involved in scientific and academic activities at the Universities: Nicolaus Copernicus University in Toruń; Dresden University of Technology (Germany); Adam Mickiewicz University, Poznań, Poland having various positions. Currently he is assistant professor at the Department of Operator Theory of the Faculty of Mathematics and Computer Science, Adam Mickiewicz University, Poznań, Poland.

General survey of the papers [K1-K6]. Motivation:

Title of the series of thematically related articles is *Maximal regularity of linear evolution equations in Banach space.*

The main goal of the series of papers [K1-K6] constituting the achievement was to develop methods and tools for the study of problems directly related to the regularity theory of abstract evolution equations in Banach spaces. The first order abstract Cauchy problem (ACP shortly) has the form:

$$f(t) = \begin{cases} \dot{u} + Au = f \ \ on \ \ [0, \infty), \\ u(t) = x, . \end{cases}$$

where A denotes a closed, linear operator on a Banach space X and $x \in X$.

The research carried out in the papers [K1-K3] is in correspondence with the extrapolation theory of Rubio de Francía. They are fulfilled jointly with R. Chill. In fact, they are connected with the following topic: studying of the extrapolation of ACP, i.e., the description of the function spaces Φ such that the maximal L^p-regularity automatically implies Φ-maximal regularity. Moreover, characterize the space X_Φ of initial states $x \in X$ such that if the operator A has Φ-maximal regularity, then it has Φ-maximal regularity for each state $x \in X_\Phi$.

The papers $[K4 - K6]$ are a part of research directions that are related to sufficient conditions for the L^p-maximal regularity of ACP.

The papers [K1-K3] are dedicated to singular integrals, while papers [K4-K6] are devoted to the study of Fourier multipliers and related subjects, e.g., estimates of the Littlewood-Paley functions. More precisely, the following topics are studied by the author: the theory of Fourier multipliers ([K4,K5]), Littlewood-Paley theory ([K5]), theory of abstract second-order Cauchy problems ([K6]), theory of singular integral operators ([K1,K3]), Rubio de Francía extrapolation theory ([K1,K3]), interpolation theory ([K2]), the regularity theory of the abstract evolution equations ([K1,K3,K6]). All of these topics are in interest of the researchers of modern analysis, more precisely, Harmonic Analysis. problems are one of the conceptual problems of modern analysis.

Detailed survey of the papers [K1-K6]:

I underline some of the results of the candidate from the Harmonic Analysis viewpoint:

In 1986 E. Sawyer established the one-weight inequality for one–sided maximal operators under the one–sided Muckenhoupt conditions A_p^+ and A_p^-. One–sided Muckenhoupt classes are larger than classical Muckenhupt classes A_p. Later, one–sided Calderón–Zygmund operators were studied by H. Aimar, L. Forzani, and F. J. Martín-Reyes from one–weighed viewpoint. The paper [K1] can be considered as a continuation of the research carried out in those articles for one–sided maximal and singular integrals satisfying Sawyer's one–sided conditions on weights. The kernels of the operators satisfy relaxed Dini conditions. They apply the weighted estimates to extrapolation of L^p maximal regularity of first order, second order and fractional order Cauchy problems to weighted rearrangement-invariant Banach function spaces. In particular, Dr. Król provides extensions as well as a unification of recent results due to Auscher and Axelsson, and Chill and Fiorenza.

[K2] is dedicated to the study real interpolation spaces on the basis of general Banach function spaces and, in particular, weighted rearrangement invariant Banach function spaces. The authors showed equivalence of the trace method and the K-method, identify real interpolation spaces between a Banach space and the domain of a sectorial operator, and reprove an extension of Dores theorem on the boundedness of H^∞ functional calculus to this general setting.

The paper [K3] is devoted to extrapolation of L^p maximal regularity for second order Cauchy problems. In particular, the authors showed that if the second order problem $\ddot{u} + B\dot{u} + Au = f$ has L^p- maximal regularity for some $p \in (0, \infty)$, then it has E_w- maximal regularity for every rearrangement invariant Banach function space E with Boyd indices p_E, q_E from $(1, \infty)$ and for every Muckenhoupt weight $w \in A_{p_E}$.

It should be emphasized that the general extrapolation theorem for singular integral operators with operator-valued kernels (see Theorem 4.3 in [Chill-Fiorenza, 2014]), which goes back to [Rubio de Francía, Ruiz and Torrea, 1986], and which was used in the proof of the extrapolation of maximal regularity for the first order problem, cannot directly be applied for the second order problem formulated above. Therefore, in the mentioned joint paper the authors state and prove a new extrapolation theorem for singular integral operators with operator–valued kernels, which also slightly improves the result by Rubio de Francía, Ruiz and Torrea (1986) in the case of weighted Lebesgue spaces.

In the interesting paper [K4] Dr. Sebastian Król gives T. Hytönens embedding theorem, which allows the author to extend and unify several sufficient conditions for a function to be a Fourier multiplier on the real Hardy spaces.

Multipliers on the real Hardy spaces $H^p(\mathbf{R^n})$, $p \in (0, \infty)$, have attracted considerable interest of researchers in the last two decades (see the papers by Hytönen, Kolomoitsev and references therein) which goes back to Hörmander (1960), Calderón and Torchinsky (1977), Kurtz and Wheeden (1979), De Michel and Inglis (1980), Miyachi (1980, 1981), Baerstein and Sawyer (1985) (see also the monograph by Strömberg and Torchinsky (1989).

The obtained multiplier conditions in the mentioned work [K4] are two-fold. This corresponds to two different approaches of measuring the smoothness of multiplier functions which are applied in that paper. The diverse character is well illustrated by considering the minimal number of derivatives for checking either of the conditions. More precisely, as a consequence of the main Theorem of [K4] which is obtained by adaptation of Hytönen's (2004) ∞ -norm the author gets the following sufficient condition for a function to be a Fourier multiplier on the real Hardy spaces $H^p(\mathbf{R^n})$ for each n$\in \mathbf{N}$.

Fourier multipliers in weighted L^p spaces are studied in interesting work [K5] by Dr. Król. I would evaluate the results of this paper (like others) as **deep and quite strong**. In particular, the author provides a complement to the classical results on Fourier multipliers on L^p spaces. It is proved that prove that if $q \in (1,2)$, and a function $m : R \mapsto C$ is of bounded q-variation uniformly on the dyadic intervals in R, i.e., $m \in V_q(\mathcal{D})$, then m is a Fourier multiplier on $L^p(R, wdx)$ for every $p \geq q$ and every weight w satisfying Muckenhoupt's $A_{p/q}$-condition. The author also obtained a higher dimensional counterpart of this result as well as of a result by E. Berkson and T. A. Gillespie including the case of the $V_q(\mathcal{D})$ spaces with $q > 2$. New weighted estimates for modified Littlewood–Paley functions are also provided.

In another important work [K6] of Dr. Król new characterisations of generators of cosine functions and C_0-groups on UMD spaces and their applications to some classical problems in cosine function theory. In particular, the author proved that on UMD spaces, generators of cosine functions and C_0-groups can be characterised by means of a complex inversion formula. This allows the author to provide a strikingly elementary proof of Fattorinis result on square root reduction for cosine function generators on UMD spaces. Moreover, Dr. Król provides a cosine function analogue of McIntoshs characterisation of the boundedness of the H^∞ functional calculus for sectorial operators in terms of square function estimates. Another valuable result of this paper says that the class of cosine function generators on a Hilbert space is exactly the class of operators which possess a dilation to a multiplication operator on a vector-valued L^2 space. Finally, the author proved a cosine function analogue of the Gomilko-Feng-Shi characterisation of C_0-semigroup generators and apply it to answer in the affirmative a question by Fattorini on the growth bounds of perturbed cosine functions on Hilbert spaces

Conclusion:

• Dr. Król is an active mathematician. He obtained deep and valuable results in modern analysis; especially, his results are important from the Harmonic Analysis viewpoint.
• Since 2009 Dr. Król published more than 10 papers in high rank mathematical journals.
• He was participant of many grant projects which are important in the area.
• Dr. Sebastian Król participated and delivered talks in 14 international conferences and workshops. His results were delivered in seminars of various scientific centers.
• He was awarded by Humboldt Research Fellowship for Postdoctoral Researchers (2013 - 2015); by the award of Rector of Nicolaus Copernicus University in Toruń for scientific achievements (2010/11).
• Dr. Sebastian Król's academic activity is also impressive. Since 2017 he was supervisor of 4 Master theses; since 2015 he delivered various courses of lectures in Poland.

Summarizing everything mentioned above, I have an excellent opinion oh Dr. Król's scientific and academic achievements. He made considerable contribution in the study of modern conceptual problems of modern analysis. He can be considered as an independent researcher with a very good potential in research and teaching. **I strongly recommend awarding the habilitation to Dr. Sebastian Król.**

A. Meskhi

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